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# A multigrid method based on incomplete Gaussian elimination

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**Abstract.** In this paper we introduce and analyze a new Schur complement approximation based on incomplete Gaussian elimination. The approximate Schur complement is used to develop a multigrid method. This multigrid method has an algorithmic structure that is very similar to the algorithmic structure of classical multigrid methods. The resulting method is almost purely algebraic and has interesting robustness properties with respect to variation in problem parameters.

*AMS Subject Classification:* 62N20, 65N30, 65N55.

*Key words:* multigrid, incomplete Gaussian elimination.

## 1 Introduction

In recent years extensive research has been devoted to the development and analysis of multigrid methods for elliptic boundary value problems which converge with an optimal rate independent of the regularity of the solution (cf. [17], [18] and the references therein). Clearly this is a form of robustness. Another interesting topic with respect to robustness is the dependence of the multigrid convergence rate on certain problem parameters. An example of such a problem parameter is the ratio of anisotropy in an elliptic boundary value problem. There are some recent papers (e.g. [5], [6], [9], [14]) in which multigrid methods are treated that are robust with respect to variation in this anisotropy parameter. A convection-diffusion equation is another example in which we have interesting problem parameters (convection/diffusion ratio, flow direction). In some recent work ([10], [11], [12]) we studied multigrid methods based on Schur complement approximation. An important property of these methods is a strong robustness w.r.t. variation in relevant problem parameters. In this paper we consider a multigrid method based on Schur complement approximation that is closely related to the methods in [10], [12]. However, in this paper we introduce a new and very simple Schur complement approximation. This new approximation is based on the algebraic tool of incomplete Gaussian elimination. We assume two nested grids ("coarse" and "fine") and on the fine mesh the new mesh points are ordered first and then the coarse grid points. This yields a block two by two partitioning of the fine grid stiffness matrix  $A$ :

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Based on the block LU factorization

$$A = \begin{bmatrix} I & \emptyset \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ \emptyset & \mathcal{S}_A \end{bmatrix}, \quad \mathcal{S}_A := A_{22} - A_{21}A_{11}^{-1}A_{12},$$

we consider a linear iterative method with iteration matrix

$$(1.1) \quad W = I - \left( \begin{bmatrix} I & \emptyset \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ \emptyset & \omega^{-1}\tilde{\mathcal{S}}_A \end{bmatrix} \right)^{-1} A.$$

In (1.1),  $\tilde{\mathcal{S}}_A$  is a preconditioner of the Schur complement  $\mathcal{S}_A$  and  $\omega$  a scaling parameter. We will show that the linear iterative method with iteration matrix as in (1.1) can be implemented using an algorithmic structure that is very similar to the structure of classical two-grid methods, which is now well understood. As in the classical approach we then obtain a multigrid ( $V$ - or  $W$ -) cycle if we apply one or more recursive calls for solving the coarse grid problem (with matrix  $\tilde{\mathcal{S}}_A$ ) approximately.

A main topic of this paper is the choice for  $\tilde{\mathcal{S}}_A$ . Usually, in multigrid methods based on Schur complement approximation, the approximate Schur complement  $\tilde{\mathcal{S}}_A$  is based on the coarse grid stiffness matrix (e.g. [1], [2], [9], [12]). In [10] one can find a more advanced approach (resulting in a better approximation of  $\mathcal{S}_A$ ) in which the Schur complement is approximated by the exact Schur complement of modified fine grid equations. These modified fine grid equations are obtained using information about the underlying differential equation (e.g. convection-diffusion equation). In this paper we introduce and analyze a very simple *incomplete Gaussian elimination* approach. Besides the given (fine grid) matrix this incomplete Gaussian elimination process only uses the underlying structure of a sequence of refined meshes. Thus we obtain a multigrid method which is almost purely algebraic. In a certain sense our multigrid method is a "perturbation" of the direct method of Cyclic Reduction (cf. [7], [15]), with a much larger range of applicability than the Cyclic Reduction method.

Using Fourier analysis we prove that the two-grid method has an interesting robustness property with respect to variation in problem parameters.

The remainder of this paper is organized as follows. In Section 2 we derive a few elementary algebraic properties of (approximate) block Gaussian elimination. In Section 3 we describe the Schur complement approximation based on incomplete Gaussian elimination. In Section 4 we apply Fourier analysis to the two-grid method and we derive estimates, both numerically and theoretically, for the contraction number of the two-grid method. In Section 5 we show results of numerical experiments with the multigrid  $W$ -cycle applied to a few test problems.

## 2 Block Gaussian elimination

In this section we discuss some algebraic aspects of a two-grid method based on Schur complement approximation.

We consider a second order elliptic linear boundary value problem on a plane polygonal domain  $\Omega$ . Let  $\Omega_H$  be a given "coarse" mesh on  $\Omega$  consisting of triangles or quadrilaterals. By  $\Omega_h$

we denote the corresponding "fine" mesh that results after a standard refinement of  $\Omega_H$ . The space of grid functions on  $\Omega_H(\Omega_h)$  is denoted by  $U_H(U_h)$ . In  $U_H$  and  $U_h$  we use the standard nodal basis. The ordering of the basis functions in  $U_h$  is chosen such that the basis functions corresponding to nodes in  $\Omega_h \setminus \Omega_H$  are taken first. This induces a partitioning of  $u \in U_h$  into two blocks. We assume a given finite element or finite difference discretization method on  $\Omega_h$ , resulting in a linear system

$$(2.1) \quad A_h x_h = b_h ,$$

with  $A_h : U_h \rightarrow U_h$  nonsingular. For ease of notation we drop the subscript  $h$ , i.e. we write  $A = A_h$ . The ordering of the nodes yields a block partitioning

$$(2.2) \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} ,$$

in which  $[A_{11} \ A_{12}]$  corresponds to the equations in the points of  $\Omega_h \setminus \Omega_H$ . We assume that  $A_{11}$  is nonsingular. We introduce the following notation, in which we use a block partitioning as in (2.2):

$$(2.3a) \quad J := I - \begin{bmatrix} A_{11}^{-1} & \emptyset \\ \emptyset & \emptyset \end{bmatrix} A \quad (\text{"block Jacobi"})$$

$$(2.3b) \quad S_A := A_{22} - A_{21} A_{11}^{-1} A_{12} \quad (\text{"Schur complement"})$$

$$(2.3c) \quad p := \begin{bmatrix} \emptyset \\ I \end{bmatrix} , r := [\emptyset \ I] \quad (\text{"prolongation, restriction"})$$

$$(2.3d) \quad p_A := \begin{bmatrix} -A_{11}^{-1} A_{12} \\ I \end{bmatrix} \quad (\text{"matrix dependent prolongation"}) .$$

Note the identity

$$(2.4) \quad S_A = r A p_A ,$$

i.e., the Schur complement can be obtained with a Galerkin approach if we use a suitable matrix dependent prolongation.

The nonsingularity of  $A$  implies nonsingularity of the Schur complement  $S_A$ . The Schur complement originates, in a natural way, from a block LU factorization of  $A$ :

$$A = \begin{bmatrix} I & \emptyset \\ A_{21} A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ \emptyset & S_A \end{bmatrix} .$$

We now assume a given nonsingular Schur complement preconditioner  $\tilde{\mathcal{S}}_A$ . Our choice for  $\tilde{\mathcal{S}}_A$  will be discussed in Section 3. We also use a parameter  $\omega$  that can be used to correct a possible wrong scaling of  $\tilde{\mathcal{S}}_A$  compared to  $\mathcal{S}_A$  (such that  $\omega\tilde{\mathcal{S}}_A^{-1}\mathcal{S}_A \approx I$ ). For given  $\tilde{\mathcal{S}}_A$  and  $\omega$  there is an obvious candidate for a corresponding iterative method for solving the system in (2.1), namely the linear iterative method with iteration matrix  $W$  defined by

$$(2.5) \quad W = I - \left( \begin{bmatrix} I & \emptyset \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ \emptyset & \omega^{-1}\tilde{\mathcal{S}}_A \end{bmatrix} \right)^{-1} A.$$

The choice for  $\tilde{\mathcal{S}}_A$  will be such that  $\tilde{\mathcal{S}}_A$  has properties (e.g. w.r.t. sparsity and stability) comparable to those of  $A$ . Due to this we can solve the coarse grid problem with matrix  $\tilde{\mathcal{S}}_A$  approximately using a recursive call.

The implementation of classical multigrid methods (cf. [4]) based on presmoothing, coarse grid correction, postsmoothing is now well understood. The result in (2.6a) below shows that the method based on approximate LU factorization (iteration matrix as in (2.5)) can be implemented in the classical multigrid style.

By  $\sigma(M)$  we denote the spectrum of an operator  $M$ .

