Computing 56, 117–139 (1996)

Computing

Further Analysis of the Local Defect Correction Method

P. J. J. Ferket and **A. A. Reusken**, Eindhoven

Received December 27, 1994; revised August 28, 1995

Abstract — Zusammenfassung

Further Analysis of the Local Defect Correction Method. We analyze a special case of the Local Defect Correction (LDC) method introduced in [4]. We restrict ourselves to finite difference discretizations of elliptic boundary value problems. The LDC method uses the discretization on a uniform global coarse grid and on one or more uniform local fine grids for approximating the continuous solution. We prove that this LDC method can be seen as an iterative method for solving an underlying composite grid discretization. This result makes it possible to explain important properties of the LDC method, e.g. concerning the size of the discretization error. Furthermore, the formulation of LDC as an iterative solver for a given composite grid problem makes it possible to prove a close correspondence between LDC and the Fast Adaptive Composite grid (FAC) method from [8–10].

AMS Subject Classifications: 65N22, 65N50

Key words: Local defect correction, fast adaptive composite grid method.

Weitere Analyse der lokalen Defektkorrektur-Methode. Wir analysieren einen Spezialfall der lokalen Defektkorrektur-Methode (LDC) die in [4] eingeführt wurde. Wir beschränken uns auf Finite-Differenzen-Diskretisierungen elliptischer Randwertprobleme. Die lokale Defektkorrektur-Methode verwendet Diskretisierungen auf einem globalen uniformen groben Gitter und einem oder mehreren lokalen uniformen feinen Gittern zur Approximation der stetigen Lösung. Wir beweisen, daß diese LDC-Methode als iterative Methode zur Lösung einer zugehörigen Diskretisierung auf dem zusammengesetzten Gitter betrachtet werden kann. Dieses Resultat ermöglicht es, wichtige Eigenschaften der LDC-Methode zu erklären, z.B. in Bezug auf die Größenordnung des Diskretisierungsfehlers. Außerdem ermöglicht die Formulierung der LDC-Methode als iterativer Solver für ein gegebenes Problem auf dem zusammengesetzten Gitter den Beweis eines engen Zusammenhangs zwischen LDC und der "Fast adaptive grid (FAC)" – Methode aus [8–10].

1. Introduction

Many practical boundary value problems produce solutions which contain several high activity regions. In these regions the solution varies much more rapidly than in the remaining part of the domain. This behaviour of the solution may be caused by the differential operator itself, by the forcing terms in the differential equation, by the boundary conditions or by an irregular boundary (e.g. a re-entrant corner).

If one wants to discretize such a boundary value problem on a uniform grid, then, due to the large variations of the solution in the high activity regions, a

relatively small mesh size is required to obtain a sufficiently accurate approximation of the solution. However, outside the high activity regions the behaviour of the solution is much more smooth and therefore a (much) larger mesh size seems to be sufficient in that part of the domain. So approximating the continuous solution on a single uniform grid is often computationally inefficient for boundary value problems which produce solutions that contain high activity regions.

Instead, the solution can be approximated using *several uniform grids with different mesh sizes that cover different parts of the domain* [1, 4, 9]. At least one grid should cover the entire domain. The mesh size of this *global coarse grid* is chosen in agreement with the smooth behaviour of the solution outside the high activity regions. Besides a global grid, several *local grids* are used which are also uniform. Each of them covers only a (small) part of the domain and contains a high activity region. The grid size of each of these grids is chosen in agreement with the behaviour of the solution in the corresponding high activity region. In this way every part of the domain is covered by a (locally) uniform grid whose mesh size is in agreement with the behaviour of the continuous solution in that part of the domain. This refinement strategy is known as *local uniform grid refinement*. The solution is approximated on the *composite grid* which is the union of the uniform subgrids.

In [4] Hackbusch introduced the *local defect correction method* (LDC) for approximating the continuous solution on a composite grid. In this iterative process a basic global discretization is improved by local discretizations defined in the subdomains. At every step this iterative process yields a discrete approximation of the continuous solution on the composite grid. This method is an iterative discretization and solution method. The discrete problem that is actually being solved is an implicit result of the iterative process.

In this paper we present a further analysis of the LDC method. In [4] an overlap parameter $d \geq 0$ is introduced and an analysis of the LDC method for the case $d > 0$ (independent of the mesh size) is given. In this paper we analyze the LDC method for the case with minimal overlap, i.e. $d = 0$. We show that the discrete problem that is actually being solved by this LDC method is a composite grid discretization. This result has some interesting consequences for the analysis of the LDC method. For example, using the underlying composite grid system (that is not used in the LDC algorithm) bounds for the discretization error, in a finite difference setting, can be derived. Also an expression for the iteration matrix is derived that can be used to gain understanding of the convergence properties of the LDC method.

The *fast adaptive composite grid* (*FAC*) *method* by McCormick [8–10] is an iterative method for solving a *given* discrete problem on the composite grid. Approximations of the solution of this discrete problem are computed by solving discrete problems on the global and local grids. It is often noted in literature

that LDC and FAC are very similar. Here, we give a concrete theoretical comparison of these methods for a model situation. We show that the LDC method is equivalent with the FAC method applied to the composite grid discretization resulting from the LDC method. The resulting iterates of both methods are the same (although the algorithms are different!).

The remainder of this paper is organized as follows. In Section 2 we describe a model problem and a model composite grid. In Subsection 3.1 the LDC iteration is presented. The underlying composite grid discretization and an expression for the iteration matrix are derived in Subsection 3.2. Some numerical results related to the convergence rate of the LDC method are presented. In Subsection 3.3 we present a Finite Difference based FAC method and we prove that this method is equivalent with the LDC method.

2. Model Situation

In this section we introduce notation and describe a model case with a global coarse grid and one local fine grid. In Remark 2.1 we discuss possible generalizations.

We consider *Dirichlet boundary value problems*

$$\begin{aligned} \mathcal{L}U &= f \quad \text{in } \Omega, \\ U &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

with $\Omega = (0,1) \times (0,1)$, $\partial\Omega$ the boundary of Ω and $\mathcal{L}$ a *scalar linear elliptic second-order differential operator*. Dirichlet boundary conditions are chosen for ease of presentation. We assume that problem (2.1) is such that the continuous solution varies very rapidly in some (small) part of the domain, which is contained in the region $\Omega_l \subset \Omega$. In the remaining part of the domain the continuous solution is assumed to behave much more smoothly. The boundary $\partial\Omega_l$ of Ω_l consists of two parts. A part that coincides with $\partial\Omega$ and a remaining part. The latter part is called the *interface* $\Gamma = \partial\Omega_l \setminus \partial\Omega$ (see Fig. 1). We note that we may have $\partial\Omega_l \cap \partial\Omega = \{\emptyset\}$. in which case the interface Γ coincides with $\partial\Omega_l$.

In order to compute a numerical approximation of the solution U we discretize (2.1) with respect to some discretization grid using *finite differences*. We assume that the finite difference matrices that appear in this section are all nonsingular.

We use two *uniform grids*, a *global* one and a *local* one. The *global coarse grid* Ω^H is a uniform grid with mesh size H that covers the domain Ω. The *local fine grid* Ω_l^h is a uniform grid with size h that covers the region Ω_l (see Fig. 1). The *space of grid functions* on $\Omega^H(\Omega_l^h)$ is denoted by $\mathcal{F}^H(\mathcal{F}_l^h)$. The continuous solution varies (much) more rapidly in Ω_l than in the remainder of Ω. Therefore, a (much) smaller grid size is needed in Ω_l than in the remainder of Ω to provide the required level of resolution: $h \ll H$.

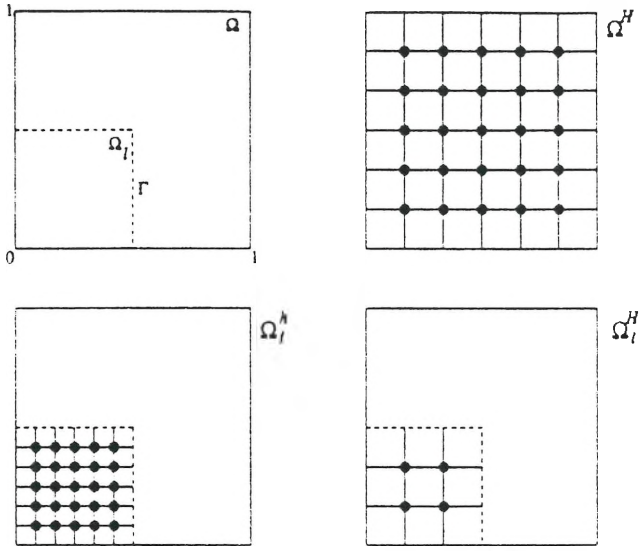

Figure 1. Examples of Ω_l, Ω^H, Ω_l^h and Ω_l^H

We assume that the interface Γ coincides with grid lines of Ω^H. Also we assume that all grid points of $\Omega^H \cap \Omega_l$ belong to Ω_l^h. We note that Ω_l^h does not contain grid points on the interface Γ (see Fig. 1). These fine interface grid points generate the *fine interface grid* Γ^h. The *coarse interface grid* Γ^H consists of all coarse interface grid points $\mathbf{x} \in \Omega^H \cap \Gamma$. The corresponding spaces of grid functions are denoted by $\mathcal{F}_\Gamma^h$ and $\mathcal{F}_\Gamma^H$ respectively.

