# The Smoothing Property for Regular Splittings

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## Summary

In this paper we discuss convergence of multigrid methods with respect to the *maximum norm* for 2D elliptic boundary value problems. Our analysis uses Hackbusch's framework based on the Smoothing Property and Approximation Property (cf. [4]). We present a rather general framework for establishing the Smoothing Property in the maximum norm. The analysis fits in nicely with the classical theory of diagonally dominant matrices and of M-matrices.

## 1. Introduction

We consider multigrid methods applied to standard linear finite element discretizations of second order elliptic boundary value problems in 2D. It is well-known that the convergence of these methods can be analyzed using Hackbusch's framework based on the Approximation Property and Smoothing Property (cf. [4]).
In this paper we study convergence with respect to the *maximum norm*. Some first results about multigrid convergence in the maximum norm are given in [8,9]. Below in §4 we briefly discuss the Approximation Property in the maximum norm. A detailed analysis concerning this subject can be found in [9]. The main topic of this paper is to present a rather general framework for establishing the Smoothing Property in the maximum norm. We show connections between nonexpansive splittings (i.e. $A = W - R$ with $\|W^{-1}R\|_\infty \leq 1$) and smoothers and between (weak) regular splittings and smoothers.
Our analysis fits in nicely with the classical theory of diagonally dominant matrices and of M-matrices. In particular our analysis can be applied to the ILU-factorization of an M-matrix. Our results differ from the results concerning the Smoothing Property in [4], [11,12] because we use the maximum norm instead of the energy norm or the euclidean norm and we do not need any symmetry conditions.

## 2. Continuous problem, discretization and two-grid method

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain. We consider the following second order variational boundary value problem with homogeneous Dirichlet boundary conditions (and $f \in L^2(\Omega)$):

$$\begin{cases} \text{find } \ \varphi^* \in H_0^1(\Omega) \ \text{ such that for all } \ \psi \in H_0^1(\Omega) \ : \\ \sum_{|\alpha|,|\beta|\leq 1} \int_\Omega a_{\alpha\beta}(x)\,(D^\alpha \varphi^*(x))\,(D^\beta \psi(x))\,dx = \int_\Omega f(x)\,\psi(x)\,dx \ . \end{cases} \tag{2.1}$$

We use the following notation

$$a(\varphi,\psi) = \sum_{|\alpha|,|\beta|\leq 1} \int_\Omega a_{\alpha\beta}(x)\,(D^\alpha \varphi)\,(D^\beta \psi)\,dx \quad (\varphi,\psi \in H_0^1(\Omega)) \ , \tag{2.2}$$

and we assume that the coefficients $a_{\alpha\beta}$ are such that this bilinear form is bounded and elliptic on $H_0^1(\Omega)$.

We use standard linear finite element discretizations on a sequence of nested quasi-uniform triangulations $\{\mathcal{T}_k \mid k \in \mathbb{N}_0\}$. This results in a sequence of nested finite dimensional function spaces

$$\Phi_0 \subset \Phi_1 \subset \ldots \subset \Phi_k \subset \ldots \subset H_0^1(\Omega) \ ,$$

with

$$\Phi_k := \{\varphi \in C(\bar{\Omega}) \mid \varphi \ \text{ is linear on every } \ T \in \mathcal{T}_k \ , \ \ \varphi = 0 \ \text{ on } \ \partial\Omega\} \ .$$

With $\mathcal{T}_k$ (or $\Phi_k$) there corresponds a mesh size parameter $h_k$. The collection of interior grid points in $\mathcal{T}_k$ is denoted by $\{x_k^i\}_{i \in J_k}$ for some indexset $J_k$ with $\# J_k =: n_k = O(h_k^{-2})$. We use the notation $U_k = \mathbb{R}^{n_k}$. The standard basis in $\Phi_k$ is given by the functions $\varphi_k^i \in \Phi_k$ which satisfy $\varphi_k^i(x_k^j) = \delta_{ij} \ \ (i,j \in J_k)$. This induces the natural bijection

$$P_k : U_k \to \Phi_k \ , \quad P_k(u) = \sum_{i \in J_k} u_i \varphi_k^i \ . \tag{2.3}$$