**Lemma 2.1.** *The following identities hold:*

$$(2.6a) \quad W = J(I - \omega p\tilde{\mathcal{S}}_A^{-1}rA)J$$

$$(2.6b) \quad W^k = p_A(I - \omega\tilde{\mathcal{S}}_A^{-1}\mathcal{S}_A)^k r \quad (k \in \mathbb{N})$$

$$(2.6c) \quad \sigma(W) = \sigma(I - \omega\tilde{\mathcal{S}}_A^{-1}\mathcal{S}_A) \cup \{0\}.$$

*Proof.* Using the relations  $J = p_A r$ ,  $J^2 = J$ ,  $\mathcal{S}_A = rA p_A$  we obtain

$$(2.7) \quad J(I - \omega p\tilde{\mathcal{S}}_A^{-1}rA)J = J^2 - \omega p_A \tilde{\mathcal{S}}_A^{-1} r A p_A r = p_A(I - \omega\tilde{\mathcal{S}}_A^{-1}\mathcal{S}_A)r.$$

On the other hand we also have

$$(2.8) \quad \begin{aligned} W &= I - \left( \begin{bmatrix} I & \emptyset \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ \emptyset & \omega^{-1}\tilde{\mathcal{S}}_A \end{bmatrix} \right)^{-1} A \\ &= I - \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ \emptyset & I \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & \emptyset \\ \emptyset & \omega\tilde{\mathcal{S}}_A^{-1} \end{bmatrix} \begin{bmatrix} I & \emptyset \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} A \\ &= I - \begin{bmatrix} I & A_{11}^{-1}A_{12}(I - \omega\tilde{\mathcal{S}}_A^{-1}\mathcal{S}_A) \\ \emptyset & \omega\tilde{\mathcal{S}}_A^{-1}\mathcal{S}_A \end{bmatrix} \\ &= \begin{bmatrix} \emptyset & -A_{11}^{-1}A_{12} \\ \emptyset & I \end{bmatrix} p(I - \omega\tilde{\mathcal{S}}_A^{-1}\mathcal{S}_A)r = p_A(I - \omega\tilde{\mathcal{S}}_A^{-1}\mathcal{S}_A)r. \end{aligned}$$

Comparison of the results in (2.7) and in (2.8) yields the identity in (2.6a). The result in (2.8) shows that (2.6b) holds for  $k = 1$ . The identity in (2.6b) for  $k > 1$  follows from a simple induction argument. Using (2.6b) with  $k = 1$  and  $rp_A = I$ , we obtain the result in (2.6c).  $\square$

Clearly, the algorithmic structure in (2.6a) is as in a classical two-grid method. As in the standard approach we obtain a multigrid ( $V$ - or  $W$ -) cycle if we apply one or more recursive calls for solving the coarse grid problem approximately. A complete description of a multigrid  $W$ -cycle algorithm is given in Section 5. We emphasize that the method with iteration matrix as in the right hand side of (2.6a) is *not* based on smoothing and coarse grid correction, but is just a special implementation of approximate block LU factorization. Using  $J^2 = J$  we see that essentially we have to solve only *one*  $A_{11}$  system per iteration of the two-grid method. In practice (cf. §5) the  $A_{11}$  system is solved approximately. In general it is rather easy to obtain a satisfactory efficient approximate solver for the  $A_{11}$  system. For the analysis in this paper we assume that the  $A_{11}$  systems are solved exactly.

*Remark 2.2.* In certain special cases the choice  $\tilde{S}_A = S_A$ ,  $\omega = 1$  is feasible. For example, in the 1D case if the matrix  $A$  corresponds to a 3-point discretization stencil. The multigrid  $V$ -cycle based on (2.6a) then results in a direct method. This multigrid  $V$ -cycle is an implementation of (exact) block Gaussian elimination which is closely related to cyclic reduction type of methods (cf. [7], [15]). Also, for certain special problems in 2D one can obtain a Schur complement  $S_A$  with acceptable fill-in. Examples of such direct block LU factorization methods are the cyclic reduction method ([7], [15]) and the total reduction method ([13]). However, for most interesting problems the choice  $\tilde{S}_A = S_A$  is not feasible due to an unacceptable amount of fill-in.

### 3 A Schur complement approximation based on incomplete Gaussian elimination

As is indicated in Remark 2.2, for most problems the choice  $\tilde{S}_A = S_A$  is not feasible. The range of applicability is much larger when we consider *approximate* Schur complement methods. Multigrid type of methods based on Schur complement approximation already exist. In these multigrid methods the Schur complement is approximated using (an approximation of) the coarse grid stiffness matrix. Examples of such methods can be found in [1], [2] and in [12]. For a discussion of the differences between these methods we refer to [12]. The method presented in [12] is based on the two-grid method in (2.6a) with  $\tilde{S}_A = A_H$ , where  $A_H$  is the coarse grid stiffness matrix. In [10] one can find a more advanced approach (resulting in a better approximation of  $S_A$ ) in which the Schur complement is approximated by the exact Schur complement of modified fine grid equations. These modified fine grid equations are obtained using information of the underlying differential equation (cf. Remark 3.2).

In this paper we propose a very simple *incomplete Gaussian elimination* approach. Besides the given (fine grid) matrix this incomplete Gaussian elimination process only uses the underlying structure of a sequence of refined meshes. Thus we obtain a multigrid method which is almost purely algebraic.

As in Section 2 we consider a coarse mesh  $\Omega_H$  and a fine mesh  $\Omega_h$  that is obtained after a standard refinement of  $\Omega_H$ .

Below we use the notion of a *directed graph* as explained in e.g. [3]. We assume that  $A$  results from a discretization method with linear finite elements on triangles or with 9-point (or 5-point, 7-point) finite differences on a square grid. Thus at a vertex  $v$  in the graph of  $A$  we have a typical graph structure as shown in Figure 1.

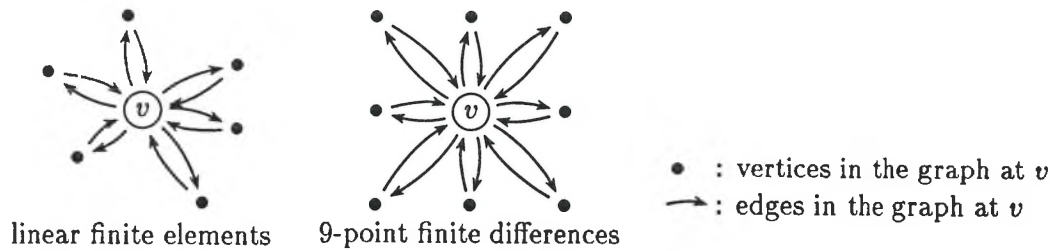


Fig. 1. Typical graph structures.

We now consider Gaussian elimination from a graph theoretical point of view. To obtain the Schur complement system we should eliminate all edges  $\overrightarrow{CF}$ , with  $C$  a vertex corresponding to a grid point in  $\Omega_H$  and  $F$  a vertex corresponding to a grid point in  $\Omega_h \setminus \Omega_H$  (note that  $\overrightarrow{CF} \neq \overrightarrow{FC}$  because we have a directed graph).

We consider an arbitrary edge  $\overrightarrow{CF}$  (cf. Figure 2a) and we perform an incomplete Gaussian elimination as follows. First we apply a standard Gaussian elimination step using the equation at  $F$  to eliminate the edge  $\overrightarrow{CF}$ . This results in fill-in edges as shown in Figure 2b. Now fill-in edges  $\overrightarrow{CG}$  with  $G$  a vertex corresponding to a grid point in  $\Omega_h \setminus \Omega_H$  are eliminated using a simple linear interpolation process. For example, an edge  $\overrightarrow{CG}$  as in Figure 2b with associated value  $\alpha$  is removed and replaced by new edges  $\overrightarrow{CD_1}, \overrightarrow{CD_2}$  with associated value  $\frac{1}{2}\alpha$ . An edge  $\overrightarrow{CG_1}$  (cf. Figure 2b) with associated value  $\alpha$  is replaced by edges  $\overrightarrow{CD_1}, \overrightarrow{CD_2}, \overrightarrow{CD_3}, \overrightarrow{CC}$  with associated value  $\frac{1}{4}\alpha$ . So after this incomplete elimination step we have removed the edge  $\overrightarrow{CF}$  and created only fill-in edges between vertices corresponding to coarse grid points (cf. Figure 2c). Applying this elimination process for all edges  $\overrightarrow{CF}$  results in a decoupling of the coarse grid unknowns from the unknowns corresponding to grid points in  $\Omega_h \setminus \Omega_H$ . For the Schur complement approximation  $\tilde{S}_A$  we use the resulting system of coarse grid equations.