Below we will also use a so called *composite grid*. The *composite grid* Ω_c is a nonuniform grid that covers the domain Ω. It is the union of the global coarse grid Ω^H and the local fine grid Ω_l^h (see Fig. 2). The space of grid functions on the composite grid is denoted by $\mathcal{F}_c$.

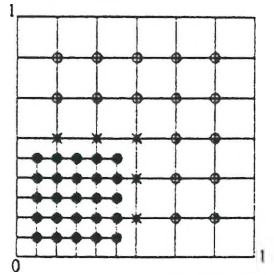

Figure 2. Partitioning of Ω_c; ○ grid point of Ω_c, ● grid point of Ω_l^h, × grid point of Γ^H

Related to these grids we now introduce discrete operators and appropriate intergrid transfer operators.

First we discretize (2.1) with respect to the global coarse grid Ω^H. At all grid points $\mathbf{x} \in \Omega^H$ the differential operator in (2.1) is replaced by a finite difference approximation. This yields the *basic coarse grid problem*:

$$L^H u^H = f^H \qquad \text{on } \Omega^H, \tag{2.2}$$

with u^H, $f^H \in \mathcal{F}^H$ and $L^H : \mathcal{F}^H \to \mathcal{F}^H$. The Dirichlet boundary values in (2.1) are incorporated in f^H.

For a given $v^H \in \mathcal{F}^H$ (e.g. $v^H = u^H$) we consider a corresponding fine grid problem on Ω_l^h. For this problem *artificial* Dirichlet boundary values are specified at all grid points $\mathbf{x} \in \Gamma^h$ using the values $v^H(\mathbf{x})$, $\mathbf{x} \in \Gamma^H$ and an interpolation operator $p_\Gamma : \mathcal{F}_\Gamma^H \to \mathcal{F}_\Gamma^h$. In practice one will use piecewise linear or quadratic interpolation with suitable modifications near the boundary $\partial\Omega$: the boundary values from (2.1) are used when interpolating between a point of $\partial\Omega$ and a grid point of Γ^H. For example, let $(H, \gamma) \in \Gamma^H$. Then, for $\mathbf{x}$ on the line segment $(1-\delta)(0,\gamma) + \delta(H,\gamma)$, $0 \le \delta \le 1$, the linear interpolation is defined by

$$\begin{aligned}\left(p_\Gamma\left(v^H_{|\Gamma^H}\right)\right)(\mathbf{x}) &= (1-\delta)\, U((0,\gamma)) + \delta\, v^H((H,\gamma)) \\ &= (1-\delta) g((0,\gamma)) + \delta\, v^H((H,\gamma)).\end{aligned} \tag{2.3}$$

At all grid points $\mathbf{x} \in \Omega_l^h$ the differential operator in (2.1) is replaced by a finite difference approximation. Using boundary values derived from v^H results in the linear system

$$L_l^h v_l^h = f_l^h - L_\Gamma^h p_\Gamma\left(v^H_{|\Gamma^H}\right) \qquad \text{on } \Omega_l^h, \tag{2.4}$$

with v_l^h, $f_l^h \in \mathcal{F}_l^h$, $L_l^h : \mathcal{F}_l^h \to \mathcal{F}_l^h$, $L_\Gamma^h : \mathcal{F}_\Gamma^h \to \mathcal{F}_l^h$. The Dirichlet boundary conditions on $\partial\Omega_l \cap \partial\Omega$ are incorporated into f_l^h. The incorporation of the *artificial* Dirichlet boundary conditions on Γ in the system is given explicitly by the term $-L_\Gamma^h p_\Gamma(v^H_{|\Gamma^H})$.

Finally we consider a discretization on the composite grid Ω_c. At all grid points $\mathbf{x} \in \Omega_c$ the differential operator in (2.1) is replaced by a certain finite difference approximation. We denote the resulting *composite grid problem* by

$$L_c u_c = f_c \qquad \text{on } \Omega_c, \tag{2.5}$$

with u_c, $f_c \in \mathcal{F}_c$ and $L_c : \mathcal{F}_c \to \mathcal{F}_c$.

The composite grid is partitioned in the following way (see Fig. 2):

$$\Omega_c = \Omega_C \cup \Gamma^H \cup \Omega_l^h. \tag{2.6}$$

We assume that at all grid points $\mathbf{x} \in \Omega_C$ the same finite difference formula is used as in the discretization process on the global coarse grid:

$$(L_c u_c)(\mathbf{x}) = \left(L^H\left(u_{c_{|\Omega^H}}\right)\right)(\mathbf{x}), \qquad \mathbf{x} \in \Omega_C, \tag{2.7a}$$

$$f_c(\mathbf{x}) = f^H(\mathbf{x}), \qquad \mathbf{x} \in \Omega_C, \tag{2.7b}$$

with L^H and f^H as in (2.2).

We assume that at all grid points $\mathbf{x} \in \Omega_l^h$ the composite grid discretization is of the following form:

$$(L_c u_c)(\mathbf{x}) = \left(L_l^h(u_{c|\Omega_l^h})\right)(\mathbf{x}) + \left(L_\Gamma^h p_\Gamma(u_{c|\Gamma^H})\right)(\mathbf{x}), \qquad \mathbf{x} \in \Omega_l^h, \quad (2.8a)$$

$$f_c(\mathbf{x}) = f_l^h(\mathbf{x}), \qquad x \in \Omega_l^h, \quad (2.8b)$$

with L_l^h, L_Γ^h, f_l^h and p_Γ as in (2.4).

There are several options for choosing the composite grid discretization at the interface points $\mathbf{x} \in \Gamma^H$. In Subsection 3.3 we will consider one particular choice which results from the analysis of the LDC method.

Remark 2.1. In [4] a more general setting is presented. For example, for the LDC method it is not necessary that the local fine grid is a refinement of the coarse grid ($(\Omega^H \cap \Omega_l) \subset \Omega_l^h$). Also the setting in [4] allows the use of a variety of discretization methods. Due to this generality the analysis in [4] uses several technical assumptions which may be hard to verify in concrete situations. In this paper we restrict ourselves to the specific situation described above. This makes it possible to give a detailed analysis of the LDC method without technical assumptions.

3. Local Defect Correction

3.1. Introduction of the LDC Method

In the LDC iteration the global coarse grid Ω^H and the local fine grid Ω_l^h are used to compute a numerical approximation of the continuous solution U of (2.1). At each iteration step a discrete problem on Ω^H and a discrete problem on Ω_l^h are solved.

We introduce the following notation. We use a *local coarse grid*

$$\Omega_l^H := \Omega_l \cap \Omega^H, \qquad (3.1)$$

and the space of grid functions on Ω_l^H is denoted by $\mathcal{F}_l^H$. The local coarse grid is a uniform grid with size H that covers the region Ω_l (see Fig. 1).

The characteristic function χ is defined by

$$(\chi w)(\mathbf{x}) := \begin{cases} w(\mathbf{x}) & \mathbf{x} \in \Omega_l^H \\ 0 & \mathbf{x} \in \Omega^H \setminus \Omega_l^H. \end{cases} \qquad (3.2)$$

In LDC one starts with solving the basic coarse grid problem (2.2). The resulting u^H is used to define boundary values for a local fine grid problem, i.e. we solve (2.4) with $v^H = u^H$, resulting in a local fine grid approximation u_l^h. By solving the local fine grid problem we aim at improving the approximation of the continuous solution U in the region Ω_l. However, the Dirichlet boundary conditions on Γ^h

result from the basic global coarse grid problem and the approximation u_l^h can be no more accurate than the approximation u^H *at the interface*. In general, local phenomena cause the approximations $u^H(\mathbf{x})$ to be relatively inaccurate at all grid points $\mathbf{x} \in \Omega^H$. Therefore, the results of this simple two step process usually do not achieve an accuracy that is in agreement with the added resolution (see e.g. [2], [4]). In the local defect correction iteration coarse and fine processing steps are reused to quickly obtain such accuracy.