On $U_k$ we use a scaled euclidean inner product:

$$< u, v >_k = h_k^2 \sum_{i \in J_k} u_i v_i \ . \tag{2.4}$$

Below adjoints are always defined with respect to this inner product on $U_k$ and the $L^2$-inner product on $\Phi_k$. The maximum norm on $U_k$ is denoted by $\|\cdot\|_\infty$. The norms $\|\cdot\|_\infty$ (on $U_k$) and $\|\cdot\|_{L^\infty}$ (on $\Phi_k$) induce associated operator norms which are denoted by $\|\cdot\|_\infty$.

Galerkin discretization results in a stiffness matrix $L_k : U_k \to U_k$ defined by

$$< L_k u, v >_k = a(P_k u, P_k v) \quad \text{for all} \quad u, v \in U_k \ . \tag{2.5}$$

In the two-grid (and multigrid) method we use a prolongation $p = p_k : U_{k-1} \to U_k$ and restriction $r = r_k : U_k \to U_{k-1}$ defined by

$$p = P_k^{-1} P_{k-1} \ , \quad r = p^* \ . \tag{2.6}$$

The iteration matrix of the smoothing method (cf. §3) is denoted by $S_k$. The standard two-grid method, with $\nu$ pre-smoothing iterations, for solving a system $L_k u_k = b_k$ then has the following iteration matrix:

$$T_k(\nu) = (I - pL_{k-1}^{-1} r L_k)\, S_k^\nu = (L_k^{-1} - pL_{k-1}^{-1} r)\, L_k\, S_k^\nu \ . \tag{2.7}$$

In §3 we give a detailed analysis of the following Smoothing Property (cf. [4]):

$$\|L_k S_k^\nu\|_\infty \le \xi_0(\nu)\, h_k^{-2} \quad \text{with} \quad \xi_0(\nu) \to 0 \quad \text{for} \quad \nu \to \infty \ . \tag{2.8}$$

In §4 we briefly discuss the following Approximation Property:

$$\|L_k^{-1} - pL_{k-1}^{-1} r\|_\infty \le C\, |\ln h_k|^2\, h_k^2 \ . \tag{2.9}$$

Combination of (2.8) and (2.9) yields a bound for the two-grid contraction number with respect to the maximum norm:

$$\|T_k(\nu)\|_\infty \le \xi_0(\nu)\ |\ln h_k|^2 \ . \tag{2.10}$$

## 3. The Smoothing Property

The usual technique for proving the Smoothing Property requires symmetry or a nearly symmetric situation, and yields results in the energy norm or in the euclidean norm. We refer to Hackbusch [4] and Wittum [11,12] where smoothing and the construction of smoothers are discussed in a general framework.
In this section we present a rather general framework for establishing the Smoothing Property with respect to the *maximum norm*. Our analysis is based on a new technique, that was first used in [7], which does not require symmetry. One of the main results of this section (cf. Corollary 3.6 and Criterion 3.9) is a connection between weak regular splittings and smoothing methods. We note that our analysis fits in nicely with the theory of diagonally dominant matrices and of M-matrices.

We recall that the approach to the Smoothing Property used by Hackbusch [4] and Wittum [11,12] is based on the following elementary lemma:

*If $A$ is symmetric and $\sigma(A) \subset [0,1]$ then*
$$\|A(I-A)^\nu\|_2 \le \eta_0(\nu) := \frac{\nu^\nu}{(\nu+1)^{\nu+1}} \quad (\sim \frac{1}{e\nu} \quad \textit{for} \quad \nu \to \infty) \ .$$

Also our analysis below is based on an elementary lemma (cf. Lemma 3.2) where instead of the condition "$A$ is symmetric and $\sigma(A) \subset [0,1]$" we use the condition "$\|A\|_\infty \le 1$".
The following function turns out to be important for our analysis:

$$\xi_0(\nu) := 2^{1-\nu} \begin{pmatrix} \nu \\ [\frac{1}{2}\nu] \end{pmatrix} \quad (\nu \in \mathbb{N}_0) . \tag{3.1}$$