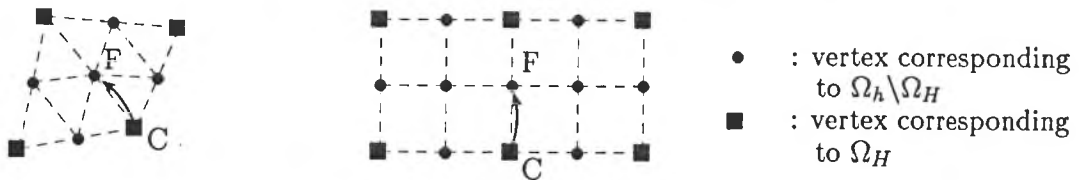


Fig. 2a. Edge  $\overrightarrow{CF}$  that has to be eliminated.

*Remark 3.1.* From Figure 2c it is clear that sparsity is preserved in the incomplete Gaussian elimination process. Also note that if the given matrix  $A$  has nonpositive off-diagonal elements and is weakly diagonally dominant, then the resulting Schur complement approximation  $\tilde{S}_A$  has these properties, too. In this sense, stability is preserved.

*Remark 3.2.* We now comment on implementation aspects of the incomplete Gaussian elimin-



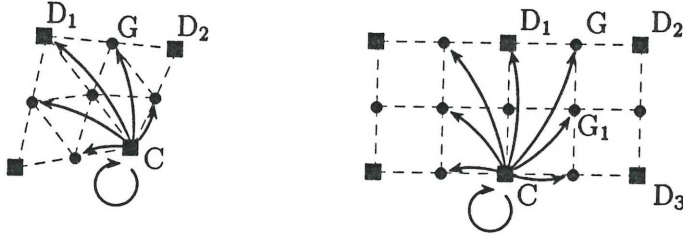


Fig. 2b. Fill-in edges after elimination  $\overrightarrow{CF}$ .

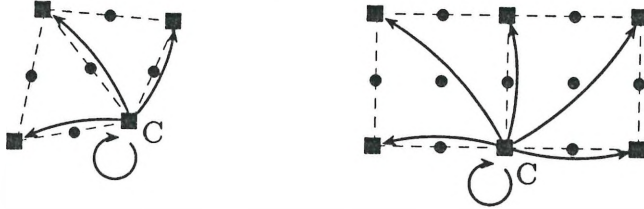


Fig. 2.c Fill-in edges after linear interpolation.

ation process. Consider the situation as shown in Figure 2 (finite difference case). Let  $\beta$  be the value associated with the edge  $\overrightarrow{CF}$ . Furthermore, at  $F$  we have a 9-point stencil denoted by

$$(3.1) \quad \begin{bmatrix} -\alpha_{NW} & -\alpha_N & -\alpha_{NE} \\ -\alpha_W & \alpha & -\alpha_E \\ -\alpha_{SW} & -\alpha_S & -\alpha_{SE} \end{bmatrix}.$$

After elimination and linear interpolation we obtain fill-in edges and associated values as shown in Figure 3.

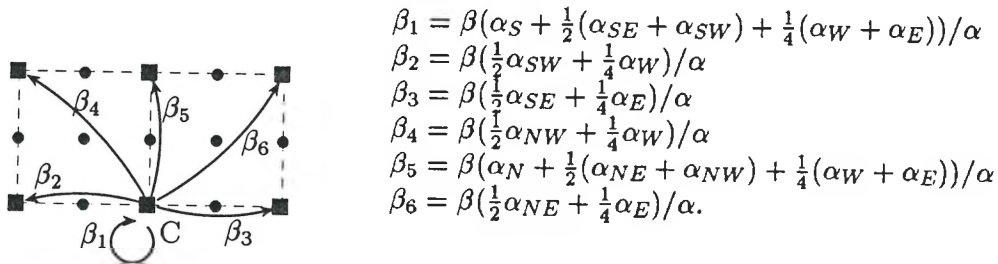


Fig. 3 Graph with associated values after incomplete Gaussian elimination.

The same result as in Figure 3 is obtained if we first *modify* the equation in the point  $F \in \Omega_h \setminus \Omega_H$  and then *eliminate* the edge  $\overrightarrow{CF}$  using this modified equation. The modification process is based on linear interpolation, i.e. the stencil as in (3.1) is replaced by the stencil

$$(3.2) \quad \begin{bmatrix} -(\frac{1}{2}\alpha_{NW} + \frac{1}{4}\alpha_W) & 0 & -(\alpha_N + \frac{1}{2}(\alpha_{NE} + \alpha_{NW}) + \frac{1}{4}(\alpha_W + \alpha_E)) & 0 & -(\frac{1}{2}\alpha_{NE} + \frac{1}{4}\alpha_E) \\ 0 & 0 & \alpha & 0 & 0 \\ -(\frac{1}{2}\alpha_{SW} + \frac{1}{4}\alpha_W) & 0 & -(\alpha_S + \frac{1}{2}(\alpha_{SE} + \alpha_{SW}) + \frac{1}{4}(\alpha_W + \alpha_E)) & 0 & -(\frac{1}{2}\alpha_{SE} + \frac{1}{4}\alpha_E) \end{bmatrix}.$$

Elimination of the edge  $\overrightarrow{CF}$  using the stencil in (3.2) yields the result shown in Figure 3. In matrix block form the modified system of equations can be represented as

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with  $\tilde{A}_{11}$  diagonal. The Schur complement approximation  $\tilde{\mathcal{S}}_A$ , based on incomplete Gaussian elimination, is precisely the exact Schur complement of  $\tilde{A}$ :

$$(3.3) \quad \tilde{\mathcal{S}}_A = \mathcal{S}_{\tilde{A}} := A_{22} - A_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12} = [0 \ I]A \begin{bmatrix} -\tilde{A}_{11}^{-1}\tilde{A}_{12} \\ I \end{bmatrix} =: rAp_{\tilde{A}}.$$

Based on (3.3) we can implement the incomplete Gaussian elimination using the Galerkin approach, that is often used in multigrid algorithms. Given  $A$ , we first determine the matrix dependent prolongation  $p_{\tilde{A}}$  (here the linear interpolation is used). Then we compute the coarse grid operator  $\mathcal{S}_{\tilde{A}} = rAp_{\tilde{A}}$ . We emphasize that this Galerkin approach is just a special (and often convenient) implementation of the incomplete Gaussian elimination described above.

## 4 Fourier analysis of the two-grid method

We consider the usual setting in which a Fourier analysis is applicable (cf. [4], [16]). On  $\Omega := (-1, 1]^2$  we introduce a uniform square grid with mesh size  $h$ :

$$(4.1) \quad \Omega_h := \{(x, y) \in \Omega \mid x = \nu h, \quad y = \mu h, \quad 1 - N \leq \nu, \mu \leq N\},$$

with  $N := 1/h$ . In  $\ell^2(\Omega_h)$  we have  $4N^2$  basis vectors  $e_h^{\nu\mu}$  defined by

$$(4.2) \quad e_h^{\nu\mu}(x, y) = \frac{1}{2}e^{\pi i(\nu x + \mu y)}, \quad (x, y) \in \Omega_h, \quad 1 - N \leq \nu, \mu \leq N.$$

We assume  $N$  to be even and introduce a coarse grid space with mesh size  $H := 2h$ ,  $N_H := N/2$ , and  $\Omega_H$  as in (4.1) with  $h$  and  $N$  replaced by  $H$  and  $N_H$  respectively. In  $\ell^2(\Omega_H)$  we use the Fourier basis

$$e_H^{\nu\mu}(x, y) = \frac{1}{2}e^{\pi i(\nu x + \mu y)}, \quad (x, y) \in \Omega_H, \quad 1 - N_H \leq \nu, \mu \leq N_H.$$

The vectors in (4.2) form an orthonormal basis with respect to a scaled Euclidean inner product, and thus the Fourier transform

$$Q_h : (\alpha_{\nu\mu})_{1-N \leq \nu, \mu \leq N} \rightarrow \sum_{\nu, \mu=1-N}^N \alpha_{\nu\mu} e_h^{\nu\mu}$$

is unitary. Every "low" frequency  $(\nu, \mu)$  with  $1 - N_H \leq \nu, \mu \leq N_H$  is associated with the "high" frequencies  $(\nu', \mu), (\nu, \mu'), (\nu', \mu')$  where  $\nu', \mu'$  are defined by

$$\nu' = \begin{cases} \nu + N & \text{if } \nu \leq 0 \\ \nu - N & \text{if } \nu > 0 \end{cases}, \quad \mu' = \begin{cases} \mu + N & \text{if } \mu \leq 0 \\ \mu - N & \text{if } \mu > 0 \end{cases}.$$