In the next step of the LDC iteration the approximation u_l^h is used to *update the global coarse grid problem* (2.2). The right hand side of (2.2) is updated at grid points that are part of Ω_l^H. The updated global coarse grid problem is given by

$$L^H \bar{u}^H = \bar{f}^H \tag{3.3a}$$

with

$$\bar{f}^H(\mathbf{x}) := \begin{cases} \left(L_l^H\left(u^h_{l|\Omega_l^H}\right)\right)(\mathbf{x}) + \left(L_\Gamma^H\left(u^H_{|\Gamma^H}\right)\right)(\mathbf{x}) & \mathbf{x} \in \Omega_l^H \\ f^H(\mathbf{x}) & \mathbf{x} \in \Omega^H \setminus \Omega_l^H. \end{cases} \tag{3.3b}$$

The operators $L_l^H : \mathcal{F}_l^H \to \mathcal{F}_l^H$ and $L_\Gamma^H : \mathcal{F}_\Gamma^H \to \mathcal{F}_l^H$ are coarse grid analogues of L_l^h and L_Γ^h in (2.4) and they satisfy:

$$(L^H w^H)(\mathbf{x}) = \left(L_l^H\left(w^H_{|\Omega_l^H}\right)\right)(\mathbf{x}) + \left(L_\Gamma^H\left(w^H_{|\Gamma^H}\right)\right)(\mathbf{x}), \qquad w^H \in \mathcal{F}^H, \qquad \mathbf{x} \in \Omega_l^H. \tag{3.4}$$

Using (3.2) we can rewrite (3.3) as follows:

$$L^H \bar{u}^H = f^H + \chi\left(L_l^H\left(u^h_{l|\Omega_l^H}\right) + L_\Gamma^H\left(u^H_{|\Gamma^H}\right) - f^H\right). \tag{3.5}$$

So the right hand side of the global coarse grid problem is *corrected* by the *defect* of a *local fine grid approximation.*

Remark 3.1. In [4] Hackbusch considers local defect correction with overlap (i.e. $d > 0$). Then the right hand side of the global coarse grid problem is not corrected at all grid points of Ω_l^H, but only at those grid points of Ω_l^H that have a distance larger than d to the interface Γ.

Once we have solved (3.5) we can update the local fine grid problem:

$$L_l^h \bar{u}_l^h = f_l^h - L_\Gamma^h p_\Gamma\left(\bar{u}^H_{|\Gamma^H}\right). \tag{3.6}$$

The approximations $\bar{u}^H$ and $\bar{u}_l^h$ of U are used to define an approximation of U on the composite grid:

$$\bar{u}_c(\mathbf{x}) := \begin{cases} \bar{u}_l^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ \bar{u}^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \setminus \Omega_l^h. \end{cases}$$

In [2] an error analysis for this approximation that results after only *one* LDC step is given.

In the LDC iteration global problems like (3.5) and local problems like (3.6) are combined in the way described above.

LDC

Start: exact solution of the global problem

$$L^H u_0^H = f^H \qquad \text{on } \Omega^H \tag{3.7a}$$

exact solution of the local problem

$$L_l^h u_{l,0}^h = f_l^h - L_\Gamma^h p_\Gamma\left(u_{0|\Gamma^H}^H\right) \qquad \text{on } \Omega_l^h \tag{3.7b}$$

computation of the composite grid approximation

$$u_{c,0}(\mathbf{x}) := \begin{cases} u_{l,0}^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ u_0^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \setminus \Omega_l^h \end{cases} \tag{3.7c}$$

$i = 1, 2 \ldots$:

a. Computation of the right hand side of the global problem

$$\tilde{f}^H := (1-\chi) f^H + \chi L_l^H\left(u_{l,i-1|\Omega_l^h}^h\right) + \chi L_\Gamma^H\left(u_{|\Gamma^H}^H\right) \tag{3.7d}$$

b. Exact solution of the global problem

$$L^H u_i^H = \tilde{f}^H \qquad \text{on } \Omega^H \tag{3.7e}$$

c. Exact solution of the local problem

$$L_l^h u_{l,i}^h = f_l^h - L_\Gamma^H p_\Gamma\left(u_{|\Gamma^H}^H\right) \qquad \text{on } \Omega_l^h \tag{3.7f}$$

d. Computation of the composite grid approximation

$$u_{c,i}(\mathbf{x}) = \begin{cases} u_{l,i}^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ u_i^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \setminus \Omega_l^h \end{cases} \tag{3.7g}$$

Remark 3.2. In the LDC iteration it is not necessary to compute the composite grid approximation explicitly. (3.7g) is added for reasons that will become clear later on.

In practice the systems in (3.7e, 3.7f) will be solved approximately by a fast iterative method. Then one can take advantage of the fact that one has to solve problems on *uniform* grids. The LDC iteration (3.7) is most naturally interpreted as an *iterative discretization and solution method* for a boundary value problem whose solution contains a high activity region. At each iteration step an approximation of the continuous solution is computed both on the global coarse grid and on the local fine grid. The approximation on the local fine grid is used to define a discrete problem on the global coarse grid. Thus the discretization process and the solution process are coupled and the discrete problem that is actually being solved is an implicit result of the iterative process. In the

following subsection we consider both the discretization aspect and the iterative solution aspect of the LDC iteration.

3.2. Properties of the LDC Iteration

Any fixed point $(\hat{u}^H, \hat{u}_l^h)$ of iterative process (3.7) is characterized by the coupled system

$$L^H \hat{u}^H - \chi L_\Gamma^H\left(\hat{u}^H_{|\Gamma^H}\right) - \chi L_l^H\left(\hat{u}^h_{|\Omega_l^H}\right) = (1-\chi) f^H \qquad \text{on } \Omega^H, \quad (3.8a)$$

$$L_l^h \hat{u}_l^h = f_l^h - L_\Gamma^h p_\Gamma\left(\hat{u}^H_{|\Gamma^H}\right) \qquad \text{on } \Omega_l^h. \quad (3.8b)$$

If $\partial\Omega_l \cap \partial\Omega \neq \{\emptyset\}$, then the second term on the right hand side of (3.8b) contains boundary values of (2.1) (see (2.3)). In order to separate data and unknowns, we rewrite (3.8b) in the following way. We introduce the interpolation operator $\hat{p}_\Gamma : \mathcal{F}_\Gamma^H \to \mathcal{F}_\Gamma^h$. Away from the boundary $\partial\Omega$ this operator is the same as the interpolation operator p_Γ from (2.4). When interpolating between a point of $\partial\Omega$ and a grid point of Γ^H, a *zero* value at the point of $\partial\Omega$ is used in $\hat{p}_\Gamma$. We recall that in that case the boundary values from (2.1) are used in the interpolation operator p_Γ. So, if $\partial\Omega_l \cap \partial\Omega = \{\emptyset\}$ or if $g \equiv 0$, then $\hat{p}_\Gamma \equiv p_\Gamma$. In general, however, this does not hold. In Fig. 3 the difference between p_Γ and $\hat{p}_\Gamma$ is illustrated in case of linear interpolation.

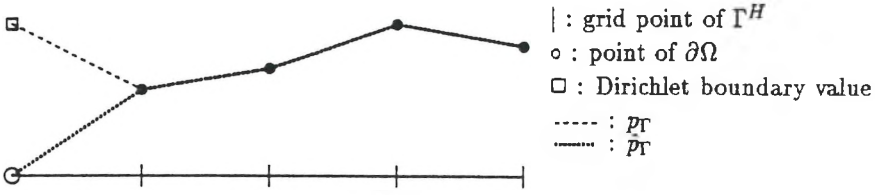

Figure 3. p_Γ and $\hat{p}_\Gamma$ in case of linear interpolation

Since

$$p_\Gamma(v+w) = p_\Gamma v + \hat{p}_\Gamma w \qquad \text{for all } v, w \in \mathcal{F}_\Gamma^H, \quad (3.9)$$

we get

$$-L_\Gamma^h p_\Gamma\left(\hat{u}^H_{|\Gamma^H}\right) = -L_\Gamma^h \hat{p}_\Gamma\left(\hat{u}^H_{|\Gamma^H}\right) - L_\Gamma^h p_\Gamma\left(\emptyset_\Gamma^H\right),$$

with $\emptyset_\Gamma^H \in \mathcal{F}_\Gamma^H$, $\emptyset_\Gamma^H(\mathbf{x}) = 0$ at all $\mathbf{x} \in \Gamma^H$.

We define

$$\hat{f}_l^h := f_l^h - L_\Gamma^h p_\Gamma\left(\emptyset_\Gamma^H\right). \quad (3.10)$$

Then (3.8b) can be replaced by

$$L_l^h \hat{u}_l^h = \hat{f}_l^h - L_\Gamma^h \hat{p}_\Gamma\left(\hat{u}^H_{|\Gamma^H}\right). \quad (3.8c)$$

In the following we show that the composite grid function corresponding to $(\hat{u}^H, \hat{u}_l^h)$ (as in (3.7g)) is the solution of a discrete problem on the composite grid

(cf. (2.5)). As stated in Section 2 we consider linear problems and we assume that the finite difference operators L^H, L_l^H, L_l^h are nonsingular. In this section we also assume that L^H, L_l^H, L_l^h correspond to 9-point stencils.

At grid points $\mathbf{x} \in \Omega_l^H$ two approximations exist: $\hat{u}^H(\mathbf{x})$ and $\hat{u}_l^h(\mathbf{x})$. We show that these approximations are identical.

