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Fourier analysis of a robust multigrid method for convection-diffusion equations

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Summary. We consider a two-grid method for solving 2D convection-diffusion problems. The coarse grid correction is based on approximation of the Schur complement. As a preconditioner of the Schur complement we use the exact Schur complement of modified fine grid equations. We assume constant coefficients and periodic boundary conditions and apply Fourier analysis. We prove an upper bound for the spectral radius of the two-grid iteration matrix that is smaller than one and independent of the mesh size, the convection/diffusion ratio and the flow direction; i.e. we have a (strong) robustness result. Numerical results illustrating the robustness of the corresponding multigrid W-cycle are given.

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1. Introduction

Concerning the theoretical analysis of multigrid methods different fields of application have to be distinguished. For selfadjoint and coercive linear elliptic boundary value problems the convergence theory has reached a mature, if not its final state, cf. [17,19]. In other areas the state of the art is (far) less advanced. For example, for convection-dominated problems the development of a satisfactory theoretic analysis is still in its infancy.

In this paper we consider a multigrid method for the 2D convection-diffusion problem

$$(1.1a)\qquad -\varepsilon\Delta u + a(x,y)u_x + b(x,y)u_y = f \quad \text{in } (-1,1)\times(-1,1)\,,$$

with suitable boundary conditions and $\varepsilon > 0$. We are mainly interested in the case $\varepsilon \ll 1$. We take a standard finite difference discretization on a square mesh (mesh size h) with upwind differences for the first order terms.

In applications, many different multigrid methods for solving convection-dominated problems are used. In general, the "standard" multigrid approach used for a diffusion problem deteriorates when applied to convection-dominated problems. For these problems modifications have been suggested, such as "robust" smoothers or smoothers that follow (roughly) the flow direction and matrix-dependent prolongations and restrictions ([9, 16, 20]). Recently, some other modifications are proposed in [8]. All these modifications are based on heuristic arguments and/or empirical studies; a rigorous convergence analysis of one of these modified multigrid methods is not known to the author.

In the convergence analyses for nonsymmetric problems (e.g. as in (1.1)) the usual approach is to treat the lower order terms as perturbations of a symmetric positive definite operator and thus obtain estimates similar to the symmetric positive definite case, often with an additional restriction that h is sufficiently small, or the coarse mesh is fine enough (e.g. [3, 7, 10, 18]). This approach is not satisfactory if $\varepsilon \ll 1$, because it cannot be used to explain the behaviour of the method on meshes of practical size for the class of problems we consider here.

An interesting, and for applications very important, question is how the performance of the multigrid solver depends on h, ε and the flow direction. In particular one is interested in *robustness* of a multigrid method, i.e. a high convergence rate for a whole relevant range of the parameters. Only a few theoretical analyses concerning the subject of robustness of multigrid for convection-dominated problems have appeared. In [4, 9, 13] multigrid convergence for the 1D model convection-diffusion problem is analyzed. These analyses, however, are restricted to the 1D case. In [5] the application of the hierarchical basis multigrid method to finite element discretizations of the problem in (1.1a) is studied. The analysis there shows how the convergence rate depends on ε and on the flow direction, but the estimates are not uniform with respect to the mesh size parameter h.

In this paper we consider a particular multigrid method for convection-diffusion equations as in (1.1a). The underlying two-grid method uses an approximation of the Schur complement. Other methods based on Schur complement approximation already exist (e.g. [1, 2, 12]). An important difference between these approaches and the Schur complement approximation in this paper is the following. In the former methods the Schur complement is preconditioned by (an approximation of) the coarse grid stiffness matrix, whereas in the present case we use as a preconditioner the *exact Schur complement of modified fine grid equations*. For the type of problems as in (1.1a) the latter preconditioner appears to have some favourable properties. The Schur complement preconditioning is combined with a block Jacobi solver on the fine grid points which are not in the coarse grid. The resulting two-grid method, that is very similar to the methods discussed in [14, 15], can be classified as a multiplicative Schwarz type of method.

In the convergence analysis we consider the two-grid method applied to a discrete version of (1.1a) with periodic boundary conditions and constant coefficients; we then assume:

(1.1b) $a(x,y) = \text{constant} = a \in [0,1]$, $b(x,y) = 1-a$,
periodic boundary conditions.

By means of Fourier analysis we prove that for the spectral radius of the iteration matrix $\rho(M)$ we have $\rho(M) \leq c < 1$, with c *independent* of $h \in (0,1)$, $a \in (0,1)$, $\varepsilon \in (0,\infty)$, i.e. we have a (strong) *robustness* result. Numerical results show that in general $\rho(M) \ll 1$ holds.
Let M be the iteration matrix of the two-grid method for the system $Ax = b$ (discretization of (1.1a)), $\mathcal{S}_A$ the Schur complement of A and S the approximation of $\mathcal{S}_A^{-1}$ we use. We will prove that $\sigma(M) = \sigma(I - S.\mathcal{S}_A) \cup \{0\}$ holds. The main part of this paper is concerned with an analysis of $\sigma(S.\mathcal{S}_A)$ for the model problem (1.1a, b). This analysis is rather technical because for robustness we need estimates that are uniform in the three parameters h, ε, a.

The remainder of this paper is organized as follows. In Sect. 2 we discuss the discretization method. In Sect. 3 we give some preliminary results which are used in subsequent sections. In Sect. 4 we discuss the two-grid method that is studied in this paper. In Sect. 5 we apply Fourier analysis to this two-grid method and we derive expressions for the eigenvalues of the Schur complement ($\mathcal{S}_A$) and of the Schur complement preconditioner (S). In Sect. 6 we analyze $\sigma(S.\mathcal{S}_A)$ for the special case of pure diffusion ($\varepsilon = \infty$) and in Sect. 7 we treat the special case of pure convection ($\varepsilon = 0$). In Sect. 8 we analyze $\sigma(S.\mathcal{S}_A)$ for the general case. Finally, in Sect. 9 some numerical results for the multigrid method are presented.

2. A model convection-diffusion equation

We consider the following convection-diffusion problem with constant coefficients and periodic boundary conditions

$$\text{(2.1)} \quad \begin{cases} -\varepsilon\Delta u + au_x + (1-a)u_y = f \quad \text{in } \Omega = (-1,1)\times(-1,1) \\ \dfrac{\partial^m}{\partial x^m}\, u(-1,y) = \dfrac{\partial^m}{\partial x^m}\, u(1,y)\,, \quad m = 0,1,\ -1 < y < 1 \\ \dfrac{\partial^m}{\partial y^m}\, u(x,-1) = \dfrac{\partial^m}{\partial y^m}\, u(x,1)\,, \quad m = 0,1,\ -1 < x < 1\,. \end{cases}$$

We assume $\varepsilon > 0$, $a \in [0,1]$, $\int_\Omega f\,dx = 0$.

For discretization we use a square grid with mesh size $h = 2^{-k}$ ($k \in \mathbb{N}$):

$$\text{(2.2)} \qquad \Omega_h = \{(x,y) \in \Omega \mid x = \nu h,\ y = \mu h,\ 1-N \leq \nu,\mu \leq N\}\,,$$

with $N = 1/h$. On this grid we use a standard approximation of (2.1) with upwind discretization for the first order terms. This results in an operator $A_h : \ell^2(\Omega_h) \to \ell^2(\Omega_h)$ with a difference star of the form

$$[A_h] = \alpha_h \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} + a \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(2.3) \qquad + (1-a) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} ,$$

with $\alpha_h := \varepsilon/h \in (0, \infty)$.
The constant function with value 1 at all grid points is denoted by $\mathbf{I}_h$. The space orthogonal to $\mathbf{I}_h$ (w.r.t. Euclidean inner product) is denoted by

$$(2.4) \qquad \mathbf{I}_h^\perp = \{u \in \ell^2(\Omega_h) \mid \langle \mathbf{I}_h, u \rangle_2 = 0\} .$$

Below we consider $A_h : \mathbf{I}_h^\perp \to \mathbf{I}_h^\perp$; then A_h is regular.

3. Preliminary results

In this section we apply a standard Fourier analysis to the discrete operator A_h. We will prove several properties of the resulting (complex) eigenvalues. These results are used in the analysis in subsequent sections.

For the Fourier analysis we use a standard approach (e.g. as in [10]). In $\ell^2(\Omega_h)$, with $Nh = 1$, we introduce the $4N^2$ basis vectors $e_h^{\nu\mu}$ with

$$(3.1) \qquad e_h^{\nu\mu}(x, y) = \tfrac{1}{2} e^{\pi i(\nu x + \mu y)} , \quad (x, y) \in \Omega_h, \ 1 - N \le \nu, \mu \le N .$$

These vectors form an orthonormal basis w.r.t. a scaled Euclidean inner product, and thus the Fourier transform

$$Q_h : (\alpha_{\nu\mu})_{1-N \le \nu, \mu \le N} \to \sum_{\nu,\mu=1-N}^{N} \alpha_{\nu\mu} e_h^{\nu\mu}$$

is unitary.
Every "low" frequency (ν, μ) with $1 - \frac{1}{2}N \le \nu, \mu \le \frac{1}{2}N$ is associated with the "high" frequencies $(\nu', \mu), (\nu, \mu'), (\nu', \mu')$ where ν', μ' are defined by

$$\nu' = \begin{cases} \nu + N & \text{if } \nu \le 0 \\ \nu - N & \text{if } \nu > 0 \end{cases} , \quad \mu' = \begin{cases} \mu + N & \text{if } \mu \le 0 \\ \mu - N & \text{if } \mu > 0 . \end{cases}$$

Clearly, $\ell^2(\Omega_h)$ is a direct sum of the $N \times N$ subspaces

$$U_h^{\nu\mu} := \text{span}\{e_h^{\nu\mu}, e_h^{\nu'\mu}, e_h^{\nu\mu'}, e_h^{\nu'\mu'}\}, \quad 1 - \tfrac{1}{2}N \le \nu, \mu \le \tfrac{1}{2}N .$$

By $Q_h^{\nu\mu}$ we denote the $4N^2 \times 4$ matrix with columns these basis vectors of $U_h^{\nu\mu}$:

$$Q_h^{\nu\mu} = [e_h^{\nu\mu} \ \ e_h^{\nu'\mu} \ \ e_h^{\nu\mu'} \ \ e_h^{\nu'\mu'}] .$$

Now note that we have $(Q_h^{\nu\mu})^* A_h Q_h^{\nu\mu} = \text{diag}(d_1^{\nu\mu}, d_2^{\nu\mu}, d_3^{\nu\mu}, d_4^{\nu\mu})$ (we use the adjoint w.r.t. the scaled Euclidean inner product); a simple calculation yields the following formulas for the eigenvalues $d_j^{\nu\mu}$ $(1 - \frac{1}{2}N \le \nu, \mu \le \frac{1}{2}N)$:

(3.2a) $$d_j^{\nu\mu} = \lambda_j^{\nu\mu} + \varphi_j^{\nu\mu} , \quad j = 1,2,3,4 ,$$

(3.2b) $$\lambda_1^{\nu\mu} := 4\alpha_h(s_\nu^2 + s_\mu^2) , \quad \varphi_1^{\nu\mu} := a\psi_\nu + (1-a)\psi_\mu$$

(3.2c) $$\lambda_2^{\nu\mu} := 4\alpha_h(c_\nu^2 + s_\mu^2) , \quad \varphi_2^{\nu\mu} := a(2-\psi_\nu) + (1-a)\psi_\mu$$

(3.2d) $$\lambda_3^{\nu\mu} := 4\alpha_h(s_\nu^2 + c_\mu^2) , \quad \varphi_3^{\nu\mu} := a\psi_\nu + (1-a)(2-\psi_\mu)$$

(3.2e) $$\lambda_4^{\nu\mu} := 4\alpha_h(c_\nu^2 + c_\mu^2) , \quad \varphi_4^{\nu\mu} := a(2-\psi_\nu) + (1-a)(2-\psi_\mu)$$

with

$$s_k := \sin(\tfrac{1}{2}k\pi h) , \quad c_k := \cos(\tfrac{1}{2}k\pi h) ,$$

$$\psi_k := 1 - \exp(-i\pi kh) = 2s_k(s_k + ic_k) \quad (k \in \mathbb{Z}) .$$

For $1 - \frac{1}{2}N \le k \le \frac{1}{2}N$ we have $s_k \in [-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}]$, $c_k \in [\frac{1}{2}\sqrt{2}, 1]$, $|s_k| \le c_k$. Note that the following holds:

(3.3) $$\lambda_1^{\nu\mu} + \lambda_4^{\nu\mu} = \lambda_2^{\nu\mu} + \lambda_3^{\nu\mu} = 8\alpha_h , \quad \varphi_1^{\nu\mu} + \varphi_4^{\nu\mu} = \varphi_2^{\nu\mu} + \varphi_3^{\nu\mu} = 2 ,$$

(3.4) $$\mathrm{Re}(d_j^{\nu\mu}) \ge 0 , \quad j = 1,2,3,4 ,$$

(3.5) $$|\psi_k|^2 = 4s_k^2 ,$$

(3.6) $$\psi_k(\overline{2 - \psi_\ell}) = 4s_k c_\ell\{\sin((k-\ell)\tfrac{1}{2}\pi h) + i\cos((k-\ell)\tfrac{1}{2}\pi h)\} .$$

In Lemma 3.1 we derive some results concerning the real part of certain products and quotients of the eigenvalues $\varphi_j^{\nu\mu}$. In Lemma 3.2 we give bounds for the norm of certain quotients of the eigenvalues $d_j^{\nu\mu}$.

Lemma 3.1. *The following holds for all* $(\nu,\mu) \ne (0,0)$:

(3.7a) $\mathrm{Re}(\varphi_j^{\nu\mu}/\varphi_k^{\nu\mu}) \ge 0$ *for all* $j,k \in \{1,2,3,4\}$

(3.7b) $\mathrm{Re}(\varphi_2^{\nu\mu}\varphi_3^{\nu\mu}) \ge 0$, $\mathrm{Re}(\varphi_1^{\nu\mu}\varphi_4^{\nu\mu}) \ge 0$

(3.7c) $\mathrm{Re}(\varphi_2^{\nu\mu}\varphi_3^{\nu\mu}/\varphi_j^{\nu\mu}) \ge 0$, $\mathrm{Re}(\varphi_1^{\nu\mu}\varphi_4^{\nu\mu}/\varphi_j^{\nu\mu}) \ge 0$ *for all* $j \in \{1,2,3,4\}$.

Proof. We first prove the result in (3.7a). The result is trivial for $j = k$. Using $\mathrm{Re}(1/z) = |z|^{-2}\mathrm{Re}(z)$ it is clear that it is sufficient to consider $j < k$.
Below, we use that $\mathrm{Re}(\psi_\ell(2 - \overline{\psi}_\ell)) = 0$ (cf. (3.6)).
We start with $j = 1$. The result for $(j,k) = (1,2)$ follows from

$$\mathrm{Re}(\varphi_1^{\nu\mu}\overline{\varphi}_2^{\nu\mu}) = \mathrm{Re}[(a\psi_\nu + (1-a)\psi_\mu)(a(2-\overline{\psi}_\nu) + (1-a)\overline{\psi}_\mu)]$$

$$= a(1-a)\{\mathrm{Re}(\psi_\nu\overline{\psi}_\mu - \overline{\psi}_\nu\psi_\mu) + 2\mathrm{Re}(\psi_\mu)\} + (1-a)^2|\psi_\mu|^2$$

$$= 2a(1-a)\mathrm{Re}(\psi_\mu) + (1-a)^2|\psi_\mu|^2 \ge 0 .$$

The same argument with ν and μ exchanged and a and $1-a$ exchanged implies the result for $(j,k)=(1,3)$. For $(j,k)=(1,4)$ we note:

$$\begin{aligned}\mathrm{Re}(\varphi_1^{\nu\mu}\overline{\varphi}_4^{\nu\mu}) &= a(1-a)\mathrm{Re}((\psi_\nu(2-\overline{\psi}_\mu)+\psi_\mu(2-\overline{\psi}_\nu))\\ &= 2a(1-a)\{1-\mathrm{Re}((1-\overline{\psi}_\nu)(1-\psi_\mu))\}\\ &\geq 2a(1-a)\{1-|(1-\overline{\psi}_\nu)(1-\psi_\mu)|\}=0 \quad (\text{use } |1-\psi_\ell|=1)\ .\end{aligned}$$

We now consider $j=2$. For $(j,k)=(2,3)$ an argument as in the case $(j,k)=(1,4)$ yields

$$\begin{aligned}\mathrm{Re}(\varphi_2^{\nu\mu}\overline{\varphi}_3^{\nu\mu}) &= 2a(1-a)\{1+\mathrm{Re}((1-\overline{\psi}_\nu)(1-\psi_\mu))\}\\ &\geq 2a(1-a)\{1-|(1-\overline{\psi}_\nu)(1-\psi_\mu)|\}=0\ .\end{aligned}$$

For $(j,k)=(2,4)$ we get

$$\mathrm{Re}(\varphi_2^{\nu\mu}\overline{\varphi}_4^{\nu\mu}) = a^2|2-\psi_\nu|^2+2a(1-a)(2-\mathrm{Re}(\psi_\nu))\geq 0\ .$$

Finally, the same argument with ν,μ exchanged and a, $1-a$ exchanged yields the result for $(j,k)=(3,4)$. This completes the proof of (3.7a).