Clearly  $\ell^2(\Omega_h)$  is a direct sum of the  $N^2$  subspaces  $U_h^{\nu\mu} := \text{span}\{e_h^{\nu\mu}, e_h^{\nu'\mu}, e_h^{\nu\mu'}, e_h^{\nu'\mu'}\}$ ,  $1 - N_H \leq \nu, \mu \leq N_H$ . By  $Q_h^{\nu\mu}$  we denote the  $4N^2 \times 4$  matrix with columns these basis vectors of  $U_h^{\nu\mu}$ :

$$Q_h^{\nu\mu} := [e_h^{\nu\mu} \quad e_h^{\nu'\mu} \quad e_h^{\nu\mu'} \quad e_h^{\nu'\mu'}].$$

We consider an operator  $A = A_h : \ell^2(\Omega_h) \rightarrow \ell^2(\Omega_h)$  that can be represented by a constant 9-point difference star

$$(4.3) \quad [A] = \begin{bmatrix} -\beta_1 & -\alpha_2 & -\beta_2 \\ -\alpha_1 & 1 & -\alpha_3 \\ -\beta_4 & -\alpha_4 & -\beta_3 \end{bmatrix},$$

with  $0 \leq \alpha_i, \beta_i$ ,  $\sum_{i=1}^4 (\alpha_i + \beta_i) = 1$ . We also assume  $\alpha_1 + \alpha_3 \neq 0$  and  $\alpha_2 + \alpha_4 \neq 0$ , which guarantees that only the constant function is in the kernel of the operator  $A$ . The Fourier modes are eigenvectors of the operator  $A$ , i.e. we have  $(Q_h^{\nu\mu})^* A Q_h^{\nu\mu} = \text{diag}(d_1^{\nu\mu}, d_2^{\nu\mu}, d_3^{\nu\mu}, d_4^{\nu\mu})$  (we use the adjoint w.r.t. the scaled Euclidean inner product). For the eigenvalues  $d_j^{\nu\mu}$ ,  $1 - N_H \leq \nu, \mu \leq N_H$ , we have the following formulas

$$(4.4a) \quad d_1^{\nu\mu} = 1 - (v + w) - z$$

$$(4.4b) \quad d_2^{\nu\mu} = 1 + (v - w) + z$$

$$(4.4c) \quad d_3^{\nu\mu} = 1 - (v - w) + z$$

$$(4.4d) \quad d_4^{\nu\mu} = 1 + (v + w) - z,$$

with

$$(4.5a) \quad v = v(\nu) = \alpha_1 e^{-\pi i \nu h} + \alpha_3 e^{\pi i \nu h}$$

$$(4.5b) \quad w = w(\mu) = \alpha_2 e^{\pi i \mu h} + \alpha_4 e^{-\pi i \mu h}$$

$$(4.5c) \quad z = z(\nu, \mu) = \beta_1 e^{-\pi i (\nu - \mu) h} + \beta_2 e^{\pi i (\nu + \mu) h} + \beta_3 e^{\pi i (\nu - \mu) h} + \beta_4 e^{-\pi i (\nu + \mu) h}.$$

In the remainder we also use the notation

$$(4.6) \quad s_k = \sin(\frac{1}{2} k \pi h), \quad c_k = \cos(\frac{1}{2} k \pi h).$$

For  $(\nu, \mu) \neq (0, 0)$  we introduce the *harmonic mean* of the eigenvalues  $d_j^{\nu\mu}$ :

$$(4.7) \quad \mathcal{H}_{\nu\mu} := 4 \left( \sum_{j=1}^4 1/d_j^{\nu\mu} \right)^{-1} \quad (1 - N_H \leq \nu, \mu \leq N_H).$$

In Lemma 2.1 it is shown that for the two-grid iteration matrix  $W$  we have  $\sigma(W) = \sigma(I - \omega \tilde{S}_A^{-1} \mathcal{S}_A) \cup \{0\}$ . So the convergence (rate) of the two-grid method is determined by  $\sigma(\tilde{S}_A^{-1} \mathcal{S}_A)$ . Below we derive estimates, both numerically and theoretically, for this spectrum. We use  $\tilde{S}_A = \mathcal{S}_{\tilde{A}}$  as explained in Section 3 (cf. Remark 3.2). First we derive expressions for the eigenvalues of  $\mathcal{S}_{\tilde{A}}^{-1} \mathcal{S}_A$  (Lemma 4.1, Lemma 4.2, Corollary 4.3) and then we analyze the dependence of  $\sigma(\mathcal{S}_{\tilde{A}}^{-1} \mathcal{S}_A)$  on the coefficients  $\alpha_j, \beta_j$  in the stencil of  $A$ .

For  $\mathcal{S}_A$ , the Schur complement of  $A$ , we have the following result, that is proved in [10] (cf. also [12]):

**Lemma 4.1.** *The Fourier mode  $e_H^{\nu\mu}$  ( $1 - N_H \leq \nu, \mu \leq N_H$ ,  $(\nu, \mu) \neq (0, 0)$ ) is an eigenvector of  $\mathcal{S}_A$  with corresponding eigenvalue the harmonic mean of  $d_1^{\nu\mu}, d_2^{\nu\mu}, d_3^{\nu\mu}, d_4^{\nu\mu}$ , i.e.:*

$$(4.8) \quad \mathcal{S}_A e_H^{\nu\mu} = \mathcal{H}_{\nu\mu} e_H^{\nu\mu} .$$

To be able to apply Fourier analysis to  $\mathcal{S}_{\tilde{A}} = r A p_{\tilde{A}}$  we first introduce some notation. As discussed in Remark 3.2 we have modified equations  $[\tilde{A}_{11} \quad \tilde{A}_{12}]$  in the grid points of  $\Omega_h \setminus \Omega_H$ . The grid points of  $\Omega_h \setminus \Omega_H$  are divided in three sets:

$$\begin{aligned} \Omega_h^{(1)} &= \{(x, y) \in \Omega_h \setminus \Omega_H \mid y = kH, k \in \mathbb{Z}\} \\ \Omega_h^{(2)} &= \{(x, y) \in \Omega_h \setminus \Omega_H \mid x = kH, k \in \mathbb{Z}\} \\ \Omega_h^{(3)} &= (\Omega_h \setminus \Omega_H) \setminus (\Omega_h^{(1)} \cup \Omega_h^{(2)}) . \end{aligned}$$

Note that for given  $j \in \{1, 2, 3\}$   $\tilde{A}$  has a constant difference star in the points of  $\Omega_h^{(j)}$ , thus for a suitable  $\tau_{(j)}^{\nu\mu}$ , independent of  $(x, y) \in \Omega_h^{(j)}$  we have

$$(4.9) \quad (\tilde{A}_{11}^{-1} \tilde{A} e_h^{\nu\mu})|_{\Omega_h^{(j)}} = \tau_{(j)}^{\nu\mu} e_h^{\nu\mu}|_{\Omega_h^{(j)}} .$$

In Lemma 4.2 it is shown that the eigenvalues of  $\mathcal{S}_{\tilde{A}}$  can be expressed in terms of these  $\tau_{(j)}^{\nu\mu}$  and the eigenvalues of  $A$ . A proof of this lemma can be found in [10].

**Lemma 4.2.** *For  $(\nu, \mu) \neq (0, 0)$  with  $1 - N_H \leq \nu, \mu \leq N_H$  the following holds:*

$$(4.10) \quad \begin{aligned} \mathcal{S}_{\tilde{A}} e_H^{\nu\mu} &= \{d_1^{\nu\mu} + \frac{1}{4}(\tau_{(1)}^{\nu\mu} + \tau_{(2)}^{\nu\mu})(d_4^{\nu\mu} - d_1^{\nu\mu}) + \frac{1}{4}(\tau_{(1)}^{\nu\mu} - \tau_{(2)}^{\nu\mu})(d_2^{\nu\mu} - d_3^{\nu\mu}) \\ &\quad + \frac{1}{4}\tau_{(3)}^{\nu\mu}((d_2^{\nu\mu} + d_3^{\nu\mu}) - (d_1^{\nu\mu} + d_4^{\nu\mu}))\} e_H^{\nu\mu} . \end{aligned}$$

On  $\Omega_h^{(2)}$  we have (cf. Remark 3.2) the stencil

$$[\bar{A}] = \begin{bmatrix} -(\frac{1}{2}\beta_1 + \frac{1}{4}\alpha_1) & 0 & -(\alpha_2 + \frac{1}{2}(\beta_1 + \beta_2) + \frac{1}{4}(\alpha_1 + \alpha_3)) & 0 & -(\frac{1}{2}\beta_2 + \frac{1}{4}\alpha_3) \\ 0 & 0 & 1 & 0 & 0 \\ -(\frac{1}{2}\beta_4 + \frac{1}{4}\alpha_1) & 0 & -(\alpha_4 + \frac{1}{2}(\beta_3 + \beta_4) + \frac{1}{4}(\alpha_1 + \alpha_3)) & 0 & -(\frac{1}{2}\beta_3 + \frac{1}{4}\alpha_3) \end{bmatrix} .$$