Lemma 3.3. *Any limit value* $(\hat{u}^H, \hat{u}_l^h)$ *of the LDC method satisfies*

$$\hat{u}^H(\mathbf{x}) = \hat{u}_l^h(\mathbf{x}), \qquad \mathbf{x} \in \Omega_l^H. \tag{3.11}$$

Proof: From (3.2) and (3.8a) we obtain

$$(L^H \hat{u}^H)(\mathbf{x}) - \left(L_\Gamma^H\left(\hat{u}_{|\Gamma^H}^H\right)\right)(\mathbf{x}) - \left(L_l^H\left(\hat{u}_{l|\Omega_l^H}^h\right)\right)(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Omega_l^H.$$

From (3.4) it follows that

$$(L^H \hat{u}^H)(\mathbf{x}) = \left(L_l^H\left(\hat{u}_{|\Omega_l^H}^H\right)\right)(\mathbf{x}) + \left(L_\Gamma^H\left(\hat{u}_{|\Gamma^H}^H\right)\right)(\mathbf{x}), \qquad \mathbf{x} \in \Omega_l^H.$$

Thus

$$L_l^H\left(\hat{u}_{|\Omega_l^H}^H\right) = L_l^H\left(\hat{u}_{l|\Omega_l^H}^h\right),$$

and this is equivalent with (3.11) since L_l^H is nonsingular. □

As a consequence of Lemma 3.3 the coupled system (3.8) can be represented as a composite grid system. To show this, we first introduce some further notation related to the composite grid.

Define $\hat{u}_c \in \mathcal{F}_c$ by

$$\hat{u}_c(\mathbf{x}) := \begin{cases} \hat{u}_l^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ \hat{u}^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \setminus \Omega_l^h = \Omega^H \setminus \Omega_l^H. \end{cases} \tag{3.12}$$

Define

$$\tilde{\Gamma}^H := \left\{\mathbf{x} \in \Omega_l^H \mid \text{distance}(\mathbf{x}, \Gamma) = H\right\}.$$

The space of grid functions on $\Omega^H \setminus \Omega_l^H$ ($\tilde{\Gamma}^H$) is denoted by $\mathcal{F}_o^H$ ($\mathcal{F}_{\tilde{\Gamma}}^H$). Note that $\mathcal{F}^H = \mathcal{F}_l^H \oplus \mathcal{F}_o^H$. We introduce $L_o^H : \mathcal{F}_o^H \to \mathcal{F}_o^H$ and $L_\Gamma^H : \mathcal{F}_{\tilde{\Gamma}}^H \to \mathcal{F}_o^H$ such that (cf. (3.4))

$$(L^H w^H)(\mathbf{x}) = \left(L_o^H\left(w_{|\Omega^H \setminus \Omega_l^H}^H\right)\right)(\mathbf{x}) + \left(L_\Gamma^H\left(w_{|\tilde{\Gamma}^H}^H\right)\right)(\mathbf{x})$$

$$w^H \in \mathcal{F}^H, \mathbf{x} \in \Omega^H \setminus \Omega_l^H. \tag{3.13}$$

Further we introduce the *trivial injections* $r_{\tilde{\Gamma}} : \mathcal{F}_l^h \to \mathcal{F}_{\tilde{\Gamma}}^H$ and $r_\Gamma : \mathcal{F}_o^H \to \mathcal{F}_\Gamma^H$:

$$r_{\tilde{\Gamma}} w = w_{|\tilde{\Gamma}^H}, \tag{3.14a}$$

$$r_\Gamma u = u_{|\Gamma^H}. \tag{3.14b}$$

In the following theorem we use block partitioning corresponding to $\mathcal{F}_c = \mathcal{F}_l^h \oplus \mathcal{F}_o^H$ (see Fig. 4).

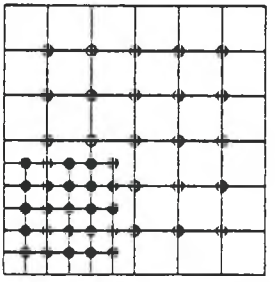

Figure 4. Partitioning of Ω_c corresponding to $\mathcal{F}_c = \mathcal{F}_l^h \oplus \mathcal{F}_o^H$

Theorem 3.4. *The composite grid approximation* $\hat{u}_c$ *from* (3.12) *satisfies*:

$$\hat{L}_c \hat{u}_c = \hat{f}_c \tag{3.15a}$$

with

$$\hat{L}_c = \begin{pmatrix} L_l^h & L_\Gamma^h \hat{p}_\Gamma r_\Gamma \\ L_\Gamma^H r_{\tilde{\Gamma}} & L_o^H \end{pmatrix}, \tag{3.15b}$$

$$\hat{u}_c = \begin{pmatrix} \hat{u}_l^h \\ \hat{u}^H_{|\Omega^H \setminus \Omega_l^H} \end{pmatrix}, \tag{3.15c}$$

$$\hat{f}_c = \begin{pmatrix} \hat{f}_l^h \\ f^H_{|\Omega^H \setminus \Omega_l^H} \end{pmatrix}. \tag{3.15d}$$

Proof: First note that

$$L_l^h \hat{u}_l^h + L_\Gamma^h \hat{p}_\Gamma r_\Gamma \left(\hat{u}^H_{|\Omega^H \setminus \Omega_l^H} \right) = \hat{f}_l^h$$

holds due to (3.8c) and (3.14b).

For $\mathbf{x} \in \Omega^H \setminus \Omega_l^H$ we have

$$\begin{aligned} \left(L_\Gamma^H r_{\tilde{\Gamma}} \hat{u}_l^h + L_o^H \left(\hat{u}^H_{|\Omega^H \setminus \Omega_l^H} \right) \right)(\mathbf{x}) &\overset{(3.14a)}{=} \left(L_\Gamma^H \left(\hat{u}^h_{|\tilde{\Gamma}^H} \right) + L_o^H \left(\hat{u}^H_{|\Omega^H \setminus \Omega_l^H} \right) \right)(\mathbf{x}) \\ &\overset{(3.11)}{=} \left(L_\Gamma^H \left(\hat{u}^H_{|\tilde{\Gamma}^H} \right) + L_o^H \left(\hat{u}^H_{|\Omega^H \setminus \Omega_l^H} \right) \right)(\mathbf{x}) \\ &\overset{(3.13)}{=} \left(L^H \hat{u}^H \right)(\mathbf{x}) \\ &\overset{(3.8a)}{=} f^H(\mathbf{x}). \end{aligned}$$

Thus $L_\Gamma^H r_{\tilde{\Gamma}} \hat{u}_l^h + L_o^H (\hat{u}^H_{|\Omega^H \setminus \Omega_l^H}) = f^H_{|\Omega^H \setminus \Omega_l^H}$. □

Remark 3.5. The composite grid problem (3.15) is such that (2.7), (2.8) hold. Furthermore, we have specified the discretization at the interface points $\mathbf{x} \in \Gamma^H$ (cf. (3.13)).

Note that existence and uniqueness of a fixed point of the LDC iteration is guaranteed if the matrix $\hat{L}_c$ in (3.15b) is nonsingular. In certain cases nonsingularity of $\hat{L}_c$ can be concluded from properties of L^H and L_l^h. For example, suppose that the finite difference matrices L^H from (2.2) and L_l^h from (2.4) have positive diagonal elements and nonpositive nondiagonal elements and that the matrices are irreducibly diagonally dominant (see e.g. [6]). If we use piecewise linear interpolation on the interface then it is easy to verify that $\hat{L}_c$ in (3.15b) also has positive diagonal elements and nonpositive nondiagonal elements and that $\hat{L}_c$ is irreducibly diagonally dominant. Hence $\hat{L}_c$ is nonsingular (it is even an M-matrix). In case of piecewise quadratic interpolation things are more complicated. In [3] nonsingularity of $\hat{L}_c$ is proven for the Poisson equation and piecewise quadratic interpolation on the interface. If both L_l^h and L^H are nonsingular then, in general, this does not imply that $\hat{L}_c$ is nonsingular. A counter-example is given in [4], Example 3.3.1.