With respect to (3.7b), (3.7c) we first note that for $z\neq 0$ we have $\mathrm{Re}(z)=|z|^2\mathrm{Re}(1/z)$ and thus $(\mathrm{Re}(z)\geq 0)\Leftrightarrow(\mathrm{Re}(1/z)\geq 0)$. Using (3.3) we have

$$\begin{aligned}\mathrm{Re}(1/(\varphi_2^{\nu\mu}\varphi_3^{\nu\mu}) &= \tfrac{1}{2}\mathrm{Re}(1/\varphi_2^{\nu\mu}+1/\varphi_3^{\nu\mu})\\ &= \tfrac{1}{2}(|\varphi_2^{\nu\mu}|^{-2}\mathrm{Re}(\varphi_2^{\nu\mu})+|\varphi_3^{\nu\mu}|^{-2}\mathrm{Re}(\varphi_3^{\nu\mu}))\geq 0\ .\end{aligned}$$

And thus $\mathrm{Re}(\varphi_2^{\nu\mu}\varphi_3^{\nu\mu})\geq 0$ holds. Similarly one can prove $\mathrm{Re}(\varphi_1^{\nu\mu}\varphi_4^{\nu\mu})\geq 0$. So the inequalities in (3.7b) hold.
We now consider (3.7c). Using (3.3) and (3.7a) we have

$$\mathrm{Re}(\varphi_j^{\nu\mu}/(\varphi_2^{\nu\mu}\varphi_3^{\nu\mu}))=\tfrac{1}{2}(\mathrm{Re}(\varphi_j^{\nu\mu}/\varphi_2^{\nu\mu})+\mathrm{Re}(\varphi_j^{\nu\mu}/\varphi_3^{\nu\mu}))\geq 0\ ,$$

and thus $\mathrm{Re}(\varphi_2^{\nu\mu}\varphi_3^{\nu\mu}/\varphi_j^{\nu\mu})\geq 0$ holds. With the same arguments one can prove that $\mathrm{Re}(\varphi_1^{\nu\mu}\varphi_4^{\nu\mu}/\varphi_j^{\nu\mu})\geq 0$ holds. □

Lemma 3.2. *The following holds for all* $(\nu,\mu)\neq(0,0)$:

(3.8a) $\quad |d_1^{\nu\mu}/d_j^{\nu\mu}|\leq\sqrt{2(2+\sqrt{2})}\ ,\quad j=2,3\ ,$

(3.8b) $\quad |d_1^{\nu\mu}/d_4^{\nu\mu}|\leq 1\ .$

Proof. First we note that for $j=2,3,4$ we have $\lambda_1^{\nu\mu}\leq\lambda_j^{\nu\mu}$, $\mathrm{Re}(\varphi_1^{\nu\mu})\leq\mathrm{Re}(\varphi_j^{\nu\mu})$, and thus

$$\begin{aligned}|d_1^{\nu\mu}/d_j^{\nu\mu}|^2 &= \frac{(\lambda_1^{\nu\mu})^2+2\lambda_1^{\nu\mu}\mathrm{Re}(\varphi_1^{\nu\mu})+|\varphi_1^{\nu\mu}|^2}{(\lambda_j^{\nu\mu})^2+2\lambda_j^{\nu\mu}\mathrm{Re}(\varphi_j^{\nu\mu})+|\varphi_j^{\nu\mu}|^2}\\ &\leq\frac{(\lambda_j^{\nu\mu})^2+2\lambda_j^{\nu\mu}\mathrm{Re}(\varphi_j^{\nu\mu})+|\varphi_1^{\nu\mu}|^2}{(\lambda_j^{\nu\mu})^2+2\lambda_j^{\nu\mu}\mathrm{Re}(\varphi_j^{\nu\mu})+|\varphi_j^{\nu\mu}|^2}\leq\max\left\{1,\frac{|\varphi_1^{\nu\mu}|^2}{|\varphi_j^{\nu\mu}|^2}\right\}\ .\end{aligned}$$

So it is sufficient to have bounds for $|\varphi_1^{\nu\mu}|/|\varphi_j^{\nu\mu}|$.
For $|\varphi_1^{\nu\mu}|$ we have

(3.9) $$|\varphi_1^{\nu\mu}|^2 = 4\{a^2 s_\nu^2 + (1-a)^2 s_\mu^2 + 2a(1-a)(s_\nu^2 s_\mu^2 + s_\nu c_\nu s_\mu c_\mu)\} .$$

We first consider $j = 2$; for $|\varphi_2^{\nu\mu}|$ we get

(3.10) $$|\varphi_2^{\nu\mu}|^2 = 4\{a^2 c_\nu^2 + (1-a)^2 s_\mu^2 + 2a(1-a)(c_\nu^2 s_\mu^2 - s_\nu c_\nu s_\mu c_\mu)\} .$$

If $s_\nu s_\mu \leq 0$ then clearly $|\varphi_1^{\nu\mu}|^2/|\varphi_2^{\nu\mu}|^2 \leq 1$. We now take $s_\nu s_\mu \geq 0$. Then $\mu, \nu \in [0, \frac{1}{2}N]$ or $\mu, \nu \in [\frac{1}{2}N - 1.0]$ and thus $|\sin(\frac{1}{2}(\mu - \nu)\pi h)| \leq \frac{1}{2}\sqrt{2}$ holds. Using this we get

$$2a(1-a)|c_\nu^2 s_\mu^2 - s_\nu c_\nu s_\mu c_\mu| = 2a(1-a)|c_\nu s_\mu|\,|\sin(\tfrac{1}{2}(\nu - \mu)\pi h)|$$
$$\leq \tfrac{1}{2}\sqrt{2}(a^2 c_\nu^2 + (1-a)^2 s_\mu^2) .$$

Using this in (3.10) results in

$$|\varphi_2^{\nu\mu}|^2 \geq 4(1 - \tfrac{1}{2}\sqrt{2})\{a^2 c_\nu^2 + (1-a)^2 s_\mu^2\} .$$

From (3.9) it is clear that

$$\begin{aligned}|\varphi_1^{\nu\mu}|^2 &= 4\{a^2 s_\nu^2 + (1-a)^2 s_\mu^2 + 2a(1-a)s_\nu s_\mu \cos(\tfrac{1}{2}(\nu - \mu)\pi h)\} \\ &\leq 8\{a^2 s_\nu^2 + (1-a)^2 s_\mu^2\} \leq 8\{a^2 c_\nu^2 + (1-a)^2 s_\mu^2\} .\end{aligned}$$

We conclude that $|\varphi_1^{\nu\mu}|^2/|\varphi_2^{\nu\mu}|^2 \leq 2/(1 - \frac{1}{2}\sqrt{2}) = 2(2 + \sqrt{2})$ holds. This completes the proof for the case $j = 2$. For the case $j = 3$ we note that for $|\varphi_3^{\nu\mu}|$ we get an expression as in (3.10), only with ν and μ exchanged and a, $1 - a$ exchanged. Thus the same arguments yield a proof for $j = 3$.
We finally consider $j = 4$. Because $\mathrm{Im}(\varphi_4^{\nu\mu}) = -\mathrm{Im}(\varphi_1^{\nu\mu})$ and $\mathrm{Re}(\varphi_4^{\nu\mu}) \geq \mathrm{Re}(\varphi_1^{\nu\mu}) \geq 0$, we immediately have $|\varphi_1^{\nu\mu}|/|\varphi_4^{\nu\mu}| \leq 1$. □

Remark 3.3. Concerning the sharpness of the bounds in (3.8) we note that the bound in (3.8b) is sharp; if we take $\alpha_h = 0$, $a = \frac{1}{2}$, $\nu = \mu = \frac{1}{2}N$ then

$$|d_1^{\nu\mu}|/|d_4^{\nu\mu}| = |\varphi_1^{\nu\mu}|/|\varphi_4^{\nu\mu}| = |1 + i|/|1 - i| = 1 .$$

The bound in (3.8a) is fairly sharp; e.g. for the case $j = 2$ we may take $\alpha_h = 0$, $\nu = \frac{1}{2}N$, $\mu = 1$, $a = \sqrt{2}\, s_\mu = \sqrt{2}\, s_1$, then for $h \downarrow 0$ we have

$$|d_1^{\nu\mu}|^2/|d_2^{\nu\mu}|^2 = |\varphi_1^{\nu\mu}|^2/|\varphi_2^{\nu\mu}|^2 = \frac{2 + \sqrt{2}}{2 - \sqrt{2}} + O(h) ,$$

and thus for $h \downarrow 0$, $|d_1^{\nu\mu}/d_2^{\nu\mu}| \to 1 + \sqrt{2} \approx 2.41$ (note that $\sqrt{2(2 + \sqrt{2})} \approx 2.61$).

In view of the analysis in subsequent sections, we give some properties of the *harmonic mean* of the eigenvalues $d_j^{\nu\mu}$. For given complex numbers $z_1, \ldots, z_k \in \mathbb{C}\backslash\{0\}$ we define the harmonic mean $H(z_1, z_2, \ldots, z_k)$ as

(3.11) $$H(z_1, z_2, \ldots, z_k) = k \left(\sum_{j=1}^{k} 1/z_j \right)^{-1} .$$

Using (3.3) is easy to see that the eigenvalues $d_j^{\nu\mu}$ have the following properties ($(\nu,\mu)\neq(0,0)$)

(3.12a) $$H(d_j^{\nu\mu},d_k^{\nu\mu})=\frac{1}{4\alpha_h+1}\,d_j^{\nu\mu}d_k^{\nu\mu}\quad\text{for } (j,k)\in\{(1,4),(2,3)\}$$

(3.12b) $$H(d_j^{\nu\mu},d_k^{\nu\mu})=d_j^{\nu\mu}+\tfrac{1}{2}\,\frac{d_j^{\nu\mu}}{4\alpha_h+1}\,(d_k^{\nu\mu}-d_j^{\nu\mu})$$

for $(j,k)\in\{(1,4),(2,3)\}$

(3.12c) $$H(d_1^{\nu\mu},d_2^{\nu\mu},d_3^{\nu\mu},d_4^{\nu\mu})=H(d_1^{\nu\mu},d_4^{\nu\mu})+\frac{H(d_1^{\nu\mu},d_4^{\nu\mu})}{H(d_1^{\nu\mu},d_4^{\nu\mu})+H(d_2^{\nu\mu},d_3^{\nu\mu})}\,\{H(d_2^{\nu\mu},d_3^{\nu\mu})-H(d_1^{\nu\mu},d_4^{\nu\mu})\}\,.$$

4. Two–grid method

In this section we discuss the specific two-grid method that will be analyzed in subsequent sections. As in the standard approach (cf. [10]) it is obvious how a multigrid algorithm can be obtained. In view of the Fourier analysis in Sects. 5–8 we explain the two-grid method for the model problem (1.1a, b). However, the same approach is directly applicable to a problem as in (1.1a) with varying coefficients and/or with Dirichlet boundary conditions (cf. Remark 4.2, Sect. 9).

We take the discrete problem as in Sect. 2 (cf. (2.3)) and use standard $h\to 2h=:H$ coarsening. We make a corresponding block partitioning of $A=A_h$ as

(4.1) $$A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},$$

in which $[A_{21}\;\;A_{22}]$ corresponds to the (fine grid) equations in the coarse-grid points.

For a block matrix $C=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}$ the Schur complement of C_{11}, i.e. $C_{22}-C_{21}C_{11}^{-1}C_{12}$ is denoted by $\mathcal{S}_C$.

We define the following "prolongations" and "restrictions" (block partitioning as in (4.1); note that A_{11} is regular)

(4.2) $$p_1=\begin{bmatrix}I\\0\end{bmatrix},\quad r_1=[I\;\;0],\quad p_2=\begin{bmatrix}-A_{11}^{-1}A_{12}\\ I\end{bmatrix},\quad r_2=[0\;\;I]\,.$$

Our two-grid method is based on the factorization

(4.3) $$(I-p_2\mathcal{S}_A^{-1}r_2A)(I-p_1A_{11}^{-1}r_1A)=0\,.$$

Remark 4.1. For the Schur complement $\mathcal{S}_A$ we have $\mathcal{S}_A=\tilde{r}_2Ap_2$ with $\tilde{r}_2:=[-A_{21}A_{11}^{-1}\;\;I]$. Note that $A\mathbf{I}_h=A^{\mathrm{T}}\mathbf{I}_h=0$, $p_2\mathbf{I}_H=\mathbf{I}_h=\tilde{r}_2^{\mathrm{T}}\mathbf{I}_H$. From this it follows that $\mathrm{Ker}(\mathcal{S}_A)=\mathbf{I}_H$, $\mathcal{R}(\mathcal{S}_A)=\mathrm{Ker}(\mathcal{S}_A^{\mathrm{T}})^{\perp}=\mathbf{I}_H^{\perp}$, and thus $\mathcal{S}_A:\ \mathbf{I}_H^{\perp}\to\mathbf{I}_H^{\perp}$ is regular (cf. Sect. 2); the inverse is denoted by $\mathcal{S}_A^{-1} : \mathbf{I}_H^\perp \to \mathbf{I}_H^\perp$.

Note that $I - p_1 A_{11}^{-1} r_1 A = p_2 r_2$ holds and thus $r_2 A(I - p_1 A_{11}^{-1} r_1 A) = r_2 A p_2 r_2 = \mathcal{S}_A r_2$. So $\mathcal{S}_A^{-1}$ in (4.3) is well-defined and indeed $(I - p_2 \mathcal{S}_A^{-1} r_2 A)(I - p_1 A_{11}^{-1} r_1 A) = p_2 r_2 - p_2 \mathcal{S}_A^{-1} \mathcal{S}_A r_2 = 0$ holds.
We use the notation

(4.4) $$P_1 := p_1 A_{11}^{-1} r_1 A \ , \quad P_2 := p_2 \mathcal{S}_A^{-1} r_2 A \ .$$

For $P_2 x$ to be well-defined we must have $r_2 A x \in \mathcal{R}(\mathcal{S}_A) = \mathbf{I}_H^\perp$, i.e. $x \in (A^{\mathrm{T}} \begin{pmatrix} 0 \\ \mathbf{I}_H \end{pmatrix})^\perp$. The following properties hold:

(4.5a) P_1 is a projection on $\mathcal{R}(p_1)$.

(4.5b) $P_2 : (A^{\mathrm{T}} \begin{pmatrix} 0 \\ \mathbf{I}_H \end{pmatrix})^\perp \to \mathcal{R}(p_2)$ is a projection on $\mathcal{R}(p_2)$;

this projection has the following orthogonality property:

$$\langle A(I - P_2)x, \, AP_2 x \rangle_2 = 0 \text{ for } x \in (A^{\mathrm{T}} \begin{pmatrix} 0 \\ \mathbf{I}_H \end{pmatrix})^\perp.$$

(4.5c) $\mathbb{R}^{4N^2} = \mathcal{R}(p_1) \oplus \mathcal{R}(p_2)$.

(4.5d) $A_{11} = r_1 A p_1, \ \mathcal{S}_A = r_2 A p_2$.

In view of these properties the factorization in (4.3) corresponds to a method that might be classified as a multiplicative Schwarz method. Note that the subspace $\mathcal{R}(p_2)$ is matrix-dependent. The condition concerning the domain of P_2 is due to the fact that A is singular. The $I - P_1$ term corresponds to a block Jacobi iteration on the points of $\Omega_h \backslash \Omega_H$. Using a basic iterative (line) method, systems with matrix A_{11} can be solved "accurately" with $O(N^2)$ flops, even if we have strong convection (cf. results in Sect. 9). For the analysis in this paper we assume that in the block Jacobi method the system with matrix A_{11} is solved exactly. In practice we will use (a few) inner iterations.
To obtain a feasible method, in P_2 we replace p_2 and $\mathcal{S}_A^{-1}$ by approximations, say $\tilde{p}_2 = \begin{bmatrix} -B \\ I \end{bmatrix}$ and S. Our choice for B and S is discussed below. First we give some results for the general case with iteration matrix

(4.6) $$M = (I - \tilde{p}_2 S r_2 A)(I - P_1) \ , \quad \tilde{p}_2 = \begin{bmatrix} -B \\ I \end{bmatrix} .$$

Lemma 4.1. *For M as in (4.6) the following properties hold:*

(4.7a) $M\mathbf{I}_h = \mathbf{I}_h$

(4.7b) $$M = \begin{bmatrix} 0 & D \\ 0 & I - S.\mathcal{S}_A \end{bmatrix}, \quad \text{with } D = B - A_{11}^{-1}A_{12} - B(I - S.\mathcal{S}_A)$$

(4.7c) $\sigma(M) = \sigma(I - S.\mathcal{S}_A) \cup \{0\}$.

Proof. The results in (4.7a, b) follow directly from the definitions. The result in (4.7c) follows from (4.7b). □

In $\sigma(M)$ we have the rather special eigenvalue 1 that orginates from $A\mathbf{I}_h = 0$. When solving a problem with $A\mathbf{I}_h = 0$ one should use a method in which errors remain in $\mathbf{I}_h^{\perp}$. Such a method can be obtained by combining the method corresponding to M with an orthogonal projection on $\mathbf{I}_h^{\perp}$. For a further discussion of this subject we refer to [10]. (Note that this special treatment of $\mathbf{I}_h$ is not needed if A is nonsingular, e.g. (1.1a) with Dirichlet boundary conditions). If errors are in $\mathbf{I}_h^{\perp}$ then the eigenvalue $1 \in \sigma(M)$ plays no role and the convergence rate is determined by $\sigma(M)\backslash\{1\} = \sigma(I - S.\mathcal{S}_A)\backslash\{1\}$. From this we see that if $\max\{|\lambda| \mid \lambda \in \sigma(I - S.\mathcal{S}_A),\ \lambda \neq 1\} \ll 1$ then we have a two-grid method with a favourable convergence property. It is a first step towards robustness that in essence only the preconditioning of $\mathcal{S}_A$ by S determines the convergence of the method corresponding to M. The main result of this paper, given in Sect. 8, is that for our choice of S, which is feasible in a practical multigrid algorithm, we have $|\lambda| \leq c < 1$ for all $\lambda \in \sigma(I - \omega S.\mathcal{S}_A)\backslash\{1\}$ with constants ω and c independent of h, ε, a. This yields a (strong) robustness result for the two-grid method of (4.6).