A straightforward computation yields that for  $\tau_{(2)}^{\nu\mu}$  as in (4.9) we have the expression

$$(4.11a) \quad \tau_{(2)}^{\nu\mu} = d_1^{\nu\mu} + \gamma_{\nu\mu}v + 2s_\nu^2z ,$$

with  $d_1^{\nu\mu}$ ,  $v$ ,  $z$ ,  $s_\nu$  as in (4.4), (4.5), (4.6) and

$$(4.12) \quad \gamma_{\nu\mu} := 2(s_\nu^2c_\mu^2 + s_\mu^2c_\nu^2) .$$

Similarly, we obtain for  $\tau_{(1)}^{\nu\mu}$  and  $\tau_{(3)}^{\nu\mu}$  the expressions

$$(4.11b) \quad \tau_{(1)}^{\nu\mu} = d_1^{\nu\mu} + \gamma_{\nu\mu}w + 2s_\mu^2z ,$$

$$(4.11c) \quad \tau_{(3)}^{\nu\mu} = d_1^{\nu\mu} + 2(s_\nu^2w + s_\mu^2v) .$$

When we substitute the results from (4.11) in (4.10), use the definitions in (4.4) and rearrange the terms in the resulting expression, we obtain

**Corollary 4.3.** *For  $(\nu, \mu) \neq (0, 0)$  with  $1 - N_H \leq \nu, \mu \leq N_H$  we have*

$$(4.12) \quad \mathcal{S}_A e_H^{\nu\mu} = \{d_1^{\nu\mu}(2 - d_1^{\nu\mu}) + 2\gamma_{\nu\mu}vw + 4z(s_\nu^2w + s_\mu^2v)\}e_H^{\nu\mu} .$$

From Lemma 4.1 and Corollary 4.3 we immediately obtain an expression for the eigenvalues of  $\mathcal{S}_A^{-1}\mathcal{S}_A$ . Using this expression we can analyze  $\sigma(\mathcal{S}_A^{-1}\mathcal{S}_A)$ . Below we first show results of numerical calculations for a few test problems, and then we derive some theoretical results.

For given coefficients  $\alpha_j, \beta_j$  ( $1 \leq j \leq 4$ ) we computed (using MATLAB) the quantities defined in (4.13), (4.14). Note that  $\sigma(\mathcal{S}_A^{-1}\mathcal{S}_A)$  does not depend on the scaling of the operator  $A$ . We always leave out the constant function (i.e.  $(\nu, \mu) = (0, 0)$ ). We take  $h = 1/64$  and define

$$(4.13) \quad \text{shift} := 2 \left( \min_{\lambda} \text{Re}(\lambda) + \max_{\lambda} \text{Re}(\lambda) \right)^{-1} ,$$

where the minimum and the maximum is taken over  $\sigma(\mathcal{S}_A^{-1}\mathcal{S}_A)$  and

$$(4.14) \quad \rho := \max\{ |1 - \text{shift} * \lambda| \mid \lambda \in \sigma(\mathcal{S}_A^{-1}\mathcal{S}_A) \} .$$

*Experiment 1* (Convection-diffusion, 5-point stencil).

We consider the difference star

$$\frac{\varepsilon}{h} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} + \cos(\varphi) \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sin(\varphi) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

In Table 1 we give the results for shift and  $\rho$  for several values of  $\varepsilon$  and  $\varphi$ .

$\varepsilon$	1	$10^{-2}$			$10^{-4}$		
$\varphi$	0	$\pi/32$	$\pi/8$	$\pi/4$	$\pi/32$	$\pi/8$	$\pi/4$
shift	0.85	0.67	0.67	0.67	0.69	0.69	0.69
$\rho$	0.19	0.33	0.33	0.33	0.40	0.40	0.40

Table 1

*Experiment 2* (Convection–diffusion, 9-point stencil).

We consider the realigned skew upwind scheme proposed in [8]. With  $c := \cos(\varphi)$ ,  $s := \sin(\varphi)$ ,  $\varphi \in [0, \frac{\pi}{2}]$ , the stencil is given by

$$\frac{\varepsilon}{h} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} + \frac{1}{c+s} \begin{bmatrix} 0 & 0 & 0 \\ -c^2 & 1+cs & 0 \\ -cs & -s^2 & 0 \end{bmatrix}.$$

Results of numerical computations are given in Table 2.

$\varepsilon$	$10^{-2}$			$10^{-4}$		
$\varphi$	$2\pi/16$	$3\pi/16$	$4\pi/16$	$2\pi/16$	$3\pi/16$	$4\pi/16$
shift	0.67	0.67	0.67	0.72	0.71	0.71
$\rho$	0.33	0.33	0.33	0.46	0.44	0.44

Table 2

*Experiment 3* (random stencil).

We generated random coefficients  $\alpha_j, \beta_j \in (0, 1)$  (uniform distribution) and computed the corresponding values for shift and for  $\rho$ . This was repeated hundred times. The results are shown in Figure 4.

In the numerical experiments above we see a strong robustness with respect to variation in the coefficients  $\alpha_j, \beta_j$ . Note that for almost all experiments we have shift  $\approx 0.7$ .

We now turn to theoretical bounds for  $\sigma(\mathcal{S}_A^{-1}\mathcal{S}_A)$ . We only consider the case of a *5-point stencil*, i.e.  $\beta_j = 0$  for  $j = 1, 2, 3, 4$ . For this case the expressions in (4.4) simplify:

$$(4.15a) \quad d_1^{\nu\mu} = 1 - (v + w)$$

$$(4.15b) \quad d_2^{\nu\mu} = 1 + (v - w)$$

$$(4.15c) \quad d_3^{\nu\mu} = 1 - (v - w)$$

$$(4.15d) \quad d_4^{\nu\mu} = 1 + (v + w).$$

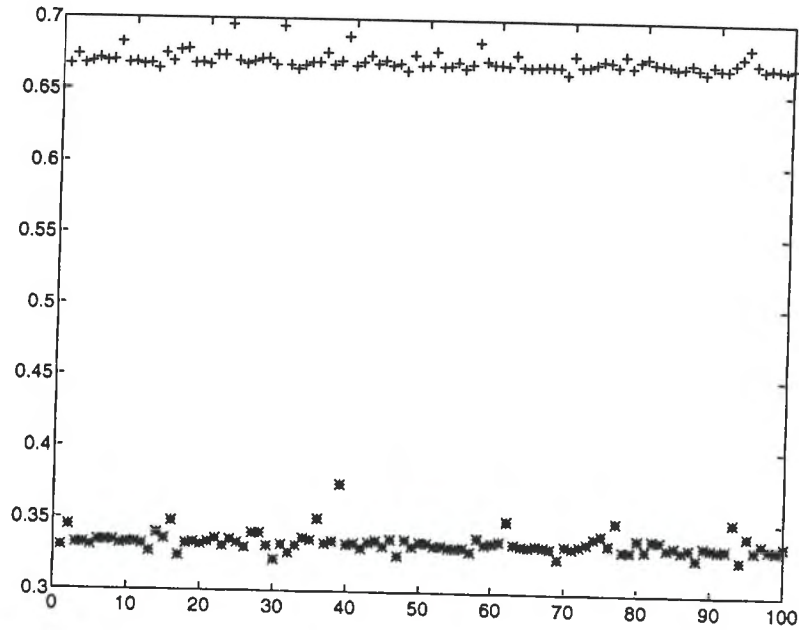


Fig. 4. +: shift, \*:  $\rho$ .

Note that for  $v, w$  we have:

$$(4.16a) \quad \operatorname{Re}(v) = (\alpha_1 + \alpha_3) \cos(\nu\pi h)$$

$$(4.16b) \quad |v| \leq \alpha_1 + \alpha_3$$

$$(4.16c) \quad \operatorname{Re}(w) = (\alpha_2 + \alpha_4) \cos(\mu\pi h)$$

$$(4.16d) \quad |w| \leq \alpha_2 + \alpha_4 .$$

First we show some properties of the eigenvalues  $d_j^{\nu\mu}$ .