An important consequence of Theorem 3.4 is that for the discretization error of the limit of the LDC iteration it is sufficient to analyze the composite grid discretization in (3.15) (which may be easier than an analysis of the coupled system in (3.8)). Note that in this discretization the treatment of the interface points is rather unusual. It turns out that, at least for the finite difference case, stability and reasonable error estimates can be proved. A detailed analysis of the finite difference discretization on composite grids for Poisson type of problems is presented in another paper [3]. Here we just give a typical error estimate from [3]. By $d_{h,H}(\mathbf{y})$ we denote the local discretization error at the grid point $\mathbf{y}$. By Γ_h^* we denote the set of grid points in Ω_l^H next to the interface Γ, i.e.

$$\Gamma_h^* = \{\mathbf{x} \in \Omega_l^h \mid dist(\mathbf{x}, \Gamma) = h\}.$$

Due to the interpolation needed on Γ the local discretization errors at points $\mathbf{y} \in \Gamma_h^*$ depend on $\sigma := H/h$. For the composite grid discretization as in (3.15) applied to a Poisson problem, the following estimates are valid:

$$\max_{\mathbf{y} \in \Omega^H \setminus \Omega_l^H} |d_{h,H}(\mathbf{y})| \leq C_1 H^2, \tag{3.16a}$$

$$\max_{\mathbf{y} \in \Omega_l^h \setminus \Gamma_h^*} |d_{h,H}(\mathbf{y})| \leq C_2 h^2, \tag{3.16b}$$

$$\max_{\mathbf{y} \in \Gamma_h^*} |d_{h,H}(\mathbf{y})| \leq C_3 \sigma^2 H^j + C_2 h^2, \tag{3.16c}$$

with $j = 0$, 1 if p_Γ corresponds to piecewise linear or piecewise quadratic interpolation respectively. The constants C_i depend on higher derivatives of U and due to the local high activity we have $C_2 \gg C_1$, $C_2 \gg C_3$. In [3] it is proved that the following global discretization error estimate holds:

$$\|\hat{u}_c - U\|_\infty \leq C\left(C_1 H^2 + C_2 h^2 + C_3 H^{j+1}\right), \tag{3.17}$$

with C a small constant that does not depend on U, h, H.

As usual in finite difference estimates, the result in (3.17) has the disadvantage that high (fourth order) derivatives are involved. However, the estimate in (3.17) nicely separates the influence of the high activity region (C_2h^2), the low activity region (C_1H^2), and the interpolation on the interface (C_3H^{j+1}). Note that all constants in (3.17) are independent of $\sigma = H/h$. We refer to [3] for numerical results related to the global discretization error bound (3.17).

In the remainder we assume that $\hat{L}_c$ is nonsingular. Thus the LDC iteration (3.7) has a unique fixed point $\hat{u}_c$ as in (3.12).

Below, in Theorem 3.6, we derive an expression for the iteration matrix of the LDC method. First we introduce two trivial injections r_c and r_{cl} and corresponding prolongations. The restrictions $r_c : \mathcal{F}_c \to \mathcal{F}^H$ and $r_{cl} : \mathcal{F}_c \to \mathcal{F}_l^h$ are defined by

$$r_c w := w_{|\Omega^H}, \tag{3.18}$$

$$r_{cl} w := w_{|\Omega_l^h}. \tag{3.19}$$

We use prolongations $r_c^T : \mathcal{F}^H \to \mathcal{F}_c$ and $r_{cl}^T : \mathcal{F}_l^h \to \mathcal{F}_c$. Below the following operators $P_i : \mathcal{F}_c \to \mathcal{F}_c$ play an important role:

$$P_1 := r_c^T (L^H)^{-1} r_c \hat{L}_c, \qquad P_2 := r_{cl}^T (L_l^h)^{-1} r_{cl} \hat{L}_c. \tag{3.20}$$

We use block partitioning corresponding to $\mathcal{F}^H = \mathcal{F}_l^H \oplus \mathcal{F}_o^H$, $\mathcal{F}_c = \mathcal{F}_l^h \oplus \mathcal{F}_o^H$. Then the restrictions r_c, r_{cl} are of the form

$$r_c = \begin{bmatrix} r_{inj} & \emptyset \\ \emptyset & I \end{bmatrix}, \tag{3.21}$$

with $r_{inj} : \mathcal{F}_l^h \to \mathcal{F}_l^H$ the trivial injection, and

$$r_{cl} = [\, I \quad \emptyset \,].$$

Theorem 3.6. *The iterates $u_{c,i}$ $(i \geq 1)$ from the LDC method* (3.7) *satisfy*

$$u_{c,i} - \hat{u}_c = M(u_{c,i-1} - \hat{u}_c), \tag{3.22a}$$

with

$$M = (I - P_2)(I - P_1). \tag{3.22b}$$

Proof: First we note that:

$$\begin{aligned} \tilde{f}^H &\overset{(3.7d)}{=} (1-\chi) f^H + \chi L_l^H \left(u_{l,i-1|\Omega_l^H}^h \right) + \chi L_\Gamma^H \left(u_{i-1|\Gamma^H}^H \right) \\ &\overset{(3.4),(3.18)}{=} (1-\chi) f^H + \chi L^H r_c u_{c,i-1} \\ &= L^H r_c u_{c,i-1} + (1-\chi)\left(f^H - L^H r_c u_{c,i-1} \right). \end{aligned} \tag{3.23}$$

Consider the term $(1-\chi)(f^H - L^H r_c u_{c,i-1})$. We will show that for all $\mathbf{x} \in \Omega^H$ the following holds:

$$\left((1-\chi)\left(f^H - L^H r_c u_{c,i-1} \right) \right)(\mathbf{x}) = \left(r_c \left(\hat{f}_c - \hat{L}_c u_{c,i-1} \right) \right)(\mathbf{x}). \tag{3.24}$$

For $\mathbf{x} \in \Omega_l^H$ the left hand side in (3.24) equals zero due to the definition of χ. On the other hand, for $\mathbf{x} \in \Omega_l^H$, we also have

$$\begin{aligned}\left(r_c\left(\hat{f}_c - \hat{L}_c u_{c,i-1}\right)(\mathbf{x})\right) &= \left(\left[r_{inj}\ \emptyset\right]\left(\hat{f}_c - \hat{L}_c u_{c,i-1}\right)\right)(\mathbf{x}) \\ &= \left(r_{inj}\left(\hat{f}_l^h - L_l^h u_{l,i-1}^h - L_\Gamma^h \hat{p}_\Gamma r_\Gamma \left(u^H_{i-1|\Omega^H \setminus \Omega_l^H}\right)\right)\right)(\mathbf{x}) \\ &= \left(r_{inj}\left(\hat{f}_l^h - L_l^h u_{l,i-1}^h - L_\Gamma^h \hat{p}_\Gamma \left(u^H_{i-1|\Gamma^H}\right)\right)\right)(\mathbf{x}) = 0,\end{aligned}$$

due to (3.7b), (3.7f). So (3.24) holds for $\mathbf{x} \in \Omega_l^H$.

For $\mathbf{x} \in \Omega_C = \Omega^H \setminus (\Omega_l^H \cup \Gamma^H)$ we have $(r_c \hat{L}_c u_c)(\mathbf{x}) = (L^H r_c u_c)(\mathbf{x})$ and thus

$$\begin{aligned}\left((1-\chi)\left(f^H - L^H r_c u_{c,i-1}\right)\right)(\mathbf{x}) &= \left(f^H - L^H r_c u_{c,i-1}\right)(\mathbf{x}) \\ &= \left(r_c\left(\hat{f}_c - \hat{L}_c u_{c,i-1}\right)\right)(\mathbf{x}).\end{aligned}$$

So (3.24) holds for $\mathbf{x} \in \Omega_C$.

Finally, for $\mathbf{x} \in \Gamma^H$, we have

$$\begin{aligned}\left((1-\chi)\left(f^H - L^H r_c u_{c,i-1}\right)\right)(\mathbf{x}) &= \left(f^H - L^H r_c u_{c,i-1}\right)(\mathbf{x}) \\ &= \left(r_c \hat{f}_c\right)(\mathbf{x}) - \left(L^H r_c u_{c,i-1}\right)(\mathbf{x}) \\ &\overset{(3.13)}{=} \left(r_c \hat{f}_c\right)(\mathbf{x}) - \left(L_o^H\left(u_{c,i-1|\Omega^H \setminus \Omega_l^H}\right)\right)(\mathbf{x}) - \\ &\quad \left(L_\Gamma^H\left(u_{c,i-1|\Gamma^H}\right)\right)(\mathbf{x}) \\ &= \left(r_c \hat{f}_c\right)(\mathbf{x}) - \left(\hat{L}_c u_{c,i-1}\right)(\mathbf{x}) \\ &\overset{(3.18)}{=} \left(r_c\left(\hat{f}_c - \hat{L}_c u_{c,i-1}\right)\right)(\mathbf{x}).\end{aligned}$$

So (3.24) also holds for $\mathbf{x} \in \Gamma^H$.