In the multigrid literature one can find methods based on approximation of the Schur complement, cf. [1, 2, 12]. These methods, and the corresponding convergence analyses, apply to symmetric positive definite problems only. The two main types of approximations S of $\mathcal{S}_A^{-1}$ are the following

(4.8a) $S = A_H^{-1}$ (A_H: coarse grid stiffness matrix)

(4.8b) $S = (I - p_k(A_H^{-1}.\mathcal{S}_A)).\mathcal{S}_A^{-1}$, where p_k is a polynomial of degree k related to a basic iterative method for solving $\mathcal{S}_A y = d$.

Note that (4.8a) is a special case of (4.8b) for the choice $p_1(t) = 1 - t$. The polynomial method in (4.8b) is introduced because the method with iteration matrix $I - \omega A_H^{-1}\mathcal{S}_A$ is too slow and has to be accelerated. Multigrid variants of (4.8a, b) exist in which A_H^{-1} is approximated using a preconditioner from coarser grids. Disadvantages of the approach in (4.8b) are that, for $k \geq 2$, we have to compute matrix–vector products with $\mathcal{S}_A$ and that we need a "suitable" polynomial based on $\sigma(A_H^{-1}.\mathcal{S}_A)$. Note that the problem of finding such a polynomial becomes more difficult if we have complex eigenvalues.

It turns out that in our approach we do not need an acceleration procedure because our preconditioner S of $\mathcal{S}_A$ is (significantly) better than A_H^{-1} and results in an iteration matrix $I - \omega S \mathcal{S}_A$ with spectral radius much smaller than one.
For S we do not take (an approximation of) A_H^{-1} but we use the *inverse of the exact Schur complement of modified fine grid equations*. The approach is the same as in [14, 15]. We take the fine grid equations as given in the matrix A in (4.1) and "discretize" these equations by replacing the equations in $[A_{11} \;\; A_{12}]$ by approximating equations $[\tilde{A}_{11} \;\; \tilde{A}_{12}]$ with $\tilde{A}_{11}$ *diagonal*. The equations in the coarse grid points are not altered. This results in a modified fine grid matrix

$$(4.9) \qquad \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ A_{21} & A_{22} \end{bmatrix} .$$

For S we take the inverse of the Schur complement of $\tilde{A}$: $S = \mathcal{S}_{\tilde{A}}^{-1} : \mathbf{I}_H^{\perp} \to \mathbf{I}_H^{\perp}$. Because $[\tilde{A}_{11} \;\; \tilde{A}_{12}]$ is meant to be an approximation of $[A_{11} \;\; A_{12}]$ an obvious choice for B in $\tilde{p}_2$ (cf. (4.6), (4.2)) is $B := \tilde{A}_{11}^{-1}\tilde{A}_{12}$, so

$$(4.10) \qquad \tilde{p}_2 := \begin{bmatrix} -\tilde{A}_{11}^{-1}\tilde{A}_{12} \\ I \end{bmatrix} .$$

With this choice we then have properties as for the "optimal" projection P_2 in (4.5b, d):

$$(4.11a) \qquad \tilde{P}_2 := \tilde{p}_2 \mathcal{S}_{\tilde{A}}^{-1} r_2 A : \; (A^{\mathrm{T}} \begin{pmatrix} 0 \\ \mathbf{I}_H \end{pmatrix})^{\perp} \to \mathcal{R}(\tilde{p}_2)$$

is a projection on $\mathcal{R}(\tilde{p}_2)$; this projection

has the following orthogonality property:

$$\langle \tilde{A}(I - \tilde{P}_2)x, \; A\tilde{P}_2 x\rangle_2 = 0 \text{ for } x \in (A^{\mathrm{T}} \begin{pmatrix} 0 \\ \mathbf{I}_H \end{pmatrix})^{\perp}.$$

$$(4.11b) \qquad \mathcal{S}_{\tilde{A}} = r_2 A \tilde{p}_2 .$$

Due to the fact that $\tilde{A}_{11}$ is diagonal we have local operators $\tilde{p}_2$, $\mathcal{S}_{\tilde{A}}$. Note that the two-grid method (4.6), with $\tilde{p}_2$ as in (4.10) and $S = \mathcal{S}_{\tilde{A}}^{-1}$ is now completely determined by the "discretization" $[A_{11} \;\; A_{12}] \to [\tilde{A}_{11} \;\; \tilde{A}_{12}]$.

We now discuss this discretization process.
Consider a grid point P of $\Omega_h \backslash \Omega_H$ (cf. Fig. 1).

The equation in P consists of a linear combination of the difference stars

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} .$$

We get a modified equation, represented in $[\tilde{A}_{11} \;\; \tilde{A}_{12}]$, by the following substitution:

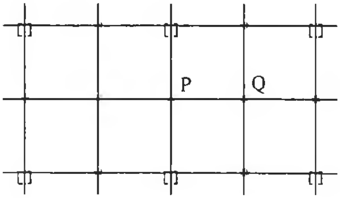


Fig. 1.

$$(4.12a)\quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1/8 & 0 & -3/4 & 0 & -1/8 \\ 0 & 0 & 2 & 0 & 0 \\ -1/8 & 0 & -3/4 & 0 & -1/8 \end{bmatrix}$$

$$(4.12b)\quad \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1/4 & 0 & -1/4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1/4 & 0 & -1/4 & 0 & 0 \end{bmatrix}.$$

The $\dfrac{\partial}{\partial y}$ difference star is not changed.

Taylor expansion shows that for smooth functions the difference between the results of the two stars in (4.12a), (4.12b) is $\mathcal{O}(h^2)$, $\mathcal{O}(h)$ respectively. In a point Q (cf. Fig. 1) we make the following substitution

$$(4.13a)\quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1/2 & 0 & -1/2 \\ 0 & 2 & 0 \\ -1/2 & 0 & -1/2 \end{bmatrix} \quad (\mathcal{O}(h^2) \text{ accurate})$$

$$(4.13b)\quad \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 0 \end{bmatrix} \quad (\mathcal{O}(h) \text{ accurate})$$

$$(4.13c)\quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & -1/2 \end{bmatrix} \quad (\mathcal{O}(h) \text{ accurate}) .$$

Clearly this approach uses information from the underlying differential equation. We may combine this with a more algebraic approach in which only the structure of the grid is used. In the latter approach a relation between unknowns in $\Omega_h \backslash \Omega_H$ is "eliminated" by a (linear) interpolation procedure (as in the hierarchical basis multigrid method). For example, if in the point $P = (x, y)$ the unknown $u(x - h,\ y - h)$ is replaced by $\frac{1}{2}(u(x - 2h,\ y - h) + u(x,\ y - h))$ (cf. Fig. 1) then the star $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}_P$ changes into $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1/2 & 0 & -1/2 & 0 & 0 \end{bmatrix}_P$. In the implementations we always used a two-step discretization process. In the first step, in a given grid point in $\Omega_h \backslash \Omega_H$, we modify the star by replacing certain finite differences, that are selected using the BVP, by other finite differences (as in (4.12), (4.13)). In the second step we eliminate remaining relations between grid points in $U_h \backslash U_H$ by an algebraic elimination process (e.g. linear interpolation).

Remark 4.2. The approximation $[A_{11} \;\; A_{12}] \to [\tilde{A}_{11} \;\; \tilde{A}_{12}]$ as described above is based on a given difference star in a given grid point and therefore also applicable to (1.1a) with varying coefficients a, b. With respect to the *multigrid* method we note that if we apply the discretization process (cf. (4.12), (4.13)) to A from (2.3) then $\tilde{p}_2$ will have a 17-point stencil and $\mathcal{S}_{\tilde{A}}$ will have a 9-point stencil. Because $\tilde{A}$ is a "discretization" of A it will be of the same type as A and its Schur complement $\mathcal{S}_{\tilde{A}}$ will also be of the same type. One can prove (cf. [14]) that if A is an M-matrix, then $\tilde{A}$ and $\mathcal{S}_{\tilde{A}}$ are M-matrices, too. So stability is preserved. The same procedure can then be applied to $\mathcal{S}_{\tilde{A}}$ (on Ω_H), etc. This results in 17-point prolongations and 9-point Schur complement preconditioners on all coarser grids. These operators are used in a standard multigrid approach based on the two-grid operator in (4.6) (cf. also Sect. 9).

Remark 4.3. For the Poisson equation with red–black coarsening (then A_{11} is diagonal!) the setting of a multiplicative Schwarz method is used by Braess in [6] (cf. also [11], Sect. 11.4.4) in an algebraic convergence analysis of a two-grid method. He takes $S = A_H^{-1}$ and then $\rho(I - A_H^{-1}\mathcal{S}_A) \leq \frac{1}{2}$ holds, which can be proved using a strengthened Cauchy–Schwarz inequality.

5. Fourier analysis of the two-grid method

In this section we derive expressions for the eigenvalues of $\mathcal{S}_{\tilde{A}}^{-1}\mathcal{S}_A$. These expressions will be used in Sects. 6, 7, 8 to obtain bounds on $\sigma(\mathcal{S}_{\tilde{A}}^{-1}\mathcal{S}_A)$.

We start with a result concerning the eigenvalues of the Schur complement $\mathcal{S}_A$. In the coarse grid space $\ell^2(\Omega_H)$ we use the standard Fourier basis (cf. (3.1)). We define $N_H := N/2$ ($N = 1/h$) and

(5.1) $$e_H^{\nu\mu}(x,y) = \tfrac{1}{2}e^{\pi i(\nu x + \mu y)} \quad (x,y) \in \Omega_H \;, \quad 1 - N_H \leq \nu, \mu \leq N_H \;.$$

Lemma 5.1. *The Fourier mode* $e_H^{\nu\mu}$ $(1 - N_H \leq \nu, \mu \leq N_H,\ (\nu,\mu) \neq (0,0))$ *is an eigenvector of* $\mathcal{S}_A$ *with corresponding eigenvalue the harmonic mean of* $d_1^{\nu\mu}, d_2^{\nu\mu}, d_3^{\nu\mu}, d_4^{\nu\mu}$ *(cf. (3.11)), i.e.:*

(5.2) $$\mathcal{S}_A e_H^{\nu\mu} = H(d_1^{\mu\mu}, d_2^{\nu\mu}, d_3^{\nu\mu}, d_4^{\nu\mu})e_H^{\nu\mu} \;.$$

Proof. If $A : \ell^2(\Omega_h) \to \ell^2(\Omega_h)$ would be nonsingular then the formula

$$A^{-1} = \begin{bmatrix} -A_{11}^{-1}A_{12} \\ I \end{bmatrix} \mathcal{S}_A^{-1}[-A_{21}A_{11}^{-1} \quad I] + \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

immediately yields $\mathcal{S}_A^{-1} = [0 \quad I]A^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix}$. Fourier transformation then results in the harmonic mean as in (5.2). However, A is singular with $\mathrm{Ker}(A) = \mathbf{I}_h$ and a special treatment of the vector $\mathbf{I}_h$ is needed. Define $\tilde{r}_2 := [-A_{21}^{-1}A_{11}^{-1} \quad I]$ and note that $\tilde{r}_2^T\mathbf{I}_H = \mathbf{I}_h$ (cf. Remark 4.1) and therefore $\tilde{r}_2(\mathbf{I}_h^\perp) \subset \mathbf{I}_H^\perp$. We use the (generalized) inverse $\mathcal{S}_A^{-1} : \mathbf{I}_H^\perp \to \mathbf{I}_H^\perp$ (cf. Remark 4.1). Now define $W : \mathbf{I}_h^\perp \to \ell^2(\Omega_h)$ by

$$W := p_2\mathcal{S}_A^{-1}\tilde{r}_2 + p_1A_{11}^{-1}r_1$$

(p_2, r_1, p_1 as in (4.2)). Note that W is well-defined due to $\tilde{r}_2(\mathbf{I}_h^\perp) \subset \mathbf{I}_H^\perp$. A simple calculation shows that $AW = I_{\mathbf{I}_h^\perp}$.

Using $\begin{bmatrix} 0 \\ I \end{bmatrix}\mathbf{I}_H^\perp \subset \mathbf{I}_h^\perp$ we see that $W\begin{bmatrix} 0 \\ I \end{bmatrix} : \mathbf{I}_H^\perp \to \ell^2(\Omega_h)$ is well-defined. From the definition of W we now conclude that $[0 \quad I]W\begin{bmatrix} 0 \\ I \end{bmatrix}\Big|_{\mathbf{I}_H^\perp} = \mathcal{S}_A^{-1}$. Note that

$$[0 \quad I]e_h^{\nu\mu} = [0 \quad I]e_h^{\nu'\mu} = [0 \quad I]e_h^{\nu\mu'} = [0 \quad I]e_h^{\nu'\mu'} = e_H^{\nu\mu} \tag{5.3}$$

$$(1 - N_H \leq \nu, \mu \leq N_H)\ .$$

We take a Fourier mode $e_H^{\nu\mu} \in \mathbf{I}_H^\perp$, i.e. $(\nu, \mu) \neq (0,0)$. Using $\begin{bmatrix} 0 \\ I \end{bmatrix} = [0 \quad I]^T = \frac{1}{4}[0 \quad I]^*$ and (5.3) we see that

$$\begin{bmatrix} 0 \\ I \end{bmatrix} e_H^{\nu\mu} = \tfrac{1}{4}(e_h^{\nu\mu} + e_h^{\nu'\mu} + e_h^{\nu\mu'} + e_h^{\nu'\mu'})\ . \tag{5.4}$$

From $AW = I_{\mathbf{I}_h^\perp}$ we have $AWe_h^{\nu\mu} = e_h^{\nu\mu}$ and thus $We_h^{\nu\mu} = 1/d_1^{\nu\mu}e_h^{\nu\mu} + \alpha_{\nu\mu}\mathbf{I}_h$ for a certain $\alpha_{\nu\mu} \in \mathbb{C}$. Similar relations hold for $e_h^{\nu'\mu}, e_h^{\nu\mu'}$ and $e_h^{\nu'\mu'}$. Combining this with (5.4) we get

$$W\begin{bmatrix} 0 \\ I \end{bmatrix} e_H^{\nu\mu} = \tfrac{1}{4}(1/d_1^{\nu\mu}e_h^{\nu\mu} + 1/d_2^{\nu\mu}e_h^{\nu'\mu} + 1/d_3^{\nu\mu}e_h^{\nu\mu'} + 1/d_4^{\nu\mu}e_h^{\nu'\mu'}) + \beta_{\nu\mu}\mathbf{I}_h$$

for a certain $\beta_{\nu\mu} \in \mathbb{C}$. Using (5.3) and $[0 \quad I]\mathbf{I}_h = \mathbf{I}_H$ we have

$$[0 \quad I]W\begin{bmatrix} 0 \\ I \end{bmatrix} e_H^{\nu\mu} = \left(\tfrac{1}{4}\sum_{j=1}^{4} 1/d_j^{\nu\mu}\right) e_H^{\nu\mu} + \beta_{\nu\mu}\mathbf{I}_H\ .$$

Finally, because $[0 \quad I]W\begin{bmatrix} 0 \\ I \end{bmatrix}\Big|_{\mathbf{I}_H^\perp} = \mathcal{S}_A^{-1} : \mathbf{I}_H^\perp \to \mathbf{I}_H^\perp$, we conclude that $\beta_{\nu\mu} = 0$ and

$$\mathcal{S}_A^{-1}e_H^{\nu\mu} = \left(\tfrac{1}{4}\sum_{j=1}^{4} 1/d_j^{\nu\mu}\right) e_H^{\nu\mu}\ . \quad \square$$

Remark 5.2. In the multigrid literature there are other approaches in which the Schur complement plays an important role (cf. Sect. 4). In these approaches the Schur complement $\mathcal{S}_A = r_2 A p_2$ (cf. (4.5d)) is approximated by the coarse grid stiffness matrix. In Fourier space this means that the harmonic average $H(d_1^{\nu\mu}, d_2^{\nu\mu}, d_2^{\nu\mu}, d_4^{\nu\mu})$ is approximated by (an approximation of) $d_1^{\nu\mu}$. In our approach we approximated $\mathcal{S}_A$ by $\tilde{\mathcal{S}}_A = r_2 A \tilde{p}_2$ (cf. (4.11b)), thus using information from $A_{|\text{span}\{e_h^{\nu\mu}, e_h^{\nu'\mu}, e_h^{\nu\mu'}, e_h^{\nu'\mu'}\}}$. This leads to a better approximation of the harmonic average as can be seen from Fig. 2, 4, 5.