**Lemma 4.4.** For  $(\nu, \mu) \neq (0, 0)$  with  $1 - N_H \leq \nu, \mu \leq N_H$  we have

$$(4.17a) \quad \operatorname{Re}(d_j^{\nu\mu}) > 0 \text{ for } j = 1, 2, 3, 4$$

$$(4.17b) \quad \operatorname{Re}(d_j^{\nu\mu}/d_k^{\nu\mu}) \geq 0 \text{ for all } j, k \in \{1, 2, 3, 4\} .$$

*Proof.* The result in (4.17a) is a direct consequence of the definition of  $d_j^{\nu\mu}$  and the properties in (4.16). The result in (4.17b) is trivial for  $j = k$ . Using  $\operatorname{Re}(1/z) = |z|^{-2} \operatorname{Re}(z)$  it follows that it is sufficient to consider  $j < k$ .

We start with the case  $j = 1$ . For  $(j, k) = (1, 2)$  the result in (4.17b) follows from

$$\begin{aligned} \operatorname{Re}(d_1^{\nu\mu} \bar{d}_2^{\nu\mu}) &= \operatorname{Re}[(1 - (v + w))(1 + \bar{v} - \bar{w})] = \operatorname{Re}[\{(1 - w) - v\}(\overline{1 - w + v})] \\ &= |1 - w|^2 - |v|^2 \geq (1 - |w|)^2 - |v|^2 \\ &= (1 - |w| - |v|)(1 - |w| + |v|) \geq 0 \quad (\text{use (4.16b,d) and } \Sigma\alpha_i = 1) . \end{aligned}$$

The same argument with  $v$  and  $w$  exchanged implies the result for the case  $(j, k) = (1, 3)$ . For  $(j, k) = (1, 4)$  we have:

$$\begin{aligned} \operatorname{Re} (d_1^{\nu\mu} \bar{d}_4^{\nu\mu}) &= \operatorname{Re} [(1 - (v + w))(1 + \overline{(v + w)})] \\ &= 1 - |v + w|^2 \geq 1 - (|v| + |w|)^2 \geq 1 - \left( \sum_{i=1}^4 \alpha_i \right)^2 = 0. \end{aligned}$$

We now consider  $j = 2$ . For  $(j, k) = (2, 3)$  we obtain

$$\begin{aligned} \operatorname{Re} (d_2^{\nu\mu} \bar{d}_3^{\nu\mu}) &= \operatorname{Re} [(1 + (v - w))(1 - \overline{(v - w)})] \\ &= 1 - |v - w|^2 \geq 1 - (|v| + |w|)^2 \geq 1 - \left( \sum_{i=1}^4 \alpha_i \right)^2 = 0. \end{aligned}$$

For  $(j, k) = (2, 4)$  we have (cf.  $(j, k) = (1, 2)$  above):

$$\begin{aligned} \operatorname{Re} (d_2^{\nu\mu} \bar{d}_4^{\nu\mu}) &= \operatorname{Re} [(1 + (v - w))(1 + \overline{(v + w)})] \\ &= \operatorname{Re} [(1 + v - w)(\overline{1 + v + w})] \\ &= |1 + v|^2 - |w|^2 \geq (1 - |v|)^2 - |w|^2 \\ &= (1 - |v| - |w|)(1 - |v| + |w|) \geq 0. \end{aligned}$$

Finally, the same argument as for the case  $(j, k) = (2, 4)$ , but with  $v$  and  $w$  exchanged, implies the result for  $(j, k) = (3, 4)$ .  $\square$

From Lemma 4.1 and Corollary 4.3 we obtain the following expressions for the eigenvalues of  $\mathcal{S}_A$  and  $\mathcal{S}_{\bar{A}}$ , denoted by  $\xi^{\nu\mu}(\mathcal{S}_A)$  and  $\xi^{\nu\mu}(\mathcal{S}_{\bar{A}})$  respectively:

$$(4.18a) \quad \xi^{\nu\mu}(\mathcal{S}_A) = \mathcal{H}_{\nu\mu} = 4 \left( \sum_{j=1}^4 1/d_j^{\nu\mu} \right)^{-1}$$

$$(4.18b) \quad \xi^{\nu\mu}(\mathcal{S}_{\bar{A}}) = d_1^{\nu\mu} (2 - d_1^{\nu\mu}) + 2\gamma_{\nu\mu} v w,$$

with  $\gamma_{\nu\mu}$  as in (4.12). Using (4.4) the expression for  $\xi^{\nu\mu}(\mathcal{S}_{\bar{A}})$  can be rewritten in the form

$$(4.18c) \quad \xi^{\nu\mu}(\mathcal{S}_{\bar{A}}) = d_1^{\nu\mu} d_4^{\nu\mu} + \frac{1}{2} \gamma_{\nu\mu} (d_2^{\nu\mu} d_3^{\nu\mu} - d_1^{\nu\mu} d_4^{\nu\mu}).$$

In Lemma 4.5 and Lemma 4.6 we give estimates for  $\xi^{\nu\mu}(\mathcal{S}_{\bar{A}})/\xi^{\nu\mu}(\mathcal{S}_A)$ .

**Lemma 4.5.** *For  $(\nu, \mu) \neq (0, 0)$  with  $1 - N_H \leq \nu, \mu \leq N_H$  we have*

$$(4.19) \quad \operatorname{Re} (\xi^{\nu\mu}(\mathcal{S}_{\bar{A}})/\xi^{\nu\mu}(\mathcal{S}_A)) \geq \frac{1}{2}.$$



*Proof.* First we note that the equalities

$$1/d_1^{\nu\mu} + 1/d_4^{\nu\mu} = 2/(d_1^{\nu\mu} d_4^{\nu\mu}) \quad \text{and} \quad 1/d_2^{\nu\mu} + 1/d_3^{\nu\mu} = 2/(d_2^{\nu\mu} d_3^{\nu\mu})$$

hold. Using this and the results in Lemma 4.4 we obtain for  $j = 2, 3$ :

$$(4.20) \quad \begin{aligned} \operatorname{Re} (d_1^{\nu\mu} d_4^{\nu\mu} / d_j^{\nu\mu}) &= |d_1^{\nu\mu} d_4^{\nu\mu} / d_j^{\nu\mu}|^2 \operatorname{Re} (d_j^{\nu\mu} / (d_1^{\nu\mu} d_4^{\nu\mu})) \\ &= \frac{1}{2} |d_1^{\nu\mu} d_4^{\nu\mu} / d_j^{\nu\mu}|^2 (\operatorname{Re} (d_j^{\nu\mu} / d_1^{\nu\mu}) + \operatorname{Re} (d_j^{\nu\mu} / d_4^{\nu\mu})) \geq 0. \end{aligned}$$

With a similar argument we obtain for  $j = 1, 4$ :

$$(4.21) \quad \operatorname{Re} (d_2^{\nu\mu} d_3^{\nu\mu} / d_j^{\nu\mu}) \geq 0.$$

Combination of the results in (4.18a), (4.18c), (4.20), (4.21) yields

$$\begin{aligned} \operatorname{Re} (\xi^{\nu\mu}(\mathcal{S}_{\bar{A}}) / \xi^{\nu\mu}(\mathcal{S}_A)) &= \operatorname{Re} \left[ \left( (1 - \frac{1}{2}\gamma_{\nu\mu}) d_1^{\nu\mu} d_4^{\nu\mu} + \frac{1}{2}\gamma_{\nu\mu} d_2^{\nu\mu} d_3^{\nu\mu} \right) \frac{1}{4} \left( \sum_{j=1}^4 1/d_j^{\nu\mu} \right) \right], \\ &= \frac{1}{2} \operatorname{Re} \left[ \left( (1 - \frac{1}{2}\gamma_{\nu\mu}) d_1^{\nu\mu} d_4^{\nu\mu} + \frac{1}{2}\gamma_{\nu\mu} d_2^{\nu\mu} d_3^{\nu\mu} \right) \left( (d_1^{\nu\mu} d_4^{\nu\mu})^{-1} + (d_2^{\nu\mu} d_3^{\nu\mu})^{-1} \right) \right] \\ &= \frac{1}{2} \operatorname{Re} \left[ 1 + (1 - \frac{1}{2}\gamma_{\nu\mu}) d_1^{\nu\mu} d_4^{\nu\mu} / (d_2^{\nu\mu} d_3^{\nu\mu}) + \frac{1}{2}\gamma_{\nu\mu} d_2^{\nu\mu} d_3^{\nu\mu} / (d_1^{\nu\mu} d_4^{\nu\mu}) \right] \\ &= \frac{1}{2} + \frac{1}{4} (1 - \frac{1}{2}\gamma_{\nu\mu}) \operatorname{Re} [d_1^{\nu\mu} d_4^{\nu\mu} (1/d_2^{\nu\mu} + 1/d_3^{\nu\mu})] \\ &\quad + \frac{1}{4}\gamma_{\nu\mu} \operatorname{Re} [d_2^{\nu\mu} d_3^{\nu\mu} (1/d_1^{\nu\mu} + 1/d_4^{\nu\mu})] \\ &\geq \frac{1}{2} \quad (\text{note that } \gamma_{\nu\mu} \in [0, 1] \text{ holds}). \end{aligned}$$