Combination of (3.23), (3.24) yields

$$\bar{f}^H = L^H r_c u_{c,i-1} + r_c\left(\hat{f}_c - \hat{L}_c u_{c,i-1}\right). \tag{3.25}$$

For $u_{c,i}$ in the LDC method we have the following (where we use block partitioning corresponding to $\mathcal{F}_c = \mathcal{F}_l^h \oplus \mathcal{F}_o^H$):

$$\begin{aligned}u_{c,i} - \hat{u}_c &= \begin{bmatrix}\emptyset & \emptyset \\ \emptyset & I\end{bmatrix} r_c^T u_i^H + r_{cl}^T u_{l,i}^h - \hat{u}_c \\ &\overset{(3.7f)}{=} \begin{bmatrix}\emptyset & \emptyset \\ \emptyset & I\end{bmatrix} r_c^T u_i^H - r_{cl}^T \left(L_l^h\right)^{-1} L_\Gamma^h p_\Gamma\left(u^H_{i|\Gamma^H}\right) + r_{cl}^T \left(L_l^h\right)^{-1} f_l^h - \hat{u}_c \\ &\overset{(3.9),(3.10)}{=} \begin{bmatrix}\emptyset & \emptyset \\ \emptyset & I\end{bmatrix} r_c^T u_i^H - r_{cl}^T \left(L_l^h\right)^{-1} L_\Gamma^h \hat{p}_\Gamma\left(u^H_{i|\Gamma^H}\right) + r_{cl}^T \left(L_l^h\right)^{-1} \hat{f}_l^h - \hat{u}_c \\ &\overset{(3.14b)}{=} \begin{bmatrix}\emptyset & -\left(L_l^h\right)^{-1} L_\Gamma^h \hat{p}_\Gamma r_\Gamma \\ \emptyset & I\end{bmatrix} r_c^T u_i^H + r_{cl}^T \left(L_l^h\right)^{-1} r_{cl} \hat{L}_c \hat{u}_c - \hat{u}_c\end{aligned}$$

$$
\begin{aligned}
&\overset{(3.7e)}{=}\begin{bmatrix} \emptyset & -\left(L_l^h\right)^{-1} L_\Gamma^h \hat{p}_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} r_c^T \left(L^H\right)^{-1} \bar{f}^H - (I-P_2)\hat{u}_c \\
&\overset{(3.25)}{=}\begin{bmatrix} \emptyset & -\left(L_l^h\right)^{-1} L_\Gamma^h \hat{p}_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} r_c^T r_c u_{c,i-1} - (I-P_2)\hat{u}_c \\
&\quad + \begin{bmatrix} \emptyset & -\left(L_l^h\right)^{-1} L_\Gamma^h \hat{p}_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} r_c^T \left(L^H\right)^{-1} r_c \left(\hat{f}_c - \hat{L}_c u_{c,i-1}\right) \\
&\overset{(3.21)}{=}\begin{bmatrix} \emptyset & -\left(L_l^h\right)^{-1} L_\Gamma^h \hat{p}_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} u_{c,i-1} - (I-P_2)\hat{u}_c \\
&\quad + \begin{bmatrix} \emptyset & -\left(L_l^h\right)^{-1} L_\Gamma^h \hat{p}_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} P_1(\hat{u}_c - u_{c,i-1}). \qquad (3.26)
\end{aligned}
$$

Now note that

$$
\begin{aligned}
\begin{bmatrix} \emptyset & -\left(L_l^h\right)^{-1} L_\Gamma^h \hat{p}_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} &= I - \begin{bmatrix} I & \left(L_l^h\right)^{-1} L_\Gamma^h \hat{p}_\Gamma r_\Gamma \\ \emptyset & \emptyset \end{bmatrix} \\
&= I - \begin{bmatrix} I \\ \emptyset \end{bmatrix} \left(L_l^h\right)^{-1} \begin{bmatrix} L_l^h & L_\Gamma^h \hat{p}_\Gamma r_\Gamma \end{bmatrix} \\
&= I - r_{cl}^T \left(L_l^h\right)^{-1} r_{cl} \hat{L}_c \\
&= I - P_2. \qquad (3.27)
\end{aligned}
$$

Substituting (3.27) in (3.26) yields

$$
\begin{aligned}
u_{c,i} - \hat{u}_c &= (I-P_2)u_{c,i-1} - (I-P_2)\hat{u}_c - (I-P_2)P_1(u_{c,i-1} - \hat{u}_c) \\
&= (I-P_2)(I-P_1)(u_{c,i-1} - \hat{u}_c). \qquad \square
\end{aligned}
$$

In the proof of Theorem 3.6 we have used the structure of the starting procedure in the LDC method. For the vector $u_{c,0}$, resulting from the starting procedure in the LDC method, we have the following result:

Lemma 3.7. *The initial approximation* $u_{c,0}$ *in* (3.7*c*) *satisfies*

$$u_{c,0} - \hat{u}_c = (I-P_2)(u_c^* - \hat{u}_c) \qquad (3.28a)$$

with

$$u_c^* = r_c^T \left(L^H\right)^{-1} f^H. \qquad (3.28b)$$

Proof: For $u_{c,0}$ in the LDC method we have:

$$
\begin{aligned}
u_{c,0} - \hat{u}_c &= \begin{bmatrix} \emptyset & \emptyset \\ \emptyset & I \end{bmatrix} r_c^T u_0^H + r_{cl}^T u_{l,0}^h - \hat{u}_c \\
&\overset{(3.7b)}{=}\begin{bmatrix} \emptyset & \emptyset \\ \emptyset & I \end{bmatrix} r_c^T u_0^H - r_{cl}^T \left(L_l^h\right)^{-1} L_\Gamma^h p_\Gamma \left(u_{0|\Gamma^H}^H\right) + r_{cl}^T \left(L_l^h\right)^{-1} f_l^h - \hat{u}_c
\end{aligned}
$$

$$\overset{(3.9)(3.10)}{=} \begin{bmatrix} \emptyset & \emptyset \\ \emptyset & I \end{bmatrix} r_c^T u_0^H - r_{cl}^T \left(L_l^h\right)^{-1} L_\Gamma^h \hat{p}_\Gamma \left(u_{0|_{\Gamma^H}}^H\right) + r_{cl}^T \left(L_l^h\right)^{-1} \hat{f}_l^h - \hat{u}_c$$

$$\overset{(3.14b)}{=} \begin{bmatrix} \emptyset & -\left(L_l^h\right)^{-1} L_\Gamma^h \hat{p}_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} r_c^T u_0^H + r_{cl}^T \left(L_l^h\right)^{-1} r_{cl} \hat{L}_c \hat{u}_c - \hat{u}_c$$

$$\overset{(3.7a)}{=} \begin{bmatrix} \emptyset & \left(-L_l^h\right)^{-1} L_\Gamma^h \hat{p}_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} r_c^T \left(L^H\right)^{-1} f^H - (I - P_2)\hat{u}_c.$$

Using (3.27) we obtain (3.28). □

Substituting (3.28a) in (3.22a) and using the fact that $(I - P_2)^2 = (I - P_2)$, yields the following expression for the LDC iterates.

Corollary 3.8. *The iterates $u_{c,i}$ $(i \geq 1)$ from the LDC method satisfy*

$$u_{c,i} - \hat{u}_c = \left((I - P_2)(I - P_1)(I - P_2)\right)^i (u_c^* - \hat{u}_c). \tag{3.29}$$

From Theorem 3.4 and Corollary 3.8 we conclude that the LDC method in (3.7) is a linear iterative method for solving the composite grid system in (3.15) and that the rate of convergence of the LDC method is determined by the operator

$$\hat{M} = (I - P_2)(I - P_1)(1 - P_2). \tag{3.30}$$

From the definition of the LDC method it is clear that, in a certain sense, this method can be seen as a multiplicative Schwarz domain decomposition type of method. In mathematical terms this is made precise by the expression for the error propagation operator $\hat{M}$ in (3.30). Using the expression for the error propagation operator $\hat{M}$ we can show a close relation between the LDC method and the FAC method of [10]. This will be discussed in Section 3.3 below.

Unfortunately, we are not able to derive satisfactory bounds for the norm (or spectral radius) of $\hat{M}$. With respect to this we note that (almost) all convergence analyses of related methods (e.g. FAC applied to a FVE discretization as in [10]) use a variational setting, whereas it is not clear to us how the discrete operator $\hat{L}_c$ (and thus P_1 and P_2) can be put in such a variational setting. So a satisfactory convergence analysis of the LDC method with $d = 0$ (no overlap) is still lacking, although we have been able to prove that the rate of convergence is determined by the matrix $\hat{M}$ (i.e. we need bounds for $\|\hat{M}\|$, $\rho(\hat{M})$). Below we give an indication of convergence properties of the LDC method by means of numerical results for two (model) problems. We consider the boundary value problems:

Case 1. The Poisson problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega = (0,1) \times (0,1), \\ U &= g \quad \text{on } \partial\Omega, \end{aligned}$$

with f, g such that the solution U is given by

$$U(x, y) = \tfrac{1}{2}\left\{\tanh\left(25\left(x + y - \tfrac{1}{8}\right)\right) + 1\right\}. \tag{3.31}$$

Case 2. The elliptic problem with variable coefficients

$$-\left(2+\sin\left(\frac{\pi x}{3}\right)\right)U_{xx}-\exp(xy)U_{yy}+\cos\left(\frac{\pi x}{5}\right)U_x+(1+x)\exp(y)U_y=f \quad \text{in } \Omega=(0,1)\times(0,1),$$

$$U=g \quad \text{on } \partial\Omega,$$

with f, g such that the solution U is given by (3.31).

The solution U in (3.31) is shown in Fig. 5. The solution varies very rapidly in a small part of the domain and is smooth in the remaining part of the domain. For Ω_l we take $\Omega_l=\{(x,y)\in\Omega | x<1/4\wedge y<1/4\}$. Both on the uniform global coarse grid and on the uniform local fine grid we use a standard discretization. We take central difference approximations both for the second order and the first order derivatives. We consider piecewise linear interpolation and piecewise quadratic interpolation on the interface. Numerical results related to the composite grid discretization errors for these problems are given in [3].