To be able to apply Fourier analysis to $\tilde{\mathcal{S}}_A = r_2 A \tilde{p}_2$ we first introduce some notation. As discussed in Sect. 4 we have modified equations $[\bar{A}_{11} \ \ \bar{A}_{12}]$ in the grid points of $\Omega_h \backslash \Omega_H$. The grid points of $\Omega_h \backslash \Omega_H$ are divided in three sets:

(5.5a) $$\Omega_h^{(1)} = \{(x,y) \in \Omega_h \backslash \Omega_H \mid y = kH,\ k \in \mathbb{Z}\}$$

(5.5b) $$\Omega_h^{(2)} = \{(x,y) \in \Omega_h \backslash \Omega_H \mid x = kH,\ k \in \mathbb{Z}\}$$

(5.5c) $$\Omega_h^{(3)} = (\Omega_h \backslash \Omega_H) \backslash (\Omega_h^{(1)} \cup \Omega_h^{(2)}) .$$

Note that for given $j \in \{1,2,3\}$ $\bar{A}$ has a constant difference star in the points of $\Omega_h^{(j)}$, thus for a suitable $\tau_{(j)}^{\nu\mu}$, independent of $(x,y) \in \Omega_h^{(j)}$ we have

(5.6) $$(\bar{A}_{11}^{-1} \bar{A} e_h^{\nu\mu})_{|\Omega_h^{(j)}} = \tau_{(j)}^{\nu\mu} e_h^{\nu\mu}{}_{|\Omega_h^{(j)}} .$$

In Lemma 5.3 it is shown that the eigenvalues of $\tilde{\mathcal{S}}_A$ can be expressed in terms of these $\tau_{(j)}^{\nu\mu}$ and the eigenvalues of A.

Lemma 5.3. *For $(\nu,\mu) \neq (0,0)$ with $1 - N_H \leq \nu, \mu \leq N_H$ the following holds*

(5.7) $$\tilde{\mathcal{S}}_A e_H^{\nu\mu} = \{d_1^{\nu\mu} + \tfrac{1}{4}(\tau_{(1)}^{\nu\mu} + \tau_{(2)}^{\nu\mu})(d_4^{\nu\mu} - d_1^{\nu\mu}) + \tfrac{1}{4}(\tau_{(1)}^{\nu\mu} - \tau_{(2)}^{\nu\mu})(d_2^{\nu\mu} - d_3^{\nu\mu})\} e_H^{\nu\mu} .$$

Proof. We use the Galerkin property $\tilde{\mathcal{S}}_A = r_2 A \tilde{p}_2$, with $r_2 = [0 \ \ I]$ and $\tilde{p}_2 = \begin{bmatrix} -\bar{A}_{11}^{-1}\bar{A}_{12} \\ I \end{bmatrix}$. The Fourier transform of $\tilde{p}_2$ is equal to the transpose of the Fourier transform of $\frac{1}{4}[-\bar{A}_{12}^T \bar{A}_{11}^{-1} \ \ I]$. The latter restriction operator has a constant (17-point) difference star. From this we see that (cf. [10] Sect. 8.1.2)

$$\tilde{p}_2 e_H^{\nu\mu} \in \text{span}\{e_h^{\nu\mu}, e_h^{\nu'\mu}, e_h^{\nu\mu'}, e_h^{\nu'\mu'}\} \quad (1 - N_H \leq \nu, \mu \leq N_H) .$$

We take a fixed $(\nu,\mu) \neq (0,0)$ with $1 - N_H \leq \nu, \mu \leq N_H$ and write $\tilde{p}_2 e_H^{\nu\mu} = \alpha_1 e_h^{\nu\mu} + \alpha_2 e_h^{\nu'\mu} + \alpha_3 e_h^{\nu\mu'} + \alpha_4 e_h^{\nu'\mu'}$.

Note that $e_h^{\nu'\mu}{}_{|\Omega_H} = e_h^{\nu\mu}{}_{|\Omega_H}$, $e_h^{\nu'\mu}{}_{|\Omega_h^{(j)}} = (-1)^j e_h^{\nu\mu}{}_{|\Omega_h^{(j)}}$, $j = 1,2,3$.

Similar relations hold for $e_h^{\nu\mu'}$ and $e_h^{\nu'\mu'}$. This yields, with $\Omega_h^{(0)} := \Omega_H$ and $\Omega_h^{(j)}$ as in (5.5), the following:

(5.8a) $$(\tilde{p}_2 e_H^{\nu\mu})_{|\Omega_h^{(j)}} = (\alpha_1 + (-1)^j \alpha_2 + \alpha_3 + (-1)^j \alpha_4)(e_h^{\nu\mu}{}_{|\Omega_h^{(j)}}), \quad j = 0,1$$

(5.8b) $$(\tilde{p}_2 e_H^{\nu\mu})_{|\Omega_h^{(j)}} = (\alpha_1 + (-1)^j \alpha_2 - \alpha_3 - (-1)^j \alpha_4)(e_h^{\nu\mu}{}_{|\Omega_h^{(j)}}), \quad j = 2,3 .$$

On the other hand, we have

$$(\tilde{p}_2 e_H^{\nu\mu})_{|\Omega_h^{(j)}} = \left(\begin{bmatrix} -\tilde{A}_{11}^{-1}\tilde{A}_{12} \\ I \end{bmatrix} e_H^{\nu\mu} \right)_{|\Omega_h^{(j)}} .$$

For $j = 0$ ($\Omega_h^{(0)} = \Omega_H$) this yields $(\tilde{p}_2 e_H^{\nu\mu})_{|\Omega_h^{(0)}} = e_H^{\nu\mu} = e_h^{\nu\mu}{}_{|\Omega_h^{(0)}}$, and for $j = 1,2,3$ we get the following, with $\Omega_h^c := \Omega_h \backslash \Omega_H$:

$$\begin{aligned}
(\tilde{p}_2 e_H^{\nu\mu})_{|\Omega_h^{(j)}} &= (-\tilde{A}_{11}^{-1}\tilde{A}_{12} e_H^{\nu\mu})_{|\Omega_h^{(j)}} \\
&= (-\tilde{A}_{11}^{-1}\{-\tilde{A}_{11}(e_h^{\nu\mu}{}_{|\Omega_h^c}) + \tilde{A}_{11}(e_h^{\nu\mu}{}_{|\Omega_h^c}) + \tilde{A}_{12}(e_h^{\nu\mu}{}_{|\Omega_H})\})_{|\Omega_h^{(j)}} \\
&= e_h^{\nu\mu}{}_{|\Omega_h^{(j)}} - (\tilde{A}_{11}^{-1}\tilde{A} e_h^{\nu\mu})_{|\Omega_h^{(j)}} = e_h^{\nu\mu}{}_{|\Omega_h^{(j)}} - \tau_{(j)}^{\nu\mu} e_h^{\nu\mu}{}_{|\Omega_h^{(j)}} \quad \text{(cf.(5.6))} \\
&= (1 - \tau_{(j)}^{\nu\mu}) e_h^{\nu\mu}{}_{|\Omega_h^{(j)}} .
\end{aligned}$$

Combining this with (5.8) yields the following equations for the unknowns $\alpha_1, \alpha_2, \alpha_3, \alpha_4$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 - \tau_{(1)}^{\nu\mu} \\ 1 - \tau_{(2)}^{\nu\mu} \\ 1 - \tau_{(3)}^{\nu\mu} \end{bmatrix} .$$

The matrix above has orthogonal columns. Inverting the matrix yields

$$\text{(5.9)} \qquad \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 - (\tau_{(1)}^{\nu\mu} + \tau_{(2)}^{\nu\mu} + \tau_{(3)}^{\nu\mu}) \\ \tau_{(1)}^{\nu\mu} - \tau_{(2)}^{\nu\mu} + \tau_{(3)}^{\nu\mu} \\ -\tau_{(1)}^{\nu\mu} + \tau_{(2)}^{\nu\mu} + \tau_{(3)}^{\nu\mu} \\ \tau_{(1)}^{\nu\mu} + \tau_{(2)}^{\nu\mu} - \tau_{(3)}^{\nu\mu} \end{bmatrix} .$$

The Fourier transformation yields

$$\mathcal{S}_{\tilde{A}} e_H^{\nu\mu} = [0 \;\; I] A \tilde{p}_2 e_H^{\nu\mu} = [1 \; 1 \; 1 \; 1] \begin{bmatrix} d_1^{\nu\mu} & & & \emptyset \\ & d_2^{\nu\mu} & & \\ & & d_3^{\nu\mu} & \\ \emptyset & & & d_4^{\nu\mu} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} e_H^{\nu\mu}$$

$$=: \xi^{\nu\mu} e_H^{\nu\mu} ,$$

with eigenvalue $\xi^{\nu\mu} = \sum_{i=1}^{4} d_i^{\nu\mu} \alpha_i$. Using (5.9) we get

$$\begin{aligned}
\xi^{\nu\mu} = d_1^{\nu\mu} \quad & + \tfrac{1}{4}(\tau_{(1)}^{\nu\mu} + \tau_{(2)}^{\nu\mu})(d_4^{\nu\mu} - d_1^{\nu\mu}) + \tfrac{1}{4}(\tau_{(1)}^{\nu\mu} - \tau_{(2)}^{\nu\mu})(d_2^{\nu\mu} - d_3^{\nu\mu}) \\
& + \tfrac{1}{4}\tau_{(3)}^{\nu\mu}((d_2^{\nu\mu} + d_3^{\nu\mu}) - (d_1^{\nu\mu} + d_4^{\nu\mu})) .
\end{aligned}$$

Finally, note that the last term in the right hand side is equal to zero due to (3.3). □

Expressions for the $\tau_{(j)}^{\nu\mu}$, $j = 1,2$, used in Lemma 5.3 are given in Lemma 5.4 below.

Lemma 5.4. *If* $[\bar{A}_{11} \;\; \bar{A}_{12}]$ *is obtained by a "discretization" process as in* (4.12), (4.13) *then the following holds. With*

$$\gamma_{\nu\mu} := 1 - \cos(\nu\pi h)\cos(\mu\pi h) = 2(s_\mu^2 c_\nu^2 + s_\nu^2 c_\mu^2) \in [0, 1] \tag{5.10}$$

we have the following expressions for $\tau_{(j)}^{\nu\mu}$ $(j = 1, 2)$:

$$\tau_{(1)}^{\nu\mu} = \{4\alpha_h(s_\nu^2 + s_\mu^2 c_\mu^2(c_\nu^2 - s_\nu^2)) + a\psi_\nu \tag{5.11a}$$

$$+(1-a)(\psi_\mu + \gamma_{\nu\mu}(1-\psi_\mu))\}/(2\alpha_h + 1)$$

$$\tau_{(2)}^{\nu\mu} = \{4\alpha_h(s_\mu^2 + s_\nu^2 c_\nu^2(c_\mu^2 - s_\mu^2)) \tag{5.11b}$$

$$+a(\psi_\nu + \gamma_{\nu\mu}(1-\psi_\nu)) + (1-a)\psi_\mu\}/(2\alpha_h + 1)\ .$$

Proof. The result in (5.11b) is a direct consequence of the definition of $\bar{A}_{|\Omega_h^{(2)}}$ as in (4.12) and the following equalities:

$$\begin{bmatrix} -1/8 & 0 & -3/4 & 0 & -1/8 \\ 0 & 0 & 2 & 0 & 0 \\ -1/8 & 0 & -3/4 & 0 & -1/8 \end{bmatrix} e_h^{\nu\mu} = 4(s_\mu^2 + s_\nu^2 c_\nu^2(c_\mu^2 - s_\mu^2))e_h^{\nu\mu}\ ,$$

$$\begin{bmatrix} -1/4 & 0 & -1/4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1/4 & 0 & -1/4 & 0 & 0 \end{bmatrix} e_h^{\nu\mu} = (\psi_\nu + \gamma_{\nu\mu}(1-\psi_\nu))e_h^{\nu\mu}\ ,$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} e_h^{\nu\mu} = \psi_\mu e_h^{\nu\mu}\ , \quad \bar{A}_{11|\Omega_h^{(2)}} = \mathrm{diag}(2\alpha_h + 1)\ .$$

Using similar arguments the result in (5.11a) can be proved. □

In Lemma 5.1 and Lemma 5.3 we derived expressions for the eigenvalues of $\mathcal{S}_A$ and $\mathcal{S}_{\bar{A}}$ respectively. The influence of the discretization approach, i.e. replacing $[A_{11} \;\; A_{12}]$ by $[\bar{A}_{11} \;\; \bar{A}_{12}]$, is expressed through the eigenvalues $\tau_{(j)}^{\nu\mu}$, $j = 1, 2$, in Lemma 5.3. In Lemma 5.4 expressions for these $\tau_{(j)}^{\nu\mu}$ are given that correspond to our particular discretization strategy as in (4.12), (4.13). As is shown in Lemma 4.1, there is a direct relation between $\sigma(\mathcal{S}_{\bar{A}}^{-1}\mathcal{S}_A)$ and the convergence of our two-grid method. Clearly, expressions for the eigenvalues of $\mathcal{S}_{\bar{A}}^{-1}\mathcal{S}_A$ are obtained by combining the results of Lemmas 5.1, 5.3, and 5.4. Estimates concerning $\sigma(\mathcal{S}_{\bar{A}}^{-1}\mathcal{S}_A)$ will be given in the next three sections. In Sect. 6 and Sect. 7 we consider special cases, namely pure diffusion (Sect. 6) and pure convection (Sect. 7). In Sect. 8 we consider the general situation.

6. $\sigma(\mathscr{S}_{\bar{A}}^{-1}\mathscr{S}_A)$: the special case of pure diffusion

In this section we analyze the case with $\varepsilon = \infty$ (cf. (2.3)). So the parameters a and ε vanish and only the mesh size parameter h remains. Moreover, we have a symmetric operator A_h, corresponding to the standard 5-point stencil of the Laplacian, and thus a real spectrum.

In Lemma 6.1 below we derive expressions for $\tau_{(1)}^{\nu\mu} \pm \tau_{(2)}^{\nu\mu}$ (cf. (5.11)) that are valid for the general case of convection-diffusion, i.e. $\varepsilon \in (0,\infty)$. We will use the following rescaled real eigenvalues (cf. (3.2)):

$$\bar{\lambda}_j^{\nu\mu} := \lambda_j^{\nu\mu}/\alpha_h \quad j = 1,2,3,4 \ . \tag{6.1}$$

Lemma 6.1. *For $\tau_{(j)}^{\nu\mu}$ as in (5.11) the following holds ($\gamma_{\nu\mu}$ as in (5.10)):*

$$\tfrac{1}{2}(\tau_{(1)}^{\nu\mu} + \tau_{(2)}^{\nu\mu}) = \{d_1^{\nu\mu} + \tfrac{1}{4}\gamma_{\nu\mu}(d_4^{\nu\mu} - d_1^{\nu\mu}) - \tfrac{1}{2}\lambda_1^{\nu\mu} - \tfrac{1}{8}\gamma_{\nu\mu}(\lambda_4^{\nu\mu} - \lambda_1^{\nu\mu})\}/(2\alpha_h + 1) \tag{6.2a}$$

$$\tfrac{1}{2}(\tau_{(1)}^{\nu\mu} - \tau_{(2)}^{\nu\mu}) = -\tfrac{1}{4}\gamma_{\nu\mu}\{d_2^{\nu\mu} - d_3^{\nu\mu} - \tfrac{1}{2}(\lambda_2^{\nu\mu} - \lambda_3^{\nu\mu})\}/(2\alpha_h + 1) \ . \tag{6.2b}$$

Proof. From (5.11) we have that

$$\begin{aligned}\tfrac{1}{2}(2\alpha_h + 1)(\tau_{(1)}^{\nu\mu} + \tau_{(2)}^{\nu\mu}) &= 2\alpha_h\{s_\nu^2 + s_\mu^2 + s_\mu^2 c_\mu^2(c_\nu^2 - s_\nu^2) + s_\nu^2 c_\nu^2(c_\mu^2 - s_\mu^2)\} \\ &\quad + a\psi_\nu + (1-a)\psi_\mu + \tfrac{1}{2}\gamma_{\nu\mu}\{(1-a)(1-\psi_\mu) + a(1-\psi_\nu)\} \\ &= d_1^{\nu\mu} - \tfrac{1}{2}\lambda_1^{\nu\mu} + 2\alpha_h\{s_\mu^2 c_\mu^2(c_\nu^2 - s_\nu^2) + s_\nu^2 c_\nu^2(c_\mu^2 - s_\mu^2)\} \\ &\quad + \tfrac{1}{2}\gamma_{\nu\mu}\{(1-a)(1-\psi_\mu) + a(1-\psi_\nu)\} \ .\end{aligned} \tag{6.3}$$

Straightforward computations show that

$$(1-a)(1-\psi_\mu) + a(1-\psi_\nu) = \tfrac{1}{2}(\varphi_4^{\nu\mu} - \varphi_1^{\nu\mu})$$

and

$$2\alpha_h\{s_\mu^2 c_\mu^2(c_\nu^2 - s_\nu^2) + s_\nu^2 c_\nu^2(c_\mu^2 - s_\mu^2)\} = \tfrac{1}{8}\gamma_{\nu\mu}(\lambda_4^{\nu\mu} - \lambda_1^{\nu\mu}) \ .$$