□

**Lemma 4.6.** For  $(\nu, \mu) \neq (0, 0)$  with  $1 - N_H \leq \nu, \mu \leq N_H$  we have

$$(4.22) \quad |1 - \xi^{\nu\mu}(\mathcal{S}_{\bar{A}}) / \xi^{\nu\mu}(\mathcal{S}_A)| \leq 1.$$

*Proof.* We introduce the notation  $H_{14} := d_1^{\nu\mu} d_4^{\nu\mu}$ ,  $H_{23} := d_2^{\nu\mu} d_3^{\nu\mu}$ . Using (4.18a), (4.18c) we obtain

$$(4.23) \quad \begin{aligned} &|1 - \xi^{\nu\mu}(\mathcal{S}_{\bar{A}}) / \xi^{\nu\mu}(\mathcal{S}_A)| \\ &= |1 - (H_{14} + \frac{1}{2}\gamma_{\nu\mu}(H_{23} - H_{14})) \frac{1}{4} \left( \sum_{j=1}^4 1/d_j^{\nu\mu} \right)| \\ &= |1 - \frac{1}{2} \left( (1 - \frac{1}{2}\gamma_{\nu\mu}) H_{14} + \frac{1}{2}\gamma_{\nu\mu} H_{23} \right) (H_{14}^{-1} + H_{23}^{-1})| \\ &= \frac{1}{2} | (1 - \frac{1}{2}\gamma_{\nu\mu})(1 - H_{14} H_{23}^{-1}) + \frac{1}{2}\gamma_{\nu\mu}(1 - H_{23} H_{14}^{-1}) | \\ &\leq \frac{1}{2} (1 - \frac{1}{2}\gamma_{\nu\mu}) |1 - H_{14} H_{23}^{-1}| + \frac{1}{4}\gamma_{\nu\mu} |1 - H_{23} H_{14}^{-1}|. \end{aligned}$$

We consider the term  $|1 - H_{14}H_{23}^{-1}|$ . Using

$$H_{14}H_{23}^{-1} = \frac{d_1^{\nu\mu}d_4^{\nu\mu}}{d_2^{\nu\mu}d_3^{\nu\mu}} = \frac{1 - (v+w)^2}{1 - (v-w)^2} = 1 - 4\frac{vw}{1 - (v-w)^2}$$

we obtain

$$(4.24) \quad |1 - H_{14}H_{23}^{-1}| = 4\frac{|v||w|}{|1 - (v-w)^2|}.$$

We use the notation  $\delta := \alpha_1 + \alpha_3$ . From (4.16a,d) we get

$$(4.25) \quad |v||w| \leq \delta(1 - \delta).$$

For  $\gamma_{\nu\mu}$  we have the identity

$$(4.26) \quad 1 - \gamma_{\nu\mu} = \cos(\nu\pi h) \cos(\mu\pi h).$$

The results in (4.16a,d), (4.26) yield:

$$(4.27) \quad \begin{aligned} |1 - (v-w)^2| &= |1 - (v-w)||1 + (v-w)| \\ &\geq \operatorname{Re}(1 - (v-w)) \operatorname{Re}(1 + (v-w)) = 1 - (\operatorname{Re}(v) - \operatorname{Re}(w))^2 \\ &= 1 - (\delta \cos(\nu\pi h) - (1 - \delta) \cos(\mu\pi h))^2 \\ &= 1 - (\delta^2 \cos^2(\nu\pi h) + (1 - \delta)^2 \cos^2(\mu\pi h) - 2\delta(1 - \delta)(1 - \gamma_{\nu\mu})) \\ &\geq 1 - \delta^2 - (1 - \delta)^2 + 2\delta(1 - \delta)(1 - \gamma_{\nu\mu}) = 4\delta(1 - \delta)(1 - \frac{1}{2}\gamma_{\nu\mu}). \end{aligned}$$

Combination of the results in (4.24), (4.25), (4.27) yields

$$(4.28) \quad |1 - H_{14}H_{23}^{-1}| \leq (1 - \frac{1}{2}\gamma_{\nu\mu})^{-1}.$$

We now treat the term  $|1 - H_{23}H_{14}^{-1}|$  (cf. (4.23)). Using

$$H_{23}H_{14}^{-1} = \frac{d_2^{\nu\mu}d_3^{\nu\mu}}{d_1^{\nu\mu}d_4^{\nu\mu}} = \frac{1 - (v-w)^2}{1 - (v+w)^2} = 1 - 4\frac{vw}{1 - (v+w)^2}$$

and the result in (4.25), we obtain

$$(4.29) \quad |1 - H_{23}H_{14}^{-1}| \leq \frac{4\delta(1 - \delta)}{|1 - (v+w)^2|}.$$

We also have (cf. (4.26)):

$$\begin{aligned}
|1 - (v + w)^2| &= |1 - (v + w)||1 + (v + w)| \\
&\geq \operatorname{Re}(1 - (v + w)) \operatorname{Re}(1 + (v + w)) = 1 - (\operatorname{Re}(v) + \operatorname{Re}(w))^2 \\
&= 1 - (\delta \cos(\nu\pi h) + (1 - \delta) \cos(\mu\pi h))^2 \\
&\geq 1 - \delta^2 - (1 - \delta)^2 - 2\delta(1 - \delta)(1 - \gamma_{\nu\mu}) = 2\delta(1 - \delta)\gamma_{\nu\mu}.
\end{aligned}$$

Using the latter result in (4.29) yields

$$(4.30) \quad |1 - H_{23}H_{14}^{-1}| \leq 2\gamma_{\nu\mu}^{-1}.$$

Using the inequalities (4.28), (4.30) in (4.23) proves the estimate in (4.22).  $\square$

As a direct consequence of Lemma 4.5 and Lemma 4.6 we have the following result:

**Theorem 4.7.** *For  $(\nu, \mu) \neq (0, 0)$  with  $1 - N_H \leq \nu, \mu \leq N_H$  the following holds:*

$$\xi^{\nu\mu}(\mathcal{S}_{\bar{A}})/\xi^{\nu\mu}(\mathcal{S}_A) \in \{z \in \mathbb{C} \mid |z - 1| \leq 1 \wedge \operatorname{Re}(z) \geq \frac{1}{2}\}.$$

Clearly, Theorem 4.7 yields a strong robustness result: for *all* constant 5-point difference stars as in (4.3) (e.g. diffusion, anisotropic diffusion, convection-diffusion) the Schur complement approximation  $\mathcal{S}_{\bar{A}}$  based on incomplete Gaussian elimination yields an "optimal" preconditioner of  $\mathcal{S}_A$ .

Numerical experiments (e.g. Experiments 2,3 above) yield the claim that this robustness result even holds for constant 9-point difference stars as in (4.3). However, we have not been able to prove this claim.

*Remark 4.8.* With respect to the sharpness of the result in Theorem 4.7 we note the following. Theorem 4.7 yields that for the spectral condition number of  $\mathcal{S}_{\bar{A}}^{-1}\mathcal{S}_A$  we have the inequality

$$(4.31) \quad \left( \max_{(\nu,\mu) \neq (0,0)} |\xi^{\nu\mu}(\mathcal{S}_{\bar{A}})/\xi^{\nu\mu}(\mathcal{S}_A)| \right) \left( \min_{(\nu,\mu) \neq (0,0)} |\xi^{\nu\mu}(\mathcal{S}_{\bar{A}})/\xi^{\nu\mu}(\mathcal{S}_A)| \right)^{-1} \leq 4.$$

From numerical experiments we see that the bound in (4.31) is about a factor 2 too pessimistic. For example, for the operator with stencil

$$[A] = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}, \text{ on a grid with } h = 1/64,$$

the spectrum  $\{\xi^{\nu\mu}(\mathcal{S}_{\bar{A}})/\xi^{\nu\mu}(\mathcal{S}_A) \mid 1 - N_H \leq \nu, \mu \leq N_H, (\nu, \mu) \neq (0, 0)\}$  is shown in Figure 5. In this case the spectral condition number is 2.24. Note that from this example we see that the estimate in Lemma 4.5 is sharp.