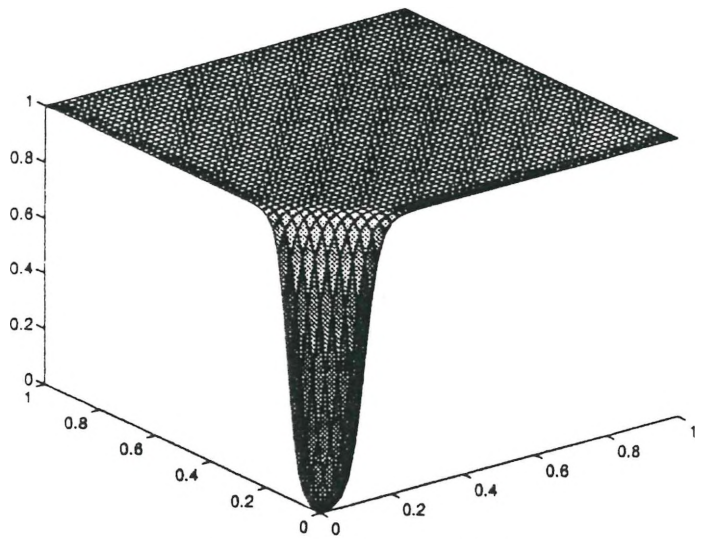

Figure 5. The solution U from (3.31)

In Table 1 we give the average error reduction per iteration (in $\|.\|_\infty$) in the first four iterations:

$$\rho=\frac{1}{4}\sum_{i=1}^{4}\frac{\|u_{c,i}-\hat{u}_c\|_\infty}{\|u_{c,i-1}-\hat{u}_c\|_\infty}.$$

We see that, both for Case 1 and Case 2, the rate of convergence is high and more or less independent of the parameters H and σ. The error reduction factors for piecewise linear and piecewise quadratic interpolation are compara-

Table 1. Average error reduction factors ρ

Case 1	$H = 1/20$			$\sigma = 2$		
	$\sigma = 2$	$\sigma = 4$	$\sigma = 8$	$H = 1/20$	$H = 1/40$	$H = 1/80$
linear	$2.2e-02$	$2.9e-02$	$3.1e-02$	$2.2e-02$	$1.5e-02$	$1.1e-02$
quadratic	$1.9e-02$	$2.2e-02$	$2.3e-02$	$1.9e-02$	$1.0e-02$	$0.7e-02$
Case 2	$H = 1/20$			$\sigma = 2$		
	$\sigma = 2$	$\sigma = 4$	$\sigma = 8$	$H = 1/20$	$H = 1/40$	$H = 1/80$
linear	$2.3e-02$	$3.1e-02$	$3.3e-02$	$2.3e-02$	$2.7e-02$	$2.0e-02$
quadratic	$2.7e-02$	$3.8e-02$	$4.1e-02$	$2.7e-02$	$2.9e-02$	$1.7e-02$

ble. So, for these (and related) test problems we observe a satisfactory convergence behaviour of the LDC method.

3.3. Correspondence to FAC

In Subsection 3.2 we have shown that the discrete problem solved by the LDC method is the composite grid problem (3.15):

$$\hat{L}_c \hat{u}_c = \hat{f}_c.$$

The *fast adaptive composite grid method* (FAC) by McCormick [8–10] is an iterative method for solving a *given* discrete problem on a composite grid. In the FAC method all actual computation (i.e. (approximately) solving discrete problems) is performed on the uniform subgrids of which the composite grid is composed. In [8, 9] the FAC is described, and convergence theory is presented, for the variational case. In [10] the FAC method for solving composite grid problems resulting from a finite volume element (FVE) discretization technique is considered. There convergence results for the FVE-based FAC are derived.

It is noted in the literature that FAC and LDC are very similar (see e.g. [8, 9]). In [7] a comparison of FAC and LDC (and FIC) is made by means of numerical experiments. In the literature we did not find any theoretical results concerning the relation between FAC and LDC. Based on the analysis of the LDC method in the foregoing sections, we can show a clear relation between FAC and LDC. We consider a Finite Difference-based FAC method for solving the composite grid problem (3.15) and we prove that this method is equivalent with the LDC method (3.7).

In the FAC method approximations of $\hat{u}_c$ from (3.15) are computed in an iterative way. At each iteration step a discrete problem on the uniform global coarse grid and a discrete problem on the uniform local fine grid are solved exactly and the resulting solutions are used to improve the current iterate.

Let $\tilde{u}_c$ be an approximation of $\hat{u}_c$. Inserting $\tilde{u}_c$ into the system $\hat{L}_c u_c - \hat{f}_c = 0$ yields the *composite grid defect*

$$d_c := \hat{f}_c - \hat{L}_c \tilde{u}_c. \tag{3.32}$$

The *correction* $v_c := \hat{u}_c - \tilde{u}_c$ satisfies

$$\hat{L}_c v_c = d_c. \tag{3.33}$$

The composite grid defect d_c is restricted to the global coarse grid in the following way:

$$d^H := \tilde{r}_c d_c, \tag{3.34}$$

with $\tilde{r}_c : \mathcal{F}_c \to \mathcal{F}^H$ a restriction operator that satisfies

$$(\tilde{r}_c w)_{|\Omega_c \cup \Gamma^H} = w_{|\Omega_c \cup \Gamma^H}. \tag{3.35}$$

So $\tilde{r}_c$ is the trivial injection outside the local region Ω_l.

The composite grid defect is restricted to the local fine grid in a trivial way:

$$d_l^h := d_{c_{|\Omega_l^h}}. \tag{3.36}$$

An approximation $v^H \in \mathcal{F}^H$ of the correction v_c is computed by solving the global coarse grid problem

$$L^H v^H = d^H, \tag{3.37}$$

with L^H as in (2.2) (cf. (2.7a)).

Also an approximation $v_l^h \in \mathcal{F}_l^h$ of v_c is computed. The approximation v^H of v_c that results from (3.37) is used to define artificial Dirichlet boundary conditions on the interface in the following local fine grid problem (cf. (2.4), (2.8))

$$L_l^h v_l^h = d_l^h - L_\Gamma^h \hat{p}_\Gamma \left(v_{|\Gamma^H}^H \right), \tag{3.38}$$

with L_l^h, L_Γ^h as in (2.4). The interpolation operator $\hat{p}_\Gamma$ has been introduced in Subsection 3.2. We recall that zero boundary values are used in $\hat{p}_\Gamma$ when interpolating between a point on $\partial\Omega$ and a grid point of Γ^H (see Fig. 3).

The approximation v_l^h from (3.38) is used to correct the approximation $\tilde{u}_c$ of $\hat{u}_c$ at grid points of Ω_l^h:

$$\bar{u}_c(\mathbf{x}) := \tilde{u}_c(\mathbf{x}) + v_l^h(\mathbf{x}), \qquad \mathbf{x} \in \Omega_l^h. \tag{3.39}$$

At grid points $\mathbf{x} \in \Omega_c \setminus \Omega_l^h$ the approximation v^H from (3.37) is used to correct the approximation $\tilde{u}_c$:

$$\bar{u}_c(\mathbf{x}) := \tilde{u}_c(\mathbf{x}) + v^H(\mathbf{x}), \qquad \mathbf{x} \in \Omega_c \setminus \Omega_l^h. \tag{3.40}$$

The FAC iteration is an iterative process that combines local and global discrete problems in the way described above.

FAC

Start: Initial composite grid approximation $u_{c,0}$ given.

$i = 1, 2, \ldots$:

a1. Computation of the composite grid defect

$$d_c := \hat{f}_c - \hat{L}_c u_{c,i-1} \tag{3.41a}$$

a2. Restriction of the composite grid defect to the global coarse grid

$$d^H := \tilde{r}_c d_c \tag{3.41b}$$

a3. Restriction of the composite grid defect to the local fine grid

$$d_l^h := d_{c|\Omega_l^h} \tag{3.41c}$$

b. Exact solution of the global problem

$$L^H v^H = d^H \qquad \text{on } \Omega^H \tag{3.41d}$$

c. Exact solution of the local problem

$$L_l^h v_l^h = d_l^h - L_\Gamma^h \hat{p}_\Gamma \left(v_{|\Gamma^H}^H \right) \qquad \text{on } \Omega_l^h \tag{3.41e}$$

d. Correction of the composite grid approximation

$$u_{c,i}(\mathbf{x}) := \begin{cases} u_{c,i-1}(\mathbf{x}) + v_l^h(\mathbf{x}) & \mathbf{x} \in \Omega_l^h \\ u_{c,i-1}(\mathbf{x}) + v^H(\mathbf{x}) & \mathbf{x} \in \Omega_c \setminus \Omega_l^h \end{cases} \tag{3.41f}$$

The FAC iteration (3.41) is written in its *delayed correction form* (cf. [10], Section 4.1). In this form the method is not applicable to nonlinear problems. In case $\hat{L}_c$ is nonlinear, the method should be used in FAS-form (see [10], Section 4.5).