Using these two equalities and $d_j^{\nu\mu} = \lambda_j^{\nu\mu} + \varphi_j^{\nu\mu}$ in (6.3) yields the result in (6.2a). With respect to (6.2b) we note that

$$\begin{aligned}\tfrac{1}{2}(2\alpha_h + 1)(\tau_{(1)}^{\nu\mu} - \tau_{(2)}^{\nu\mu}) &= 2\alpha_h\{s_\nu^2 - s_\mu^2 + s_\mu^2 c_\mu^2(c_\nu^2 - s_\nu^2) - s_\nu^2 c_\nu^2(c_\mu^2 - s_\mu^2)\} \\ &\quad + \tfrac{1}{2}\gamma_{\nu\mu}\{(1-a)(1-\psi_\mu) - a(1-\psi_\nu)\} \ ,\end{aligned}$$

and that

$$(1-a)(1-\psi_\mu) - a(1-\psi_\nu) = -\tfrac{1}{2}(\varphi_2^{\nu\mu} - \varphi_3^{\nu\mu}) \ ,$$

$$2\alpha_h(s_\nu^2 - s_\mu^2 + s_\mu^2 c_\mu^2(c_\nu^2 - s_\nu^2) - s_\nu^2 c_\nu^2(c_\mu^2 - s_\mu^2)) = -\tfrac{1}{8}\gamma_{\nu\mu}(\lambda_2^{\nu\mu} - \lambda_3^{\nu\mu}) \ . \quad \square$$

For $\varepsilon \to \infty$ we have $d_j^{\nu\mu}/(2\alpha_h + 1) \to \frac{1}{2}\bar{\lambda}_j^{\nu\mu}$, $\lambda_j^{\nu\mu}/(2\alpha_h + 1) \to \frac{1}{2}\bar{\lambda}_j^{\nu\mu}$ and for $\varepsilon \downarrow 0$ we have $d_j^{\nu\mu}/(2\alpha_h + 1) \to \varphi_j^{\nu\mu}$, $\lambda_j^{\nu\mu}/(2\alpha_h + 1) \to 0$. Using this in Lemma 6.1 results in

Corollary 6.2. From Lemma 6.1 we derive the following results on $\varepsilon \to \infty$ and for $\varepsilon \downarrow 0$:

(6.4a) $$\lim_{\varepsilon \to \infty} \tfrac{1}{2}(\tau_{(1)}^{\nu\mu} + \tau_{(2)}^{\nu\mu}) = \tfrac{1}{4}(\bar{\lambda}_1^{\nu\mu} + \tfrac{1}{4}\gamma_{\nu\mu}(\bar{\lambda}_4^{\nu\mu} - \bar{\lambda}_1^{\nu\mu})) =: \tau_\infty^+$$

(6.4b) $$\lim_{\varepsilon \to \infty} \tfrac{1}{2}(\tau_{(1)}^{\nu\mu} - \tau_{(2)}^{\nu\mu}) = -\tfrac{1}{16}\gamma_{\nu\mu}(\bar{\lambda}_2^{\nu\mu} - \bar{\lambda}_3^{\nu\mu}) =: \tau_\infty^-$$

(6.4c) $$\lim_{\varepsilon \downarrow 0} \tfrac{1}{2}(\tau_{(1)}^{\nu\mu} + \tau_{(2)}^{\nu\mu}) = \varphi_1^{\nu\mu} + \tfrac{1}{4}\gamma_{\nu\mu}(\varphi_4^{\nu\mu} - \varphi_1^{\nu\mu}) =: \tau_0^+$$

(6.4d) $$\lim_{\varepsilon \downarrow 0} \tfrac{1}{2}(\tau_{(1)}^{\nu\mu} - \tau_{(2)}^{\nu\mu}) = -\tfrac{1}{4}\gamma_{\nu\mu}(\varphi_2^{\nu\mu} - \varphi_3^{\nu\mu}) =: \tau_0^- \ .$$

In the remainder of this section we consider $\sigma(\mathcal{S}_{\tilde{A}}^{-1}\mathcal{S}_A)$ for the case $\varepsilon = \infty$. For convenience we rescale the eigenvalues $d_j^{\nu\mu}$ with a factor α_h^{-1}; then $\lim_{\varepsilon \to \infty} d_j^{\nu\mu} = \bar{\lambda}_j^{\nu\mu}$ holds ($\bar{\lambda}_j^{\nu\mu}$ as in (6.1)). From Lemma 5.1, 5.3 we have that $e_H^{\nu\mu}$ ($(\nu, \mu) \neq (0, 0)$) is an eigenvector of $\mathcal{S}_A$ and of $\mathcal{S}_{\tilde{A}}$ with eigenvalue

(6.5a) $$H(\bar{\lambda}_1^{\nu\mu}, \bar{\lambda}_2^{\nu\mu}, \bar{\lambda}_3^{\nu\mu}, \bar{\lambda}_4^{\nu\mu}) \quad \text{and}$$

(6.5b) $$\bar{\lambda}_1^{\nu\mu} + \tfrac{1}{2}\tau_\infty^+(\bar{\lambda}_4^{\nu\mu} - \bar{\lambda}_1^{\nu\mu}) + \tfrac{1}{2}\tau_\infty^-(\bar{\lambda}_2^{\nu\mu} - \bar{\lambda}_3^{\nu\mu})$$

respectively. Here $\tau_\infty^+, \tau_\infty^-$ are defined in (6.4a, b) and we recall that $\bar{\lambda}_1^{\nu\mu} = 4(s_\nu^2 + s_\mu^2)$, $\bar{\lambda}_2^{\nu\mu} = 4(c_\nu^2 + s_\mu^2)$, $\bar{\lambda}_3^{\nu\mu} = 8 - \bar{\lambda}_2^{\nu\mu}$, $\bar{\lambda}_4^{\nu\mu} = 8 - \bar{\lambda}_1^{\nu\mu}$.

Lemma 6.3. *The Fourier mode $e_H^{\nu\mu}$, $(\nu, \mu) \neq (0, 0)$, is an eigenvector of $\mathcal{S}_{\tilde{A}}$ with eigenvalue*

(6.6) $$H(\bar{\lambda}_1^{\nu\mu}, \bar{\lambda}_4^{\nu\mu}) + \tfrac{1}{2}\gamma_{\nu\mu}(H(\bar{\lambda}_2^{\nu\mu}, \bar{\lambda}_3^{\nu\mu}) - H(\bar{\lambda}_1^{\nu\mu}, \bar{\lambda}_4^{\nu\mu})) \ .$$

Proof. Substitution of (6.4a, b) in (6.5b) yields the following expression for the eigenvalue:

(6.7) $$\bar{\lambda}_1^{\nu\mu} + \tfrac{1}{8}(\bar{\lambda}_1^{\nu\mu} + \tfrac{1}{4}\gamma_{\nu\mu}(\bar{\lambda}_4^{\nu\mu} - \bar{\lambda}_1^{\nu\mu}))(\bar{\lambda}_4^{\nu\mu} - \bar{\lambda}_1^{\nu\mu}) - \tfrac{1}{32}\gamma_{\nu\mu}(\bar{\lambda}_2^{\nu\mu} - \bar{\lambda}_3^{\nu\mu})^2$$

$$= \bar{\lambda}_1^{\nu\mu} + \tfrac{1}{8}\bar{\lambda}_1^{\nu\mu}(\bar{\lambda}_4^{\nu\mu} - \bar{\lambda}_1^{\nu\mu}) + \tfrac{1}{2}\gamma_{\nu\mu}\ \tfrac{1}{16}((\bar{\lambda}_4^{\nu\mu} - \bar{\lambda}_1^{\nu\mu})^2 - (\bar{\lambda}_2^{\nu\mu} - \bar{\lambda}_3^{\nu\mu})^2) \ .$$

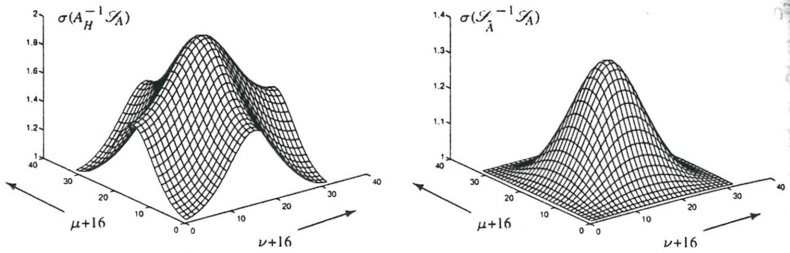
Note (cf. (3.12a, b)) that

(6.8) $$H(\bar{\lambda}_1^{\nu\mu}, \bar{\lambda}_4^{\nu\mu}) = \bar{\lambda}_1^{\nu\mu} + \tfrac{1}{8}\bar{\lambda}_1^{\nu\mu}(\bar{\lambda}_4^{\nu\mu} - \bar{\lambda}_1^{\nu\mu}) \ ,$$

and also $H(\bar{\lambda}_1^{\nu\mu}, \bar{\lambda}_4^{\nu\mu}) = \frac{1}{4}\bar{\lambda}_1^{\nu\mu}\bar{\lambda}_4^{\nu\mu}$, $H(\bar{\lambda}_2^{\nu\mu}, \bar{\lambda}_3^{\nu\mu}) = \frac{1}{4}\bar{\lambda}_2^{\nu\mu}\bar{\lambda}_3^{\nu\mu}$. Therefore

(6.9) $$\tfrac{1}{16}((\bar{\lambda}_4^{\nu\mu} - \bar{\lambda}_1^{\nu\mu})^2 - (\bar{\lambda}_2^{\nu\mu} - \bar{\lambda}_3^{\nu\mu})^2) + (H(\bar{\lambda}_1^{\nu\mu}, \bar{\lambda}_4^{\nu\mu}) - H(\bar{\lambda}_2^{\nu\mu}, \bar{\lambda}_3^{\nu\mu}))$$

$$= \tfrac{1}{\ }((\bar{\lambda}^{\nu\mu} + \bar{\lambda}^{\nu\mu})^2 - (\bar{\lambda}^{\nu\mu} + \bar{\lambda}^{\nu\mu})^2) = 0 \quad \text{(use (3.3))}$$



$\sigma(A_H^{-1}\mathcal{S}_A)$ $\sigma(\mathcal{S}_{\tilde{A}}^{-1}\mathcal{S}_A)$

Fig. 2.

Combination of (6.7), (6.8), (6.9) yields the result in (6.6). □

We see that the approximation of $\mathcal{S}_A$ by $\mathcal{S}_{\tilde{A}}$ corresponds, in terms of eigenvalues, to the approximation of the harmonic mean $H(\bar{\lambda}_1^{\nu\mu}, \bar{\lambda}_2^{\nu\mu}, \bar{\lambda}_3^{\nu\mu}, \bar{\lambda}_4^{\nu\mu})$ by the eigenvalue as in (6.6). Note the similarity between (6.6) and the expression for the harmonic mean in (3.12c). An alternative approach would be to approximate $\mathcal{S}_A$ by the discretization of the differential operator on the coarse grid (=: A_H). In the setting here this yields the standard 5-point star of the Laplace operator, with eigenvalues $2(\sin^2(\nu\pi h) + \sin^2(\mu\pi h)) = 8(s_\nu^2 c_\nu^2 + s_\mu^2 c_\mu^2)$.

Summarizing we have the following relevant eigenvalues, denoted by $\xi^{\nu\mu}(\cdot)$:

(6.10a) $\quad \xi^{\nu\mu}(\mathcal{S}_A) = H(\bar{\lambda}_1^{\nu\mu}, \bar{\lambda}_2^{\nu\mu}, \bar{\lambda}_3^{\nu\mu}, \bar{\lambda}_4^{\nu\mu})$

(6.10b) $\quad \xi^{\nu\mu}(\mathcal{S}_{\tilde{A}}) = H(\bar{\lambda}_1^{\nu\mu}, \bar{\lambda}_4^{\nu\mu}) + \frac{1}{2}\gamma_{\nu\mu}(H(\bar{\lambda}_2^{\nu\mu}, \bar{\lambda}_3^{\nu\mu}) - H(\bar{\lambda}_1^{\nu\mu}, \bar{\lambda}_4^{\nu\mu}))$

(6.10c) $\quad \xi^{\nu\mu}(A_H) = 8(s_\nu^2 c_\nu^2 + s_\mu^2 c_\mu^2)$.

and we are interested in $\xi^{\nu\mu}(\mathcal{S}_A)/\xi^{\nu\mu}(\mathcal{S}_{\tilde{A}})$ and $\xi^{\nu\mu}(\mathcal{S}_A)/\xi^{\nu\mu}(A_H)$ $((\nu,\mu) \neq (0,0))$. The strengthened CBS inequality as in [11] yields the following:

(6.11) $\quad \xi^{\nu\mu}(\mathcal{S}_A)/\xi^{\nu\mu}(A_H) \in [1,2]$.

Theorem 6.4. *For* $(\nu,\mu) \neq (0,0)$ $\;\xi^{\nu\mu}(\mathcal{S}_A)/\xi^{\nu\mu}(\mathcal{S}_{\tilde{A}}) \in [1, \frac{4}{3}]$ *holds.*

Proof. We take $(\nu,\mu) \neq (0,0)$ and introduce the notation $\gamma := \gamma_{\nu\mu} = 2(s_\nu^2 c_\mu^2 + s_\mu^2 c_\nu^2)$, $\beta := 2(s_\nu^2 c_\nu^2 + s_\mu^2 c_\mu^2)$. Note that $0 \leq \beta \leq \gamma \leq 1$ holds. Also $h_{ij} := H(\bar{\lambda}_i^{\nu\mu}, \bar{\lambda}_j^{\nu\mu}) = \frac{1}{4}\bar{\lambda}_i^{\nu\mu}\bar{\lambda}_j^{\nu\mu}$. A straightforward calculation shows that $h_{14} = 2(\gamma + \beta)$, $h_{23} - h_{14} = 4(1-\gamma)$ and $h_{14}/(h_{23} + h_{14}) = \frac{1}{2}(\beta+\gamma)/(\beta+1)$. Using the results in (6.10a, b) and (3.12c) we get

$$\begin{aligned}\xi^{\nu\mu}(\mathscr{S}_A)/\xi^{\nu\mu}(\mathscr{S}_{\tilde A}) &= 1 + (\xi^{\nu\mu}(\mathscr{S}_A) - \xi^{\nu\mu}(\mathscr{S}_{\tilde A}))/\xi^{\nu\mu}(\mathscr{S}_{\tilde A}) \\ &= 1 + \left(\frac{h_{14}}{h_{23}+h_{14}} - \tfrac{1}{2}\gamma\right)(h_{23}-h_{14})/(h_{14}+\tfrac{1}{2}\gamma(h_{23}-h_{14})) \\ &= 1 + \frac{\beta(1-\gamma)^2}{(\beta+1)(\beta+\gamma(2-\gamma))} =: 1 + f(\gamma,\beta)\ .\end{aligned}$$

An elementary analysis shows that for $0 \le \beta \le \gamma \le 1,\ 0 \le f(\gamma,\beta) \le f(\gamma,\gamma) \le \frac{1}{3}$ holds. □

With optimal damping we have that $\rho(I - \omega_{\text{opt}} A_H^{-1}.\mathscr{S}_A) \le \frac{1}{3}$ and $\rho(I - \omega_{\text{opt}}.\mathscr{S}_{\tilde A}^{-1}.\mathscr{S}_A) \le \frac{1}{7}$ and these bounds are sharp for $h \downarrow 0$. In Fig. 2 we show $\sigma(A_H^{-1}.\mathscr{S}_A)$ and $\sigma(\mathscr{S}_{\tilde A}^{-1}.\mathscr{S}_A)$ for $h = 1/32$, i.e. $N_H = 16$ and $-15 \le \nu, \mu \le 16$ $((\nu,\mu) \ne (0,0))$.

From the results above we conclude that $\sigma(.\mathscr{S}_{\tilde A}^{-1}.\mathscr{S}_A)$ is more favourable than $\sigma(A_H^{-1}.\mathscr{S}_A)$ in two respects. Firstly, $\rho(I - \omega_{\text{opt}}.\mathscr{S}_{\tilde A}^{-1}.\mathscr{S}_A) \approx 1/7$ is (significantly) smaller than $\rho(I - \omega_{\text{opt}} A_H^{-1}.\mathscr{S}_A) \approx 1/3$ and secondly, we observe a clustering of the eigenvalues in $\sigma(.\mathscr{S}_{\tilde A}^{-1}.\mathscr{S}_A)$ close to 1. This can be seen from Fig. 4, too. For the eigenvalues in Fig. 2 we have mean$\{\sigma(.\mathscr{S}_{\tilde A}^{-1}.\mathscr{S}_A)\} = 1.07$ and mean$\{\sigma(A_H^{-1}.\mathscr{S}_A)\} = 1.42$.

7. $\sigma(.\mathscr{S}_{\tilde A}^{-1}.\mathscr{S}_A)$: the special case of pure convection

In this section we analyze the case with $\varepsilon = 0,\ h \in (0,1],\ a \in (0,1)$ (cf. (2.3)). We will prove a robustness result for $\sigma(.\mathscr{S}_{\tilde A}^{-1}.\mathscr{S}_A)$ with respect to variation in h and a. Also, to give some further indication of the quality of $.\mathscr{S}_{\tilde A}$ as a preconditioner for $.\mathscr{S}_A$, we have computed $\sigma(.\mathscr{S}_{\tilde A}^{-1}.\mathscr{S}_A)$ for $h = 1/32$ and for several values $a \in (0,1)$.