**Lemma 3.1.** *The following holds for* $\nu \geq 1$*:*

$$\sqrt{2} \leq \sqrt{\nu}\, \xi_0(\nu) < 2\sqrt{\frac{2}{\pi}} \quad \textit{if } \nu \textit{ is even} \tag{3.2.a}$$

$$\sqrt{2} \leq \sqrt{\nu+1}\, \xi_0(\nu) < 2\sqrt{\frac{2}{\pi}} \quad \textit{if } \nu \textit{ is uneven} \tag{3.2.b}$$

$$\lim_{\nu\to\infty} \sqrt{\nu}\, \xi_0(\nu) = 2\sqrt{\frac{2}{\pi}} . \tag{3.2.c}$$

*Proof.* Define the following sequences for $k \in \mathbb{N}_0$:

$$a_k = \begin{pmatrix} 2k \\ k \end{pmatrix} \sqrt{k}\, 2^{-2k} , \quad b_k = \int_0^{\frac{\pi}{2}} \sin^k(x)\, dx . \tag{3.3}$$

Elementary analysis yields that $(b_k)_{k\geq 0}$ is monotonically decreasing and

$$b_k = \frac{k-1}{k} b_{k-2} , \quad b_{2k+1} = (2^k k!)^2 / (2k+1)! , \quad b_{2k} = \tfrac{1}{2}\pi (2k)! / (2^k k!)^2 .$$

From this it follows that for $k \geq 1$

$$1 < \frac{b_{2k}}{b_{2k+1}} = \frac{2k+1}{2k} \frac{b_{2k}}{b_{2k-1}} < 1 + \frac{1}{2k} ,$$

and thus $\lim_{k\to\infty} \dfrac{b_{2k}}{b_{2k+1}} = 1$. Using $\frac{b_{2k}}{b_{2k+1}} = \pi(1+\frac{1}{2k})\, a_k^2$ for $k \geq 1$ yields

$$\lim_{k\to\infty} a_k = \frac{1}{\sqrt{\pi}} . \tag{3.4}$$

Furthermore $(a_k)_{k\geq 0}$ is monotonically increasing, so

$$\tfrac{1}{2} \leq a_k < \frac{1}{\sqrt{\pi}} \quad \text{for all } k \geq 1 . \tag{3.5}$$

Also

$$\sqrt{\nu}\, \xi_0(\nu) = 2\sqrt{2}\, a_{\frac{1}{2}\nu} \quad \text{if } \nu \text{ is even} \tag{3.6}$$

and

$$\sqrt{\nu+1}\, \xi_0(\nu) = 4\sqrt{\nu+1}\, 2^{-(1+\nu)} \tfrac{1}{2} \begin{pmatrix} \nu+1 \\ \frac{1}{2}(\nu+1) \end{pmatrix} = 2\sqrt{2}\, a_{\frac{1}{2}(\nu+1)} \quad \text{if } \nu \text{ is uneven} . \tag{3.7}$$

From (3.5), (3.6), (3.7) the results in (3.2.a–c) easily follow. □

**Lemma 3.2.** *Let $B$ be a square matrix with $\|B\|_\infty \leq 1$. Then the following holds for $\nu \in \mathbb{N}_0$:*

$$\|(I-B)\,(\tfrac{1}{2}(I+B))^\nu\|_\infty \leq \xi_0(\nu)\,. \tag{3.8}$$

*Proof.* Note that

$$(I-B)\,(I+B)^\nu = (I-B)\sum_{k=0}^{\nu}\binom{\nu}{k}B^k = I - B^{\nu+1} + \sum_{k=1}^{\nu}\left(\binom{\nu}{k}-\binom{\nu}{k-1}\right)B^k\,.$$

So

$$\|(I-B)\,(I+B)^\nu\|_\infty \leq 2 + \sum_{k=1}^{\nu}\left|\binom{\nu}{k}-\binom{\nu}{k-1}\right|.$$