A simple computation shows that

$$\left(\hat{f}_c - \hat{L}_c u_{c,i} \right)(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Omega_c \setminus \Gamma^H, \qquad i \geq 1.$$

So, after the first FAC step we have that $d_l^h \equiv 0$ and that $d^H_{|\Omega^H \setminus \Gamma^H} \equiv 0$. Using this, one can easily show that $\hat{u}_c$ from (3.15) is the unique fixed point of (3.41).

Below, in Theorem 3.9, we derive an expression for the iteration matrix of the FAC method. The notation is as in Subsection 3.2.

We use block partitioning corresponding to $\mathcal{F}^H = \mathcal{F}_l^H \oplus \mathcal{F}_o^H$, $\mathcal{F}_c = \mathcal{F}_l^h \oplus \mathcal{F}_o^H$. Then the restriction operator $\tilde{r}_c$ from (3.35) is of the form

$$\tilde{r}_c = \begin{bmatrix} \tilde{r} & \emptyset \\ \emptyset & I \end{bmatrix}, \tag{3.42}$$

with $\tilde{r} : \mathcal{F}_l^h \to \mathcal{F}_l^H$.

Below, the operators P_2 from (3.20) and $\bar{P}_1 : \mathcal{F}_c \to \mathcal{F}_c$, defined by

$$\bar{P}_1 := \bar{r}_c^T (L^H)^{-1} \bar{r}_c \hat{L}_c, \tag{3.43}$$

play an important role.

Theorem 3.9. *The iterates $u_{c,i}$ $(i \geq 1)$ from the FAC iteration* (3.41) *satisfy*

$$(u_{c,i} - \hat{u}_c) = \bar{M}(u_{c,i-1} - \hat{u}_c) \tag{3.44a}$$

with

$$\bar{M} = (I - P_2)(I - \bar{P}_1). \tag{3.44b}$$

Proof: For $u_{c,i}$ in the FAC method we have:

$$\begin{aligned}
u_{c,i} - \hat{u}_c &= u_{c,i-1} - \hat{u}_c + r_{cl}^T v_l^h + \begin{bmatrix} \emptyset & \emptyset \\ \emptyset & I \end{bmatrix} \bar{r}_c^T v^H \\
&\overset{(3.41e)}{=} u_{c,i-1} - \hat{u}_c + r_{cl}^T (L_l^h)^{-1} d_l^h - (L_l^h)^{-1} L_\Gamma^h \hat{p}_\Gamma (v_{|\Gamma^H}^H) + \begin{bmatrix} \emptyset & \emptyset \\ \emptyset & I \end{bmatrix} \bar{r}_c^T v^H \\
&\overset{(3.41a,c)}{=} u_{c,i-1} - \hat{u}_c + r_{cl}^T (L_l^h)^{-1} r_{cl} \hat{L}_c (\hat{u}_c - u_{c,i-1}) \\
&\quad + \begin{bmatrix} \emptyset & -(L_l^h)^{-1} L_\Gamma^h \hat{p}_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} \bar{r}_c^T v^H \\
&\overset{(3.41d)}{=} (I - P_2)(u_{c,i-1} - \hat{u}_c) + \begin{bmatrix} \emptyset & -(L_l^h)^{-1} L_\Gamma^h \hat{p}_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} \bar{r}_c^T (L^H)^{-1} d^H \\
&\overset{(3.41a,b)}{=} (I - P_2)(u_{c,i-1} - \hat{u}_c) \\
&\quad + \begin{bmatrix} \emptyset & -(L_l^h)^{-1} L_\Gamma^h \hat{p}_\Gamma r_\Gamma \\ \emptyset & I \end{bmatrix} \bar{r}_c^T (L^H)^{-1} \bar{r}_c \hat{L}_c (\hat{u}_c - u_{c,i-1}).
\end{aligned}$$

Substituting (3.27) and (3.43) yields (3.44). □

The iteration matrix $\bar{M}$ of the FAC iteration depends on the restriction operator $\bar{r} : \mathcal{F}_l^h \to \mathcal{F}_l^H$ from (3.42). So the iteration matrix of the FAC iteration differs from the iteraton matrix of the LDC iteration if $\bar{r} \neq r_{inj}$. Due to the fact that $(I - P_2)^2 = (I - P_2)$, the error propagation of the FAC method is determined by the operator $(I - P_2)(I - \bar{P}_1)(I - P_2)$:

$$\begin{aligned}
u_{c,i} - \hat{u}_c &= \bar{M}^i (u_{c,0} - \hat{u}_c) \\
&= \left((I - P_2)(I - \bar{P}_1)(I - P_2) \right)^{i-1} (I - P_2)(I - \bar{P}_1)(u_{c,0} - \hat{u}_c),
\end{aligned} \tag{3.45}$$

for $i \geq 1$.

Lemma 3.10. *The operator $(I - P_2)(I - \bar{P}_1)(I - P_2)$ is independent of the choice of $\bar{r}$ in* (3.42).

Proof: First note that $I - P_2$ does not depend on $\bar{r}$. The operator $I - P_2$ is of the form $\begin{bmatrix} \emptyset & \star \\ \emptyset & \star \end{bmatrix}$ so $(I - P_2)\bar{r}_c^T$ does not depend on $\bar{r}$.

Further we note that $\hat{L}_c(I - P_2)$ is of the form $\begin{bmatrix} \emptyset & \emptyset \\ \emptyset & \star \end{bmatrix}$ so $\bar{r}_c \hat{L}_c(I - P_2)$ does not depend on $\bar{r}$. □

Since $(I - P_2)(I - \bar{P}_1)(I - P_2)$ does not depend on $\bar{r}$, we have that

$$(I - P_2)\big(I - \bar{P}_1\big)(I - P_2) = (I - P_2)(I - P_1)(I - P_2) \tag{3.46}$$

with P_1 as in (3.20).

We note that in the FAC method the initial approximation is not specified yet. A possible choice for this initial approximation is the approximation that results from the starting procedure in the LDC method (3.7a–c). By Lemma 3.7 we have that this initial approximation satisfies

$$u_{c,0} - \hat{u}_c = (I - P_2)(u_c^* - \hat{u}_c).$$

Using this initial approximation, the FAC iterates satisfy

$$\begin{aligned} u_{c,i} - \hat{u}_c &= \bar{M}^i(u_{c,0} - \hat{u}_c) \\ &= \big((I - P_2)\big(I - \bar{P}_1\big)(I - P_2)\big)^i(u_c^* - \hat{u}_c). \end{aligned} \tag{3.47}$$

The equivalence of LDC and FAC follows from (3.29), (3.46) and (3.47).

Corollary 3.11. *The FAC method* (3.41) *with initial approximation from* (3.7*a–c*) *and the LDC method* (3.7) *yield identical iterates.*

Acknowledgement

The authors wish to thank the referees for suggesting significant improvements in style and contents.

References

[1] Brandt, A., Bai, D.: Local mesh refinement multilevel techniques. SIAM J. Sci. Statist. Comput. *8*, 109–134 (1987).
[2] Ferket, P. J. J.: Coupling of a global coarse discretization and local fine discretizations. In: numerical treatment of coupled systems (Hackbusch, W., Wittum, G., eds) pp. 47–58. Braunschweig: Vieweg 1995 (Notes on Numerical Fluid Mechanics, Vol 51.
[3] Ferket, P. J. J., Reusken, A. A.: A finite difference discretization method on composite grids. Computing *55* (1996) [to appear].
[4] Hackbusch, W.: Local defect correction method and domain decomposition techniques. Computing [Suppl.] *5*, 89–113 (1984).
[5] Hackbusch, W.: Multigrid methods and applications. Berlin, Heidelberg, New York: Springer 1985.
[6] Hackbusch, W.: Elliptic differential equations. Theory and numerical treatment. Berlin, Heidelberg, New York, Tokyo: Springer 1992.

[7] Khadra, K., Angot, P., Caltagirone, J.-P.: A comparison of locally adaptive multigrid methods: LDC, FAC, and FIC. In: Proceedings of the Sixth Copper Mountain Conference on Multigrid Methods, April 4–9, 1993, NASA Conference Publication 3224, pp. 275–292.
[8] McCormick, S.: Fast adaptive composite grid (FAC) methods: theory for the variational case. Computing [Suppl.] *5*, 115–121 (1984).
[9] McCormick, S., Thomas, J.: The fast adaptive composite grid (FAC) method for elliptic equations. Math. Comp. *46*, 439–456 (1986).
[10] McCormick, S. F.: Multilevel adaptive methods for partial differential equations. Philadelphia: SIAM 1989.

P. J. J. Ferket and A. A. Reusken,
Department of Mathematics
and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven
The Netherlands
e-mail:peterf@win.tue.nl
wsanar@win.tue.nl