We start by noting that for $\varepsilon \downarrow 0$ we have $d_j^{\nu\mu} \to \varphi_j^{\nu\mu}$. From Lemma 5.1, 5.3 and Corollary 6.2 we see that $e_H^{\nu\mu}$ $((\nu,\mu) \ne (0,0))$ is an eigenvector of $.\mathscr{S}_A$ and of $.\mathscr{S}_{\tilde A}$ with eigenvalue

(7.1a) $\quad H(\varphi_1^{\nu\mu}, \varphi_2^{\nu\mu}, \varphi_3^{\nu\mu}, \varphi_4^{\nu\mu})$ and

(7.1b) $\quad \varphi_1^{\nu\mu} + \frac{1}{2}\tau_0^+(\varphi_4^{\nu\mu} - \varphi_1^{\nu\mu}) + \frac{1}{2}\tau_0^-(\varphi_2^{\nu\mu} - \varphi_3^{\nu\mu})$

respectively. Here τ_0^+ and τ_0^- are as in (6.4c, d).

Lemma 7.1. *The Fourier mode $e_H^{\nu\mu}$, $(\nu,\mu) \ne (0,0)$, is an eigenvector of $.\mathscr{S}_{\tilde A}$ with eigenvalue*

(7.2) $\quad H(\varphi_1^{\nu\mu}, \varphi_4^{\nu\mu}) + \frac{1}{2}\gamma_{\nu\mu}(H(\varphi_2^{\nu\mu}, \varphi_3^{\nu\mu}) - H(\varphi_1^{\nu\mu}, \varphi_4^{\nu\mu}))\ .$

Proof. Substitution of (6.4c, d) in (7.1b) yields the following expression for the eigenvalue:

$$\varphi_1^{\nu\mu} + \tfrac{1}{2}(\varphi_1^{\nu\mu} + \tfrac{1}{4}\gamma_{\nu\mu}(\varphi_4^{\nu\mu} - \varphi_1^{\nu\mu}))(\varphi_4^{\nu\mu} - \varphi_1^{\nu\mu}) - \tfrac{1}{8}\gamma_{\nu\mu}(\varphi_2^{\nu\mu} - \varphi_3^{\nu\mu})^2$$

$$= \varphi_1^{\nu\mu} + \tfrac{1}{2}\varphi_1^{\nu\mu}(\varphi_4^{\nu\mu} - \varphi_1^{\nu\mu}) + \tfrac{1}{2}\gamma_{\nu\mu}\tfrac{1}{4}((\varphi_4^{\nu\mu} - \varphi_1^{\nu\mu})^2 - (\varphi_2^{\nu\mu} - \varphi_3^{\nu\mu})^2) .$$

Note that (cf. (3.12a, b))

$$H(\varphi_1^{\nu\mu}, \varphi_4^{\nu\mu}) = \varphi_1^{\nu\mu} + \tfrac{1}{2}\varphi_1^{\nu\mu}(\varphi_4^{\nu\mu} - \varphi_1^{\nu\mu}) ,$$

$$H(\varphi_2^{\nu\mu}, \varphi_3^{\nu\mu}) = \varphi_2^{\nu\mu}\varphi_3^{\nu\mu} , \quad H(\varphi_1^{\nu\mu}, \varphi_4^{\nu\mu}) = \varphi_1^{\nu\mu}\varphi_4^{\nu\mu} ,$$

and

$$\tfrac{1}{4}((\varphi_4^{\nu\mu} - \varphi_1^{\nu\mu})^2 - (\varphi_2^{\nu\mu} - \varphi_3^{\nu\mu})^2) + (H(\varphi_1^{\nu\mu}, \varphi_4^{\nu\mu}) - H(\varphi_2^{\nu\mu}, \varphi_3^{\nu\mu}))$$

$$= \tfrac{1}{4}((\varphi_1^{\nu\mu} + \varphi_4^{\nu\mu})^2 - (\varphi_2^{\nu\mu} + \varphi_3^{\nu\mu})^2) = 0 \quad \text{(use (3.3))} . \qquad \square$$

Note the similarity between the expressions in (6.6) and in (7.2).
Below we compare $\xi^{\nu\mu}(\mathscr{S}_A) := H(\varphi_1^{\nu\mu}, \varphi_2^{\nu\mu}, \varphi_3^{\nu\mu}, \varphi_4^{\nu\mu})$ with

$$\xi^{\nu\mu}(\tilde{\mathscr{S}}_A) := H(\varphi_1^{\nu\mu}, \varphi_4^{\nu\mu}) + \tfrac{1}{2}\gamma_{\nu\mu}(H(\varphi_2^{\nu\mu}, \varphi_3^{\nu\mu}) - H(\varphi_1^{\nu\mu}, \varphi_4^{\nu\mu})) .$$

We are interested in estimates for $\xi^{\nu\mu}(\mathscr{S}_A)/\xi^{\nu\mu}(\tilde{\mathscr{S}}_A)$ that are uniform in h and a. Such estimates are given in Theorem 7.2 below, where (for ease) we consider the inverse eigenvalues $\xi^{\nu\mu}(\tilde{\mathscr{S}}_A)/\xi^{\nu\mu}(\mathscr{S}_A)$.

Theorem 7.2. *The following holds for* $(\nu, \mu) \neq (0, 0)$:

(7.3a) $\quad \mathrm{Re}(\xi^{\nu\mu}(\tilde{\mathscr{S}}_A)/\xi^{\nu\mu}(\mathscr{S}_A)) \geq \tfrac{1}{2}$

(7.3b) $\quad |\xi^{\nu\mu}(\tilde{\mathscr{S}}_A)/\xi^{\nu\mu}(\mathscr{S}_A)| \leq 2$.

Proof. If we substitute $H(\varphi_i^{\nu\mu}, \varphi_j^{\nu\mu}) = \varphi_i^{\nu\mu}\varphi_j^{\nu\mu}$, $(i, j) \in \{(2, 3), (1, 4)\}$, in the expression for $\xi^{\nu\mu}(\tilde{\mathscr{S}}_A)$ above, then we get

$$q^{\nu\mu} := \xi^{\nu\mu}(\tilde{\mathscr{S}}_A)/\xi^{\nu\mu}(\mathscr{S}_A) = \tfrac{1}{4}((1 - \tfrac{1}{2}\gamma_{\nu\mu})\varphi_1^{\nu\mu}\varphi_4^{\nu\mu} + \tfrac{1}{2}\gamma_{\nu\mu}\varphi_2^{\nu\mu}\varphi_3^{\nu\mu}) \sum_{j=1}^{4} 1/\varphi_j^{\nu\mu} .$$

Note that $\varphi_1^{\nu\mu} + \varphi_4^{\nu\mu} = \varphi_2^{\nu\mu} + \varphi_3^{\nu\mu} = 2$, $\gamma_{\nu\mu} \in [0, 1]$; using (3.7c) we see that

$$\begin{aligned} \mathrm{Re}(q^{\nu\mu}) &= \tfrac{1}{2}\mathrm{Re}\{((1 - \tfrac{1}{2}\gamma_{\nu\mu})\varphi_1^{\nu\mu}\varphi_4^{\nu\mu} + \tfrac{1}{2}\gamma_{\nu\mu}\varphi_2^{\nu\mu}\varphi_3^{\nu\mu})((\varphi_1^{\nu\mu}\varphi_4^{\nu\mu})^{-1} \\ &\quad + (\varphi_2^{\nu\mu}\varphi_3^{\nu\mu})^{-1})\} \\ &= \tfrac{1}{2}\mathrm{Re}\{1 + (1 - \tfrac{1}{2}\gamma_{\nu\mu})\varphi_1^{\nu\mu}\varphi_4^{\nu\mu}(\varphi_2^{\nu\mu}\varphi_3^{\nu\mu})^{-1} \\ &\quad + \tfrac{1}{2}\gamma_{\nu\mu}\varphi_2^{\nu\mu}\varphi_3^{\nu\mu}(\varphi_1^{\nu\mu}\varphi_4^{\nu\mu})^{-1}\} \\ &= \tfrac{1}{2} + \tfrac{1}{4}(1 - \tfrac{1}{2}\gamma_{\nu\mu})\mathrm{Re}\{\varphi_1^{\nu\mu}\varphi_4^{\nu\mu}(1/\varphi_2^{\nu\mu} + 1/\varphi_3^{\nu\mu})\} \\ &\quad + \tfrac{1}{8}\gamma_{\nu\mu}\mathrm{Re}\{\varphi_2^{\nu\mu}\varphi_3^{\nu\mu}(1/\varphi_1^{\nu\mu} + 1/\varphi_4^{\nu\mu})\} \geq \tfrac{1}{2} . \end{aligned}$$

This proves (7.3a). We now consider (7.3b):

$$
\begin{aligned}
|q^{\nu\mu}| &= \tfrac{1}{2}|(\varphi_1^{\nu\mu}\varphi_4^{\nu\mu} + \tfrac{1}{2}\gamma_{\nu\mu}(\varphi_2^{\nu\mu}\varphi_3^{\nu\mu} - \varphi_1^{\nu\mu}\varphi_4^{\nu\mu}))(\varphi_1^{\nu\mu}\varphi_4^{\nu\mu})^{-1} \\
&\quad +(\varphi_2^{\nu\mu}\varphi_3^{\nu\mu} + (1 - \tfrac{1}{2}\gamma_{\nu\mu})(\varphi_1^{\nu\mu}\varphi_4^{\nu\mu} - \varphi_2^{\nu\mu}\varphi_3^{\nu\mu}))(\varphi_2^{\nu\mu}\varphi_3^{\nu\mu})^{-1}| \\
&\le 1 + \tfrac{1}{4}\gamma_{\nu\mu}|\varphi_2^{\nu\mu}\varphi_3^{\nu\mu} - \varphi_1^{\nu\mu}\varphi_4^{\nu\mu}|\, |\varphi_1^{\nu\mu}\varphi_4^{\nu\mu}|^{-1} \\
&\quad +\tfrac{1}{2}(1 - \tfrac{1}{2}\gamma_{\nu\mu})|\varphi_1^{\nu\mu}\varphi_4^{\nu\mu} - \varphi_2^{\nu\mu}\varphi_3^{\nu\mu}|\, |\varphi_2^{\nu\mu}\varphi_3^{\nu\mu}|^{-1} .
\end{aligned} \tag{7.4}
$$

Also we have

$$
\begin{aligned}
|\varphi_2^{\nu\mu}\varphi_3^{\nu\mu} - \varphi_1^{\nu\mu}\varphi_4^{\nu\mu}| &= |\varphi_2^{\nu\mu}(2 - \varphi_2^{\nu\mu}) - \varphi_1^{\nu\mu}(2 - \varphi_1^{\nu\mu})| \\
&= |2(\varphi_2^{\nu\mu} - \varphi_1^{\nu\mu})(1 - \tfrac{1}{2}(\varphi_1^{\nu\mu} + \varphi_2^{\nu\mu}))| \\
&= |4a(1-a)(1-\psi_\nu)(1-\psi_\mu)| = 4a(1-a) .
\end{aligned} \tag{7.5}
$$

For the denominators in (7.4) we use that

$$
\begin{aligned}
|\varphi_1^{\nu\mu}\varphi_4^{\nu\mu}| &= |\varphi_1^{\nu\mu}|\, |\varphi_4^{\nu\mu}| \ge \mathrm{Re}(\varphi_1^{\nu\mu})\mathrm{Re}(\varphi_4^{\nu\mu}) \\
&= 4(as_\nu^2 + (1-a)s_\mu^2)(ac_\nu^2 + (1-a)c_\mu^2) \\
&\ge 4a(1-a)(s_\nu^2 c_\mu^2 + s_\mu^2 c_\nu^2) = 2a(1-a)\gamma_{\nu\mu} ,
\end{aligned} \tag{7.6}
$$

and

$$
\begin{aligned}
|\varphi_2^{\nu\mu}\varphi_3^{\nu\mu}| &\ge \mathrm{Re}(\varphi_2^{\nu\mu})\mathrm{Re}(\varphi_3^{\nu\mu}) = 4(ac_\nu^2 + (1-a)s_\mu^2)(as_\nu^2 + (1-a)c_\mu^2) \\
&\ge 4a(1-a)(c_\nu^2 c_\mu^2 + s_\nu^2 s_\mu^2) = 4a(1-a)(1 - \tfrac{1}{2}\gamma_{\nu\mu}) .
\end{aligned} \tag{7.7}
$$

Using (7.5), (7.6) and (7.7) in (7.4) yields the result in (7.3b). □

From Theorem 7.2 it follows that $\sigma(\mathscr{S}_{\tilde{A}}^{-1}\mathscr{S}_A)$ lies in a bounded domain in the complex right half–plane away from the imaginary axis. Moreover, this domain is independent of the parameters h and a, i.e. we have a robustness result w.r.t. variation in h and a.

In Fig. 3 we show $\sigma(\mathscr{S}_{\tilde{A}}^{-1}\mathscr{S}_A)$, in the complex plane, for $h = 1/32$ and for several values $a \in (0,1)$; due to symmetry it is sufficient to consider $a \in (0, \frac{1}{2}]$. With respect to the results in Fig. 3 we remark the following. Because $\mathscr{S}_{\tilde{A}}^{-1}\mathscr{S}_A$ is real there is symmetry w.r.t. the real axis. For $a = \frac{1}{2}$ eigenvalues coincide due to symmetry. From the results in Fig. 3 it is clear that the estimate in (7.3a) is sharp.

We briefly comment on the two clusters of eigenvalues for a small ($a = 10^{-3}, 10^{-2}$). For $a = 0$ $\mathscr{S}_A$ has kernel span$\{e_H^{\nu\mu} \mid \mu = 0\}$, and it is easy to verify that for $\mu \neq 0$ we have $\xi^{\nu\mu}(\mathscr{S}_A) = \xi^{\nu\mu}(\mathscr{S}_{\tilde{A}})$. For a small the cluster of eigenvalues with real part ≈ 1 (≈ 2) corresponds to the eigenfunctions $e_H^{\nu\mu}$ with $\mu \neq 0$ ($\mu = 0$). As might be expected, the approximation of $\xi^{\nu\mu}(\mathscr{S}_A)$ by $\xi^{\nu\mu}(\mathscr{S}_{\tilde{A}})$ is worse for the eigenvalues which are perturbations (for $a \downarrow 0$) of the zero eigenvalues of $\mathscr{S}_{A|a=0}$.

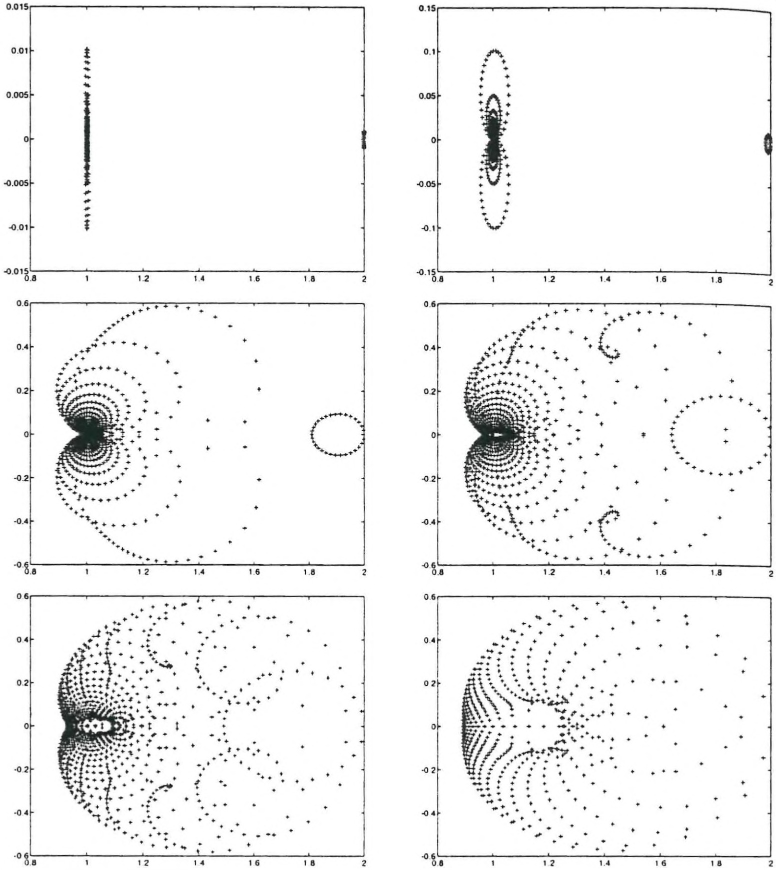


Fig. 3. $\sigma(\mathcal{S}_{\tilde{A}}^{-1}\mathcal{S}_A)$, $\varepsilon = 0$. Top left: $a = 0.001$, top right: $a = 0.01$; middle left: $a = 0.1$, middle right: $a = 0.2$; bottom left: $a = 0.3$, bottom right: $a = 0.5$

Finally we note that in all cases in Fig. 3 we observe a clustering of eigenvalues in some (small) neighbourhood of 1 (as in Sect. 6). Further calculations show that for $a = 0.3, 0.4, 0.5$ about half of the eigenvalues lies in the domain $[0.85, 1.2] \times [-0.2, 0.2]$ with area 0.14, whereas the convex hull of $\sigma(\mathcal{S}_{\tilde{A}}^{-1}\mathcal{S}_A)$ has an area ≈ 1.1.

8. $\sigma(\mathcal{S}_{\tilde{A}}^{-1}\mathcal{S}_A)$: the general case

In this section we analyze the general situation with $\alpha_h \in (0, \infty)$, $h \in (0, 1]$, $a \in (0, 1)$. In principle we follow the approach as in Sects. 6, 7. However, the analysis is more technical because our estimates here have to be uniform in *three* parameters.