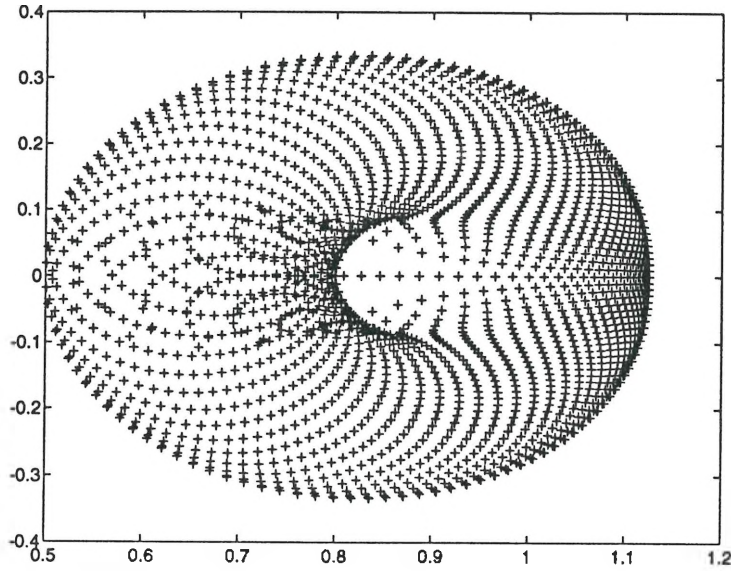


Fig. 5.  $\sigma(\mathcal{S}_{\tilde{A}}\mathcal{S}_{\tilde{A}}^{-1})$  in the complex plane.

## 5 Numerical experiments

In this section we apply a multigrid  $W$ -cycle based on (2.6a) to the following two test problems

$$(5.1) \quad \begin{cases} -\varepsilon\Delta u + a(x, y)u_x + b(x, y)u_y = f & \text{in } \Omega = (0, 1)^2 \\ u = g & \text{on } \partial\Omega \end{cases}$$

$$(5.2) \quad \begin{cases} -e^{\alpha(1-\frac{1}{x})}u_{xx} - u_{yy} = f & \text{in } \Omega = (0, 1)^2 \\ u = g & \text{on } \partial\Omega, \end{cases}$$

with  $\varepsilon > 0$ ,  $\alpha > 0$  problem parameters. The problem in (5.1) is convection-dominated (in a part of the domain) if  $\varepsilon/a \ll 1$  or  $\varepsilon/b \ll 1$  (in a part of the domain). The diffusion problem in (5.2) is strongly anisotropic in a part of the domain. We use standard finite difference discretization on a square mesh  $\Omega_h$ , resulting in a 5-point stencil and a discrete problem  $A_h x_h = b_h$ . For the first order derivatives in (5.1) we use the full upwind discretization, and thus the matrix  $A_h$  is an  $M$ -matrix.

In the experiments below the finest mesh always corresponds to  $h = 1/128$  and the coarsest mesh size is  $h = 1/4$ . For the multigrid method we use the approach as discussed in Remark 3.2. Given the matrix on the finest grid, coarse-grid operators  $(\mathcal{S}_{\tilde{A}_h})$  are computed using a Galerkin approach with matrix dependent prolongations.

We now discuss the approximation used in the block Jacobi method with iteration matrix  $J = I - \begin{bmatrix} A_{11}^{-1} & \emptyset \\ \emptyset & \emptyset \end{bmatrix} A_h$ . In general the matrix  $A_{11}$  has a condition number  $O(1)$  and then, in

principle, any basic iterative method for solving  $A_{11}y = z$  can be used. However, if we have strong alignment then  $\text{cond}(A_{11})$  deteriorates. Our main interest in this paper is on robustness, so we should use a robust solver for the  $A_{11}$  systems. Probably the ILU method will yield a good compromise between robustness and efficiency. In the method we implemented some efficiency has been sacrificed and we used a simple line Jacobi method. One iteration of this method consists of a sweep over the "odd" horizontal lines followed by a sweep over the "odd" vertical lines (these lines together form the pattern of  $\Omega_h \setminus \Omega_H$ ). The result of  $\mu$  iterations of such a line Jacobi method with starting vector 0 applied to  $A_{11}y = z$  is denoted by  $\mathcal{J}^\mu(A_{11}; 0; z)$ .

Below we use the notation  $\Omega_h^c := \Omega_h \setminus \Omega_H$  (i.e. "new" nodes). The two-grid method based on (2.6a) is as follows:

1. a)  $d_{|\Omega_h^c} := (A_h x_h - b_h)|_{\Omega_h^c}$  : compute defect on  $\Omega_h^c$ .  
 b)  $\hat{x}_h := \mathcal{J}^\mu(A_{11}; 0; d_{|\Omega_h^c})$  : line Jacobi for solving  $A_{11}$  system.  
 c)  $x_{h|\Omega_h^c} := x_{h|\Omega_h^c} - \hat{x}_h$  : add correction on  $\Omega_h^c$ .
2. a)  $d_{|\Omega_H} := (A_h x_h - b_h)|_{\Omega_H}$  : compute defect on  $\Omega_H$ .  
 b)  $\mathcal{S}_{\tilde{A}_h} v_H = d_{|\Omega_H}$  : solve coarse grid problem.  
 c)  $x_{h|\Omega_H} := x_{h|\Omega_H} - \omega v_H$  : add correction on  $\Omega_H$ .
3. Repeat 1a,b,c.

This algorithm has the same structure as a standard two-grid algorithm. As in the standard approach, we use two recursive calls in 2b) to obtain a multigrid  $W$ -cycle.

In the experiments below we always take the data such that the exact discrete solution is equal to zero and we take an arbitrary starting vector. As a measure for the error reduction we computed  $r := (\|e_{20}\|_2 / \|e_0\|_2)^{1/20}$ , with  $e_k$  the error in the  $k$ -th iteration. For  $\mu$ , i.e. the number of line Jacobi iterations, we take  $\mu = 3$ . Experiments have shown that this yields sufficiently accurate approximations when solving the  $A_{11}$  systems; often even  $\mu = 2$  is sufficient. Based on the Fourier analysis we take  $\omega = 0.7$  in *all* experiments.

*Experiment 1* (convection-diffusion). We apply the multigrid  $W$ -cycle to the discrete version of (5.1) with  $a(x, y) = \cos(\beta)$ ,  $b(x, y) = \sin(\beta)$ . In Table 3 the resulting  $r$  are given for several values of  $\beta$  and  $\varepsilon$ .

$\varepsilon \quad \beta$	0	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$
$10^{-1}$	0.23	0.23	0.23	0.23	0.23	0.23
$10^{-3}$	0.30	0.40	0.40	0.40	0.40	0.30
$10^{-5}$	0.37	0.35	0.42	0.42	0.35	0.37

Table 3

*Experiment 2* (rotating flow). We define  $\Omega_R := \{(x, y) \mid ((x - \frac{1}{3})^2 + (y - \frac{1}{3})^2) \leq \frac{1}{16}\}$ , and

$$\begin{cases} a(x, y) = \sin(\pi(y - \frac{1}{3})) \cos(\pi(x - \frac{1}{3})) & \text{if } (x, y) \in \Omega_R, \text{ and zero otherwise;} \\ b(x, y) = -\cos(\pi(y - \frac{1}{3})) \sin(\pi(x - \frac{1}{3})) & \text{if } (x, y) \in \Omega_R, \text{ and zero otherwise.} \end{cases}$$

We apply the multigrid  $W$ -cycle to the discrete version of (5.1) with these functions  $a, b$ . The results for  $r$  are given in Table 4. In Table 4 we also show the values of  $r$  corresponding to the two-grid method. These two-grid results are obtained by applying 5 recursive calls (instead of 2) on the coarse grid.

$\varepsilon$	$10^{-1}$	$10^{-2}$	$10^{-3}$	$10^{-4}$	$10^{-5}$
$W$ -cycle	0.23	0.25	0.32	0.34	0.34
$TG$	0.23	0.30	0.36	0.33	0.33

Table 4

*Experiment 3.* We apply the multigrid  $W$ -cycle to the discrete version of (5.2). The resulting values for  $r$  are given in Table 5. Again, as in Experiment 2 we give the values of  $r$  corresponding to the two-grid method.

$\alpha$	1	5
$W$ -cycle	0.33	0.37
$TG$	0.26	0.28

Table 5

The Fourier analysis in Section 4, which yields a strong robustness result, applies to a two-grid method, with exact  $A_{11}$ -solver, for solving a boundary value problem with periodic boundary conditions and constant coefficients. In the experiments above we observe that the multigrid  $W$ -cycle, with inexact  $A_{11}$ -solver, applied to a boundary value problem with Dirichlet boundary conditions and variable coefficients is very robust w.r.t. variation in problem parameters, too. Finally we note that if the  $A_{11}$  system is solved approximately using an ILU method then our two-grid method is a combination of two incomplete Gaussian elimination processes: ILU for solving the  $A_{11}$  system and incomplete Gaussian elimination as described in Section 3 for constructing a sparse coarse grid matrix.

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