Using $\binom{\nu}{k} \geq \binom{\nu}{k-1} \Leftrightarrow k \leq \frac{1}{2}(\nu+1)$ and $\binom{\nu}{k} = \binom{\nu}{\nu-k}$ we get

$$\sum_{k=1}^{\nu}\left|\binom{\nu}{k}-\binom{\nu}{k-1}\right| =$$

$$= \sum_{1}^{[\frac{1}{2}(\nu+1)]}\left(\binom{\nu}{k}-\binom{\nu}{k-1}\right) + \sum_{[\frac{1}{2}(\nu+1)]+1}^{\nu}\left(\binom{\nu}{k-1}-\binom{\nu}{k}\right)$$

$$= \sum_{1}^{[\frac{1}{2}\nu]}\left(\binom{\nu}{k}-\binom{\nu}{k-1}\right) + \sum_{m=1}^{[\frac{1}{2}\nu]}\left(\binom{\nu}{m}-\binom{\nu}{m-1}\right)$$

$$= 2\sum_{k=1}^{[\frac{1}{2}\nu]}\left(\binom{\nu}{k}-\binom{\nu}{k-1}\right) = 2\left(\binom{\nu}{[\frac{1}{2}\nu]}-\binom{\nu}{0}\right).$$

So $\|(I-B)\,(\frac{1}{2}(I+B))^\nu\|_\infty \leq 2^{-\nu}\left(2+2\left(\binom{\nu}{[\frac{1}{2}\nu]}-1\right)\right) = \xi_0(\nu)$. □

*Remark 3.3.* The estimate in (3.8) is sharp: Let $B$ be the following $n \times n$ matrix

$$B = \begin{bmatrix} 0 & 1 & & \emptyset & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & \emptyset & & \ddots & 1 \\ & & & & 0 \end{bmatrix},$$

then for $\nu \leq n-2$ equality holds in (3.8).
We also note that the analysis above holds for other norms too (cf. [7]).

*Definition 3.4.* $A = W - R$ with $W$ regular is called a *nonexpansive splitting* if $\|W^{-1}R\|_\infty \leq 1$ holds.

In the remainder we only consider splittings $A = W - R$ for which $W$ is regular.

**Theorem 3.5.** *Let $A = W - R$ be a nonexpansive splitting. Then the following holds for $\theta \in ]0, \frac{1}{2}]$:*

$$\|A(I - \theta W^{-1}A)^\nu\|_\infty \leq \frac{1}{2\theta}\, \xi_0(\nu)\, \|W\|_\infty \quad (\nu \in \mathbb{N}_0)\,.$$

*Proof.* Let $B := I - 2\theta W^{-1}A = 2\theta W^{-1}R + (1-2\theta)\,I$. Then $\|B\|_\infty \leq 1$ and using Lemma 3.2 we get

$$\|A(I - \theta W^{-1}A)^\nu\|_\infty = \|\frac{1}{2\theta}\, W(I-B)\, (\tfrac{1}{2}(I+B))^\nu\|_\infty \leq \frac{1}{2\theta}\, \xi_0(\nu)\, \|W\|_\infty\,. \qquad \square$$

**Corollary 3.6.** If we apply Theorem 3.5 to the stiffness matrices $L_k (k \geq 0)$ of §2 this yields the following. Let $L_k = W_k - R_k$ be such that for every $k$:

(a) we have a nonexpansive splitting; (3.9.a)
(b) $\|W_k\|_\infty \leq C h_k^{-2}$ with $C$ independent of $k$. (3.9.b)

Then for $\theta \in ]0, \frac{1}{2}]$ the following Smoothing Property holds:

$$\|L_k(I - \theta W_k^{-1} L_k)^\nu\|_\infty \leq \frac{C}{2\theta}\, \xi_0(\nu)\, h_k^{-2}\,.$$

Usually for the splittings used as a smoother in multigrid methods the condition (3.9.b) is fulfilled. The condition (3.9.a) is more severe. Below we give some criteria for nonexpansive splittings.

First we recall some definitions. A matrix $A = (a_{ij})$ is called *weakly diagonally dominant* if $\sum_{j \neq i} |a_{ij}| \leq |a_{ii}|$ for all $i$. A splitting $A = W - R$ is called a *regular splitting* if $W^{-1} \geq 0$ and $R \geq 0$ hold (with "$\geq$" entrywise ordering); $A = W - R$ is called a *weak regular splitting* if $W^{-1} \geq 0$ and $W^{-1}R \geq 0$.