Also, to illustrate the dependence of $\sigma(\mathcal{S}_{\tilde{A}}^{-1}\mathcal{S}_A)$ on the convection/diffusion ratio we have computed $\sigma(\mathcal{S}_{\tilde{A}}^{-1}\mathcal{S}_A)$ for $h = 1/32$, $a = 0.4$ and for several values $\alpha_h \in (0, \infty)$.

From Lemma 5.1, 5.3 we see that $e_H^{\nu\mu}$, $(\nu, \mu) \neq (0, 0)$, is an eigenvector of $\mathcal{S}_A$ and of $\mathcal{S}_{\tilde{A}}$ with eigenvalue

(8.1a) $\xi^{\nu\mu}(\mathcal{S}_A) := H(d_1^{\nu\mu}, d_2^{\nu\mu}, d_3^{\nu\mu}, d_4^{\nu\mu})$ and

(8.1b) $\xi^{\nu\mu}(\mathcal{S}_{\tilde{A}}) := d_1^{\nu\mu} + \frac{1}{4}(\tau_{(1)}^{\nu\mu} + \tau_{(2)}^{\nu\mu})(d_4^{\nu\mu} - d_1^{\nu\mu}) + \frac{1}{4}(\tau_{(1)}^{\nu\mu} - \tau_{(2)}^{\nu\mu})(d_2^{\nu\mu} - d_3^{\nu\mu})$

respectively. In (8.1b) we have $\tau_{(1)}^{\nu\mu} \pm \tau_{(2)}^{\nu\mu}$ as in Lemma 6.1.
Based on the expressions for $\tau_{(1)}^{\nu\mu} \pm \tau_{(2)}^{\nu\mu}$ we define the following:

(8.2) $$g_j^{\nu\mu} := \tfrac{1}{2}\lambda_j^{\nu\mu} + \varphi_j^{\nu\mu} = d_j^{\nu\mu} - \tfrac{1}{2}\lambda_j^{\nu\mu}, \quad j = 1, 2, 3, 4 .$$

Then we have

(8.3a) $$\tfrac{1}{2}(\tau_{(1)}^{\nu\mu} + \tau_{(2)}^{\nu\mu}) = (g_1^{\nu\mu} + \tfrac{1}{4}\gamma_{\nu\mu}(g_4^{\nu\mu} - g_1^{\nu\mu}))/(2\alpha_h + 1)$$

(8.3b) $$\tfrac{1}{2}(\tau_{(1)}^{\nu\mu} - \tau_{(2)}^{\nu\mu}) = -\tfrac{1}{4}\gamma_{\nu\mu}(g_2^{\nu\mu} - g_3^{\nu\mu})/(2\alpha_h + 1) .$$

We use (8.2), (8.3a, b) to rewrite the expression in (8.1b). A straightforward calculation then yields the following expression for $\xi^{\nu\mu}(\mathcal{S}_{\tilde{A}})$:

(8.4a) $$\xi^{\nu\mu}(\mathcal{S}_{\tilde{A}}) = T_1 + T_2 + T_3 \text{ , with}$$

(8.4b) $$T_1 = H(g_1^{\nu\mu}, g_4^{\nu\mu}) + \tfrac{1}{2}\gamma_{\nu\mu}(H(g_2^{\nu\mu}, g_3^{\nu\mu}) - H(g_1^{\nu\mu}, g_4^{\nu\mu}))$$

(8.4c) $$T_2 = \tfrac{1}{2}\lambda_1^{\nu\mu} + \tfrac{1}{4}(1 - \tfrac{1}{2}\gamma_{\nu\mu})(\lambda_4^{\nu\mu} - \lambda_1^{\nu\mu})g_1^{\nu\mu}/(2\alpha_h + 1)$$

(8.4d) $$T_3 = \tfrac{1}{8}\gamma_{\nu\mu}\{(2\alpha_h + 1)(\lambda_4^{\nu\mu} - \lambda_1^{\nu\mu}) \\ - \tfrac{1}{2}(\lambda_2^{\nu\mu} - \lambda_3^{\nu\mu})(g_2^{\nu\mu} - g_3^{\nu\mu})\}/(2\alpha_h + 1) .$$

The term T_1 is of the same form as in Lemma 6.3 and in Lemma 7.1. For $\alpha_h \downarrow 0$ the term T_1 is $O(1)$ whereas T_2, T_3 are $O(\alpha_h)$. In the Lemmas 8.2, 8.3, 8.4 below we will prove that for $(\nu, \mu) \neq (0, 0)$ we have $\text{Re}(T_j/\xi^{\nu\mu}(\mathcal{S}_A)) \geq 0$, $j = 2, 3$, and $\text{Re}(T_1/\xi^{\nu\mu}(\mathcal{S}_A)) \geq \frac{1}{8}$. Therefore we state the following

Theorem 8.1. *For* $(\nu, \mu) \neq (0, 0)$ *the following holds*:

(8.5) $$\text{Re}(\xi^{\nu\mu}(\mathcal{S}_{\tilde{A}})/\xi^{\nu\mu}(\mathcal{S}_A)) \geq \tfrac{1}{8} .$$

Proof. Direct consequence of (8.4a) and Lemma 8.2, 8.3, 8.4. □

Lemma 8.2. *With* T_2 *as in (8.4c) we have the inequality*

(8.6) $$\text{Re}(T_2/\xi^{\nu\mu}(\mathcal{S}_A)) \geq 0 \quad \textit{for } (\nu, \mu) \neq (0, 0) .$$

Proof. First notice that $\text{Re}(1/d_j^{\nu\mu}) \geq 0$ and therefore $\text{Re}(1/\xi^{\nu\mu}(\mathcal{S}_{\hat{A}})) \geq 0$. Also we have $\lambda_1^{\nu\mu} \geq 0$, $(\lambda_4^{\nu\mu} - \lambda_1^{\nu\mu}) > 0$, $(1 - \frac{1}{2}\gamma_{\nu\mu}) > 0$. From this we see that for proving the result in (8.6) it is sufficient to prove that

$$\text{(8.7)} \qquad \text{Re}(g_1^{\nu\mu}\overline{d}_j^{\nu\mu}) \geq 0 \quad \text{for } j = 1,2,3,4 \ .$$

The inequality in (8.7) follows from

$$\begin{aligned} \text{Re}(g_1^{\nu\mu}\overline{d}_j^{\nu\mu}) &= \text{Re}((\tfrac{1}{2}\lambda_1^{\nu\mu} + \varphi_1^{\nu\mu})(\lambda_j^{\nu\mu} + \overline{\varphi}_j^{\nu\mu})) \\ &> \text{Re}(\varphi_1^{\nu\mu}\overline{\varphi}_j^{\nu\mu}) \geq 0 \quad \text{(cf.(3.7a))} \ . \end{aligned}$$ □

Lemma 8.3. *With T_3 as in (8.4d) we have the inequality*

$$\text{(8.8)} \qquad \text{Re}(T_3/\xi^{\nu\mu}(\mathcal{S}_{\hat{A}})) \geq 0 \quad \textit{for } (\nu,\mu) \neq (0,0) \ .$$

Proof. We begin with rewriting T_3 as $T_3 = \frac{1}{8}\gamma_{\nu\mu}(T_3^{(1)} + T_3^{(2)})/(2\alpha_h + 1)$,

$$T_3^{(1)} := 2\alpha_h(\lambda_4^{\nu\mu} - \lambda_1^{\nu\mu}) - \tfrac{1}{4}(\lambda_2^{\nu\mu} - \lambda_3^{\nu\mu})^2$$

$$T_3^{(2)} := (\lambda_4^{\nu\mu} - \lambda_1^{\nu\mu}) - \tfrac{1}{2}(\lambda_2^{\nu\mu} - \lambda_3^{\nu\mu})(\varphi_2^{\nu\mu} - \varphi_3^{\nu\mu}) \ .$$

Because $\text{Re}(1/\xi^{\nu\mu}(\mathcal{S}_{\hat{A}})) \geq 0$ holds it is sufficient to prove

$$\text{(8.9a)} \qquad T_3^{(1)} \geq 0$$

$$\text{(8.9b)} \qquad \text{Re}\left(T_3^{(2)}\left(\sum_{j=1}^{4} 1/d_j^{\nu\mu}\right)\right) \geq 0 \ .$$

Using the definition of $\lambda_j^{\nu\mu}$ and introducing $p := c_\nu^2 - s_\nu^2$, $q := c_\mu^2 - s_\mu^2$ $(p,q \in [0,1])$ we get:

$$\begin{aligned} T_3^{(1)} &= 8\alpha_h^2(p+q) - 4\alpha_h^2(p-q)^2 \\ &\geq 4\alpha_h^2\{2(p+q) - (p+q)^2\} = 4\alpha_h^2(p+q)(2-(p+q)) \geq 0 \ . \end{aligned}$$

So (8.9a) holds. For $T_3^{(2)}$ we have

$$\begin{aligned} T_3^{(2)} &= 4\alpha_h(p+q) - 2\alpha_h(p-q)(\varphi_2^{\nu\mu} - \varphi_3^{\nu\mu}) \\ &= 2\alpha_h p(2 - (\varphi_2^{\nu\mu} - \varphi_3^{\nu\mu})) + 2\alpha_h q(2 - (\varphi_3^{\nu\mu} - \varphi_2^{\nu\mu})) \\ &= 4\alpha_h(p\varphi_3^{\nu\mu} + q\varphi_2^{\nu\mu}) \quad (\text{use } \varphi_2^{\nu\mu} + \varphi_3^{\nu\mu} = 2) \ . \end{aligned}$$

So for (8.9b) to hold it is sufficient to prove

$$\text{(8.10)} \qquad \text{Re}\left(\varphi_j^{\nu\mu}\left(\sum_{i=1}^{4} 1/d_i^{\nu\mu}\right)\right) \geq 0 \quad \text{for } j = 2,3 \ .$$

For $k \in \{1,2,3,4\}$ we have

$$\begin{aligned} \text{Re}(\varphi_j^{\nu\mu}/d_k^{\nu\mu}) &= |d_k^{\nu\mu}|^{-2}\text{Re}(\varphi_j^{\nu\mu}(\lambda_k^{\nu\mu} + \overline{\varphi}_k^{\nu\mu})) \\ &\geq |d_k^{\nu\mu}|^{-2}\text{Re}(\varphi_j^{\nu\mu}\overline{\varphi}_k^{\nu\mu}) \geq 0 \quad \text{(use (3.7a))} \ . \end{aligned}$$

From this it follows that the estimate in (8.10) holds. □

Lemma 8.4. *With T_1 as in (8.4b) we have the inequality*

(8.11) $$\mathrm{Re}(T_1/\xi^{\nu\mu}(\mathscr{S}_A)) \geq \tfrac{1}{8} \quad \textit{for } (\nu,\mu) \neq (0,0) .$$

Proof. First note that $H(g_j^{\nu\mu}, g_k^{\nu\mu}) = (2\alpha_h+1)^{-1} g_j^{\nu\mu} g_k^{\nu\mu}$ for $(j,k) \in \{(2,3),(1,4)\}$ and thus

$$T_1 = ((1-\tfrac{1}{2}\gamma_{\nu\mu}) g_1^{\nu\mu} g_4^{\nu\mu} + \tfrac{1}{2}\gamma_{\nu\mu} g_2^{\nu\mu} g_3^{\nu\mu})/(2\alpha_h+1) .$$

Using $1/d_j^{\nu\mu} + 1/d_k^{\nu\mu} = 2(4\alpha_h+1)(d_j^{\nu\mu} d_k^{\nu\mu})^{-1}$, $(j,k) \in \{(2,3),(1,4)\}$ we get

$$\begin{aligned} T_1/\xi^{\nu\mu}(\mathscr{S}_A) &= \tfrac{1}{2}\,\frac{4\alpha_h+1}{2\alpha_h+1}\left\{(1-\tfrac{1}{2}\gamma_{\nu\mu})\left(\frac{g_1^{\nu\mu} g_4^{\nu\mu}}{d_1^{\nu\mu} d_4^{\nu\mu}} + \frac{g_1^{\nu\mu} g_4^{\nu\mu}}{d_2^{\nu\mu} d_3^{\nu\mu}}\right)\right. \\ &\qquad \left. + \tfrac{1}{2}\gamma_{\nu\mu}\left(\frac{g_2^{\nu\mu} g_3^{\nu\mu}}{d_1^{\nu\mu} d_4^{\nu\mu}} + \frac{g_2^{\nu\mu} g_3^{\nu\mu}}{d_2^{\nu\mu} d_3^{\nu\mu}}\right)\right\} . \end{aligned}$$

So for (8.11) to hold it is sufficient to prove

(8.12a) $$\mathrm{Re}\left(\frac{g_j^{\nu\mu} g_k^{\nu\mu}}{d_j^{\nu\mu} d_k^{\nu\mu}}\right) \geq \tfrac{1}{4} \quad \text{for } (j,k) \in \{(2,3),(1,4)\} \text{ , and}$$

(8.12b) $$\mathrm{Re}\left(\frac{g_1^{\nu\mu} g_4^{\nu\mu}}{d_2^{\nu\mu} d_3^{\nu\mu}}\right) \geq 0 , \quad \mathrm{Re}\left(\frac{g_2^{\nu\mu} g_3^{\nu\mu}}{d_1^{\nu\mu} d_4^{\nu\mu}}\right) \geq 0 .$$

We first consider (8.12a). Take $(j,k) \in \{(2,3),(1,4)\}$ and note that $g_i^{\nu\mu} = \frac{1}{2}(d_i^{\nu\mu} + \varphi_i^{\nu\mu})$ and thus

$$\begin{aligned} \mathrm{Re}\left(\frac{g_j^{\nu\mu} g_k^{\nu\mu}}{d_j^{\nu\mu} d_k^{\nu\mu}}\right) &= \tfrac{1}{4}\mathrm{Re}\left(\frac{(d_j^{\nu\mu}+\varphi_j^{\nu\mu})(d_k^{\nu\mu}+\varphi_k^{\nu\mu})}{d_j^{\nu\mu} d_k^{\nu\mu}}\right) \\ &= \tfrac{1}{4}\left\{1 + \mathrm{Re}(\varphi_j^{\nu\mu}/d_j^{\nu\mu}) + \mathrm{Re}(\varphi_k^{\nu\mu}/d_k^{\nu\mu})\right. \\ &\qquad \left. + \mathrm{Re}\left(\frac{\varphi_j^{\nu\mu}\varphi_k^{\nu\mu}}{d_j^{\nu\mu} d_k^{\nu\mu}}\right)\right\} . \end{aligned} \tag{8.13}$$

Now use that for $i \in \{1,2,3,4\}$

$$\mathrm{Re}(\varphi_i^{\nu\mu}/d_i^{\nu\mu}) = |d_i^{\nu\mu}|^{-2}\mathrm{Re}(\varphi_i^{\nu\mu}(\lambda_i^{\nu\mu} + \overline{\varphi}_i^{\nu\mu})) \geq |d_i^{\nu\mu}|^{-2}|\varphi_i^{\nu\mu}|^2 \geq 0$$

and

$$\begin{aligned} \mathrm{Re}(\varphi_j^{\nu\mu}\varphi_k^{\nu\mu}\overline{d}_j^{\nu\mu}\overline{d}_k^{\nu\mu}) &= \lambda_j^{\nu\mu}\lambda_k^{\nu\mu}\mathrm{Re}(\varphi_j^{\nu\mu}\varphi_k^{\nu\mu}) + \lambda_j^{\nu\mu}|\varphi_k^{\nu\mu}|^2\mathrm{Re}(\varphi_j^{\nu\mu}) \\ &\quad + \lambda_k^{\nu\mu}|\varphi_j^{\nu\mu}|^2\mathrm{Re}(\varphi_k^{\nu\mu}) + |\varphi_j^{\nu\mu}|^2|\varphi_k^{\nu\mu}|^2 \geq 0 \quad \text{(use (3.7b))}. \end{aligned}$$

The latter two inequalities and (8.13) together imply the result in (8.12a).

We now consider the first inequality in (8.12b). The second result in (8.12b) can be proved similarly.

$$(8.14) \qquad \mathrm{Re}\left(\frac{g_1^{\nu\mu}g_4^{\nu\mu}}{d_2^{\nu\mu}d_3^{\nu\mu}}\right) = \tfrac{1}{2}(4\alpha_h+1)^{-1}\{|d_2^{\nu\mu}|^{-2}\mathrm{Re}(g_1^{\nu\mu}g_4^{\nu\mu}\overline{d}_2^{\nu\mu}) + |d_3^{\nu\mu}|^{-2}\mathrm{Re}(g_1^{\nu\mu}g_4^{\nu\mu}\overline{d}_3^{\nu\mu})\} .$$

Note that for $j \in \{2,3\}$ we have

$$\begin{aligned}\mathrm{Re}(g_1^{\nu\mu}g_4^{\nu\mu}\overline{d}_j^{\nu\mu}) &= \mathrm{Re}((\tfrac{1}{2}\lambda_1^{\nu\mu}+\varphi_1^{\nu\mu})(\tfrac{1}{2}\lambda_4^{\nu\mu}+\varphi_4^{\nu\mu})(\lambda_j^{\nu\mu}+\overline{\varphi}_j^{\nu\mu}))\\ &= \tfrac{1}{4}\lambda_1^{\nu\mu}\lambda_4^{\nu\mu}\lambda_j^{\nu\mu} + \tfrac{1}{4}\lambda_1^{\nu\mu}\lambda_4^{\nu\mu}\mathrm{Re}(\varphi_j^{\nu\mu}) + \tfrac{1}{2}\lambda_1^{\nu\mu}\lambda_j^{\nu\mu}\mathrm{Re}(\varphi_4^{\nu\mu})\\ &\quad +\tfrac{1}{2}\lambda_4^{\nu\mu}\lambda_j^{\nu\mu}\mathrm{Re}(\varphi_1^{\nu\mu}) + \tfrac{1}{2}\lambda_1^{\nu\mu}\mathrm{Re}(\varphi_4^{\nu\mu}\overline{\varphi}_j^{\nu\mu}) + \tfrac{1}{2}\lambda_4^{\nu\mu}\mathrm{Re}(\varphi_1^{\nu\mu}\overline{\varphi}_j^{\nu\mu})\\ &\quad +\lambda_j\mathrm{Re}(\varphi_1^{\nu\mu}\varphi_4^{\nu\mu}) + \mathrm{Re}(\varphi_1^{\nu\mu}\varphi_4^{\nu\mu}\overline{\varphi}_j^{\nu\mu}) ,\end{aligned}$$

and all the terms in the right hand side are positive due to $\lambda_i^{\nu\mu} \geq 0$ and the results of Lemma 3.1. Using this in (8.14) we see that the first inequality in (8.12b) is valid. □

The results in Lemma 8.2, 8.3, 8.4 imply Theorem 8.1 and thus we have proved that $\sigma(\mathcal{S}_A^{-1}\mathcal{S}_{\tilde{A}})$ lies in the half plane $\{z \in C \mid \mathrm{Re}(z) \geq \frac{1}{8}\}$. In Theorem 8.6 below we prove that $\sigma(\mathcal{S}_A^{-1}\mathcal{S}_{\tilde{A}})$ is bounded uniform in h, a, ε.

Lemma 8.5. *For* $(\nu,\mu) \neq (0,0)$ *the following inequality holds*:

$$(8.15) \qquad |\xi^{\nu\mu}(\mathcal{S}_{\tilde{A}})/d_1^{\nu\mu}| \leq 4 .$$

Proof. We rewrite (8.1b) as follows

$$\begin{aligned}\xi^{\nu\mu}(\mathcal{S}_{\tilde{A}}) &= d_1^{\nu\mu} + \tfrac{1}{4}\tau_{(1)}^{\nu\mu}\{(d_4^{\nu\mu}+d_2^{\nu\mu}) - (d_1^{\nu\mu}+d_3^{\nu\mu})\}\\ &\quad +\tfrac{1}{4}\tau_{(2)}^{\nu\mu}\{(d_4^{\nu\mu}+d_3^{\nu\mu}) - (d_1^{\nu\mu}+d_2^{\nu\mu})\}\\ &= d_1^{\nu\mu} + \tau_{(1)}^{\nu\mu}\{2\alpha_h(c_\nu^2-s_\nu^2) + a(1-\psi_\nu)\}\\ &\quad +\tau_{(2)}^{\nu\mu}\{2\alpha_h(c_\mu^2-s_\mu^2) + (1-a)(1-\psi_\mu)\} .\end{aligned}$$

So

$$(8.16) \qquad |\xi^{\nu\mu}(\mathcal{S}_{\tilde{A}})/d_1^{\nu\mu}| \leq 1 + |\tau_{(1)}^{\nu\mu}/d_1^{\nu\mu}|(2\alpha_h+a) + |\tau_{(2)}^{\nu\mu}/d_1^{\nu\mu}|(2\alpha_h+1-a) .$$

We first consider the term $|\tau_{(1)}^{\nu\mu}/d_1^{\nu\mu}|$. With $\hat{\lambda}_1^{\nu\mu} := 4\alpha_h(s_\nu^2+s_\mu^2c_\mu^2(c_\nu^2-s_\nu^2)) \leq \lambda_1^{\nu\mu}$ we get (cf. (5.11a) for $\tau_{(1)}^{\nu\mu}$)

$$\begin{aligned}|\tau_{(1)}^{\nu\mu}/d_1^{\nu\mu}| &= \left|\frac{\hat{\lambda}_1^{\nu\mu}+\varphi_1^{\nu\mu}+(1-a)\gamma_{\nu\mu}(1-\psi_\mu)}{\lambda_1^{\nu\mu}+\varphi_1^{\nu\mu}}\right|(2\alpha_h+1)^{-1}\\ &\leq \left(\frac{|\hat{\lambda}_1^{\nu\mu}+\varphi_1^{\nu\mu}|}{|\lambda_1^{\nu\mu}+\varphi_1^{\nu\mu}|} + \frac{(1-a)\gamma_{\nu\mu}}{\mathrm{Re}(d_1^{\nu\mu})}\right)(2\alpha_h+1)^{-1}\\ &\leq (1+(1-a)\gamma_{\nu\mu}/\mathrm{Re}(d_1^{\nu\mu}))(2\alpha_h+1)^{-1} .\end{aligned}$$

A similar computation for $|\tau_{(2)}^{\nu\mu}/d_1^{\nu\mu}|$ yields

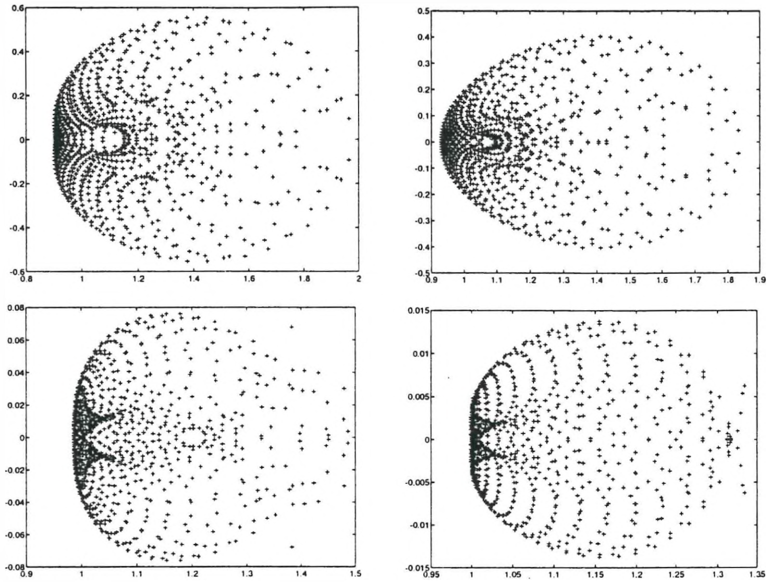


Fig. 4. $\sigma(\mathcal{S}^{-1}\mathcal{S}_A)$, $a = 0.4$. Top left: $\alpha_h = 0.01$, top right: $\alpha_h = 0.1$; bottom left: $\alpha_h = 1$, bottom right: $\alpha_h = 10$

$$|\tau_{(2)}^{\nu\mu}/d_1^{\nu\mu}| \le (1 + a\gamma_{\nu\mu}/\mathrm{Re}(d_1^{\nu\mu}))(2\alpha_h + 1)^{-1} .$$

Using these estimates in (8.16) results in

$$\begin{aligned} |\xi^{\nu\mu}(\mathcal{S}_A)/d_1^{\nu\mu}| &\le 1 + \left(1 + \frac{(1-a)\gamma_{\nu\mu}}{\mathrm{Re}(d_1^{\nu\mu})}\right)\frac{2\alpha_h + a}{2\alpha_h + 1} \\ &\quad + \left(1 + \frac{a\gamma_{\nu\mu}}{\mathrm{Re}(d_1^{\nu\mu})}\right)\frac{2\alpha_h + 1 - a}{2\alpha_h + 1} \\ &= 1 + \frac{4\alpha_h + 1}{2\alpha_h + 1} + \frac{2}{2\alpha_h + 1}\,\frac{(\alpha_h + a(1-a))\gamma_{\nu\mu}}{\mathrm{Re}(d_1^{\nu\mu})} . \end{aligned} \tag{8.17}$$

Now note that

$$(\alpha_h + a(1-a))\gamma_{\nu\mu} = 2\alpha_h(s_\nu^2 c_\mu^2 + s_\mu^2 c_\nu^2) + a(1-a)(2s_\nu^2 c_\mu^2 + 2s_\mu^2 c_\nu^2)$$

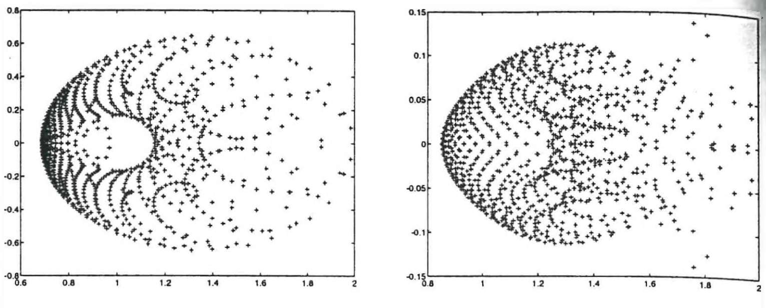
$$\le 4\alpha_h(s_\nu^2 + s_\mu^2) + 2as_\nu^2 + 2(1-a)s_\mu^2 = \mathrm{Re}(d_1^{\nu\mu}) .$$

Using this in (8.17) we get

$$|\xi^{\nu\mu}(\mathcal{S}_A)/d_1^{\nu\mu}| \le 1 + \frac{4\alpha_h + 1}{2\alpha_h + 1} + \frac{2}{2\alpha_h + 1} = \frac{6\alpha_h + 4}{2\alpha_h + 1} \le 4 . \qquad \square$$

Theorem 8.6. *For* $(\nu, \mu) \ne (0,0)$ *the following estimate holds:*

$$|\xi^{\nu\mu}(\mathcal{S}_A)/\xi^{\nu\mu}(\mathcal{S}_A)| \le 2(1 + \sqrt{2(2 + \sqrt{2})}) . \tag{8.18}$$



$\alpha_h = 0.01$ $\alpha_h = 1$

Fig. 5. $\sigma(A_H^{-1}.\mathcal{S}_A)$, $a = 0.4$

Proof. The result in (8.18) is a consequence of Lemma 3.2 and Lemma 8.5:

$$|\xi^{\nu\mu}(\mathcal{S}_A)/\xi^{\nu\mu}(\mathcal{S}_{\tilde{A}})| = \tfrac{1}{4}|\xi^{\nu\mu}(\mathcal{S}_A)(\sum_{j=1}^{4} 1/d_j^{\nu\mu})|$$

$$\leq \tfrac{1}{4}|\xi^{\nu\mu}(\mathcal{S}_A)/d_1^{\nu\mu}|(1+\sum_{j=2}^{4} |d_1^{\nu\mu}/d_j^{\nu\mu}|)$$

$$\leq \tfrac{1}{4}4(1+2\sqrt{2(2+\sqrt{2})}+1) = 2(1+\sqrt{2(2+\sqrt{2})}) \ . \qquad \square$$

From Theorem 8.1 and Theorem 8.6 it follows that $\sigma(\mathcal{S}_{\tilde{A}}^{-1}.\mathcal{S}_A)$ lies in a bounded domain in the complex right half–plane away from the imaginary axis. Moreover this domain is independent of the parameters ε, h, a, i.e. we have a robustness result w.r.t. variation in ε, h and a.
In Fig. 4 we show $\sigma(\mathcal{S}_{\tilde{A}}^{-1}.\mathcal{S}_A)$, in the complex plane, for $h = 1/32$, $a = 0.4$ and for several values of α_h $(= \varepsilon/h)$. In Fig. 5 we give analogous results for $\sigma(A_H^{-1}.\mathcal{S}_A)$ (A_H: standard coarse grid discretization).

In Fig. 4 we see that the convex hull of $\sigma(\mathcal{S}_{\tilde{A}}^{-1}.\mathcal{S}_A)$ shrinks if α_h increases (i.e. more diffusion). Also, as in Sects. 6, 7 we observe a clustering of eigenvalues in a small neighbourhood of 1.
Comparison of Fig. 4 and Fig. 5 shows that for $\alpha_h = 0.1$ and $\alpha_h = 1$ the preconditioner $\mathcal{S}_{\tilde{A}}$ is better than the preconditioner A_H.

9. Numerical experiments

The analysis in Sects. 5–8 only applies to the model problem (1.1a, b) and yields a robustness result for the two-grid method. In this section we test the robustness of the multigrid W-cycle applied to discrete versions of problems as in (1.1a), i.e. we allow Dirichlet boundary conditions and varying coefficients.
We consider the following class of convection-diffusion problems:

$$\begin{cases} -\varepsilon\Delta u + a(x,y)u_x + b(x,y)u_y = f & \text{in } \Omega = (0,1)\times(0,1) \\ u = g & \text{on } \partial\Omega \,. \end{cases}$$

We use a standard discretization as in Sect. 2. The finest mesh always has $h = 1/128$, the coarsest mesh size is $h = 1/4$. For the multigrid method we use the approach as discussed in Sect. 4. Prolongations ($\tilde{p}$) and coarse- grid operators ($\mathcal{S}_{\tilde{A}}$) are computed in a preprocessing phase (cf. Remark 4.2). A two-grid iteration on Ω_h for solving $Ax_h = b_h$ consists of the following steps (we use the notation $\Omega_h^c := \Omega_h\backslash\Omega_H$):

1. $d_{|\Omega_h^c} = (Ax_h - b_h)_{|\Omega_h^c}$: compute defect on Ω_h^c .

2. $\hat{x}_h := \mathcal{J}^{\mu}(A_{11}; 0; d_{|\Omega_h^c})$: apply μ iterations

   of a basic iterative method for solving

   $A_{11}z = d_{|\Omega_h^c}$ with starting vector 0 .

3. $x_h := x_h - \begin{bmatrix} \hat{x}_h \\ 0 \end{bmatrix}$: add correction on Ω_h^c .

4. $d_{|\Omega_H} := (Ax_h - b_h)_{|\Omega_H}$: compute defect on Ω_H .

5. solve $\mathcal{S}_{\tilde{A}} v_H = d_{|\Omega_H}$: coarse grid problem .

6. $x_h := x_h - \omega\tilde{p}_{h\leftarrow H}v_H$: add coarse grid correction .

Note that step 1, 2, 3 correspond to $I - P_1$ in (4.6) and 4, 5, 6 correspond to $I - \tilde{p}_2 S r_2 A$ in (4.6), with $S = \omega.\mathcal{S}_{\tilde{A}}^{-1}$. We use ω for convergence acceleration. With respect to the choice of $\mathcal{J}$ in step 2 we note that in general the matrix A_{11} has a condition number $O(1)$ and thus, in principle, any basic iterative method will work. However, if we have strong alignment (e.g. $\varphi = 0$ in Experiment 1 below) then cond(A_{11}) deteriorates. So, to get a robust method we take a line Jacobi method in which one iteration consists of a sweep over the "odd" horizontal lines followed by a sweep over the "odd" vertical lines (these odd lines together form the pattern of $\Omega_h\backslash\Omega_H$).
As in the standard approach, we use a recursive call in step 5 to obtain a multigrid method. Below, we use the W-cycle and we take $\mu = 2$ in step 2. Based on Figs. 2, 3, 4 we take $\omega = 1$ on coarse grids and $\omega = 0.7$ on the finest grid. In our experiments we always take the data such that the exact solution is equal to zero and we take an arbitrary starting vector. As a measure for the error reduction we computed $r := (\|e_{20}\|_2/\|e_0\|_2)^{1/20}$, with e_k the error in the k-th iteration.

Experiment 1 (standard test problem as in [16]). We take $a(x,y) = \cos\varphi$, $b(x,y) = \sin\varphi$. In Table 1 the resulting r are given for different values of φ and ε.

Experiment 2 (rotating flow). We define $\Omega_R := \{(x,y) \mid (x-\frac{1}{3})^2 + (y-\frac{1}{3})^2 \le \frac{1}{16}\}$.

$$a(x,y) = \sin(\pi(y - \tfrac{1}{3}))\cos(\pi(x - \tfrac{1}{3})) \text{ if } (x,y) \in \Omega_R \text{ , and zero otherwise ;}$$

$$b(x,y) = -\cos(\pi(y - \tfrac{1}{3}))\sin(\pi(x - \tfrac{1}{3})) \text{ if } (x,y) \in \Omega_R \text{ , and zero otherwise .}$$

The results are given in Table 2.

Experiment 3 (as in [20]). We take $a(x,y) = (2y-1)(1-x^2)$, $b(x,y) = 2xy(y-1)$. The results are given in Table 3. In Table 3 we also show results for the two-grid method (TG).

Table 1

ε φ	0	$\pi/8$	$2\pi/8$	$3\pi/8$
10^0	0.24	0.24	0.24	0.24
10^{-2}	0.24	0.24	0.24	0.24
10^{-4}	0.27	0.30	0.29	0.30
10^{-6}	0.30	0.31	0.30	0.31

Table 2

ε	r
10^0	0.25
10^{-2}	0.24
10^{-4}	0.30
10^{-6}	0.31

Table 3

ε	TG	W-cycle
10^0	0.23	0.24
10^{-2}	0.22	0.24
10^{-4}	0.30	0.30
10^{-6}	0.34	0.34

These results show the robustness of our method with respect to both the convection/diffusion ratio and the flow direction.

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