Variants of Criterion 3.7 below are well-known. In the literature one can find variants with somewhat stronger assumptions (e.g. $A$ irreducibly diagonally dominant or strictly diagonally dominant, cf. [10], [13]) which are then sufficient to yield convergence, i.e. $\rho(W^{-1}R) < 1$. Our assumptions do not imply convergence but are sufficient for $\|W^{-1}R\|_\infty \leq 1$ to hold.

**Criterion 3.7.** *Consider a system $Au = b$ with $A$ an $n \times n$ nonsingular and weakly diagonally dominant matrix. We write $A = D - L - U$ with $D = \mathrm{diag}(A)$, $L$ strictly lower triangular, $U$ strictly upper triangular and consider the following splittings:*

(a) $\quad W = D\ ,\ \ R = L + U \quad (Jacobi)\ ;$

(b) $\quad W = \frac{1}{\omega}\, D - L\ ,\ \ R = \left(\frac{1}{\omega} - 1\right) D + U \ \ with\ \ \omega \in ]0,1] \quad (SOR;\ Gauss\text{-}Seidel).$

*Then $A = W - R$ is a nonexpansive splitting.*

*Proof.* The proof for the Jacobi method is trivial (note that $W$ is regular because $A$ is regular and weakly diagonally dominant).
For the splitting in (b) the proof runs as follows. Take $v \in \mathbb{R}^n$ with $\|v\|_\infty = 1$, and define $z := W^{-1}Rv$.
From $(\frac{1}{\omega}\, D - L)\, z = ((\frac{1}{\omega} - 1)\, D + U)\, v$ it follows that for $1 \le k \le n$

$$z_k = (1-\omega)\, v_k + \omega\, \frac{1}{a_{kk}}\, \left(-\sum_{j=1}^{k-1} a_{kj} z_j + \sum_{j=k+1}^{n} a_{kj} v_j\right)\ .$$

From this it is easy to prove with induction that for $1 \le i \le n$ the following holds:

$$|z_i| \le (1-\omega) + \omega\, \frac{1}{|a_{ii}|}\, \sum_{j \ne i} |a_{ij}| \le 1\ ,$$

and thus $\|z\|_\infty = \|W^{-1}Rv\|_\infty \le 1$. □

Below we use the notation $e = (1,1,\ldots,1)^T$.

**Criterion 3.8.** *Let $A = W - R$ be a splitting such that $W^{-1}R \ge 0$. Then $A = W - R$ is a nonexpansive splitting iff $W^{-1}Ae \ge 0$.*

*Proof.* Because $W^{-1}R \ge 0$ we have that $\|W^{-1}R\|_\infty = \max_i\, (W^{-1}Re)_i$. And thus

$$\|W^{-1}R\|_\infty \le 1 \iff \max_i\, ((I - W^{-1}A)\, e)_i \le 1$$

$$\iff 1 - (W^{-1}Ae)_i \le 1 \quad \text{for all} \quad i$$

$$\iff (W^{-1}Ae)_i \ge 0 \quad \text{for all} \quad i$$

$$\iff W^{-1}Ae \ge 0\ .$$

□

An important application of Criterion 3.8 is given in Criterion 3.9 below. The latter criterion relates weak regular splittings with nonexpansive splittings (and thus with smoothing methods).

**Criterion 3.9.** *Let $A$ be such that $Ae \ge 0$. Then every weak regular splitting $A = W - R$ is a nonexpansive splitting.*

*Proof.* Because $A = W - R$ is a weak regular splitting we have $W^{-1}R \ge 0$ and $W^{-1} \ge 0$. Combined with $Ae \ge 0$ this also yields $W^{-1}Ae \ge 0$. Application of Criterion 3.8 proves that $A = W - R$ is a nonexpansive splitting. □

*Remark 3.10.* Consider $L_k = W_k - R_k$ and assume that this is a regular splitting for every $k$.

In [12] Wittum proves the following: If for every $k$ $L_k$ is symmetric, $W_k$ is symmetric positive definite and $\|W_k\|_2 \leq Ch_k^{-2}$ holds then, if we use a damping factor $\theta \in ]0,1[$, the Smoothing Property holds in the euclidean norm. Criterion 3.9 and Corollary 3.6 yield the following: If for every $k$ $L_k e \geq 0$ and $\|W_k\|_\infty \leq Ch_k^{-2}$ hold then, if we use a damping factor $\theta \in ]0, \frac{1}{2}]$, the Smoothing Property holds in the maximum norm.

We give a few examples in which Criterion 3.9 applies.
We assume that $A$ is such that $a_{ij} \leq 0$ for all $i \neq j$ and $a_{ii} > 0$ for all $i$ and that $A$ is irreducibly diagonally dominant (these conditions are easy to verify in practice). These assumptions imply that $A$ is an M-matrix and also that $Ae \geq 0$ holds.
Note that, due to the diagonal dominance, results for the Jacobi and Gauss-Seidel splittings follow already from Criterion 3.7.
As a first important application we consider an ILU factorization of $A$. It is shown in [5] that the corresponding splitting $A = W - R$ is regular. Criterion 3.9 now yields that this is a nonexpansive splitting.
Other examples are block versions of Jacobi, Gauss-Seidel or ILU. Suitable block versions correspond to regular splittings (cf. [10], [1]) and Criterion 3.9 then can be used to prove that these are nonexpansive splittings.

## 4. Approximation Property

For a multigrid convergence analysis the results of §3 should be combined with an analysis of the Approximation Property (cf. [4]). In this paper we do not present such an analysis, however, for completeness we give some results from the paper [9] in which a detailed analysis of the Approximation Property with respect to the maximum norm is given.

We outline a main result from [9] (cf. [9] for details). Assume that $\partial\Omega$ and the coefficients in the differential operator (cf. §2) are sufficiently smooth and that the principal part of this differential operator is symmetric (i.e. $a_{\alpha\beta} = a_{\beta\alpha}$ if $|\alpha| = |\beta| = 1$). Assume that the triangulations, which are quasi-uniform (cf. §2), satisfy $h_k h_{k+1}^{-1} \leq c$ with $c$ independent of $k$. Then the following holds:
There are constants $k_0$ and $C$ such that for all $k \geq k_0$:

$$\|L_k^{-1} - pL_{k-1}^{-1}r\|_\infty \leq Ch_k^2 \,|\ln h_k|^2 \,. \tag{4.1}$$

*Remark 4.1.* We briefly comment on the proof of the estimate (4.1). Two important arguments in the proof are the following. Firstly, we use that the mass matrix $P_k^* P_k$ has a condition number which is $O(1)$ $(h_k \downarrow 0)$ with respect to the maximum norm. This is a result due to Descloux [2]. Secondly, we use the following finite element asymptotic error estimate due to Rannacher and Frehse [3,6] (with $f \in L^\infty(\Omega)$ and $\varphi_k^*$ the Galerkin approximation of $\varphi^*$ in $\Phi_k$):

$$\|\varphi_k^* - \varphi^*\|_{L^\infty} \leq Ch_k^2 \,|\ln h_k|^2 \, \|f\|_{L^\infty} \,.$$

Combination of the results in §3 with the Approximation Property (4.1) immediately yields an (asymptotic) estimate for the contraction number of the two-grid iteration matrix (cf. §2):

$$\|T_k(\nu)\|_\infty \leq C \, \frac{1}{\sqrt{\nu}} \, |\ln h_k|^2 \,. \tag{4.2}$$

So instead of an "optimal" bound $C\nu^{-1}$ for the contraction number in the energy norm (or the euclidean norm) we obtain a "nearly optimal" bound $C\nu^{-\frac{1}{2}}\,|\ln h_k|^2$ if we use the maximum norm. Furthermore the bound in (4.2) is sharp in a certain sense: For a concrete (very regular) example it is shown in [9] that the contraction number with respect to the maximum norm of a standard two-grid method with a fixed number of pre-smoothing iterations is bounded from below by $C\,|\ln h_k|$.

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