CONVERGENCE ANALYSIS OF A MULTIGRID METHOD FOR A CONVECTION-DOMINATED MODEL PROBLEM

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Abstract. The paper presents a convergence analysis of a multigrid solver for a system of linear algebraic equations resulting from the disretization of a convection-diffusion problem using a finite element method. We consider piecewise linear finite elements in combination with a streamline diffusion stabilization . We analyze a multigrid method that is based on canonical inter-grid transfer operators, a "direct discretization" approach for the coarse-grid operators and a smoother of line-Jacobi type. A robust (diffusion and h-independent) bound for the contraction number of the two-grid method and the multigrid W-cycle are proved for a special class of convection-diffusion problems, namely with Neumann conditions on the outflow boundary, Dirichlet conditions on the rest of the boundary and a flow direction that is constant and aligned with gridlines. Our convergence analysis is based on modified smoothing and approximation properties. The arithmetic complexity of one multigrid iteration is optimal up to a logarithmic term.

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1. Introduction. Concerning the theoretical analysis of multigrid methods different fields of application have to be distinguished. For linear selfadjoint elliptic boundary value problems the convergence theory is well developed (cf. [5, 9, 35, 36]). In other areas the state of the art is (far) less advanced. For example, for convection-diffusion problems the development of a multigrid convergence analysis is still in its infancy. In this paper we present a convergence analysis of a multilevel method for a special class of 2D convection-diffusion problems.

An interesting class of problems for the analysis of multigrid convergence is given by

$$\begin{cases} -\varepsilon\Delta u + b\cdot\nabla u &= f \quad \text{in} \quad \Omega = (0,1)^2 \\ u &= g \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

with $\varepsilon > 0$ and $b = (\cos\phi, \sin\phi)$, $\phi \in [0, 2\pi)$. The application of a discretization method results in a large sparse linear system which depends on a mesh size parameter h_k. For a discussion of discretization methods for this problem we refer to [28, 1, 2] and the references therein. Note that in the discrete problem we have three interesting parameters: h_k (mesh size), ε (convection-diffusion ratio) and ϕ (flow direction). For the approximate solution of this type of problems *robust* multigrid methods have been developed which are efficient solvers for a large range of relevant values for the parameters h_k, ε, ϕ. To obtain good robustness properties the components in the multigrid method have to be chosen in a special way because in general the "standard" multigrid approach used for a diffusion problem does not yield satisfactory results when applied to a convection-dominated problem. To improve robustness several modifications have been proposed in the literature, such as "robust" smoothers, matrix-dependent prolongations and restrictions and semicoarsening techniques. For an explanation of these methods we refer to [9, 33, 4, 13, 14, 18, 19, 37]. These modifications are based on heuristic arguments and empirical studies and rigorous convergence analysis proving robustness is still missing for most of these modifications.

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Related to the theoretical analysis of multigrid applied to convection-diffusion problems we note the following. In the literature one finds convergence analyses of multigrid methods for nonsymmetric elliptic boundary value problems which are based on perturbation arguments [6, 9, 17, 32]. If these analyses are applied to the problem in (1.1) the constants in the estimates depend on ε and the results are not satisfactory for the case $\varepsilon \ll 1$. In [11, 25] multigrid convergence for a 1D convection-diffusion problem is analyzed. These analyses, however, are restricted to the 1D case. In [23, 26] convection-diffusion equations as in (1.1) with *periodic* boundary conditions are considered. A Fourier analysis is applied to analyze the convergence of two- or multigrid methods. In [23] the problem (1.1) with periodic boundary conditions and $\phi = 0$ is studied. For the discretization the streamline diffusion finite element method on a uniform grid is used. A bound for the contraction number of a multigrid V-cycle with point Jacobi smoother is proved which is uniform in ε and h_k provided $\varepsilon \sim h_k$ is satisfied. Note that due to the fact that a point Jacobi smoother is used one can not expect robustness of this method for $h_k \gg \varepsilon \downarrow 0$. In [26] a two-grid method for solving a first order upwinding finite difference discretization of the problem (1.1) with periodic boundary conditions is analyzed and it is proved that the two-grid contraction number is bounded by a constant smaller than one which does not depend on any of the parameters ε, h_k, ϕ. In [3] the application of the hierarchical basis multigrid method to a finite element discretization of problems as in (1.1) is studied. The analysis there shows how the convergence rate depends on ε and on the flow direction, but the estimates are not uniform with respect to the mesh size parameter h_k. In [27] the convergence of a multigrid method applied to a standard finite difference discretization of the problem (1.1) with $\phi = 0$ is analyzed. This method is based on *semicoarsening* and a matrix-dependent prolongation and restriction. It is proved that the multigrid W-cycle has a contraction number smaller than one independent of h_k and ε. The analysis in [27] is based on linear algebra arguments only and is not applicable in a finite element setting. Moreover, the case with *standard coarsening*, which will be treated in the present paper, is not covered by the analysis in [27].

In the present paper we consider the convection-diffusion problem

$$\begin{aligned} -\varepsilon\Delta u + u_x &= f \quad \text{in} \quad \Omega := (0,1)^2 \\ \frac{\partial u}{\partial x} &= 0 \quad \text{on} \quad \Gamma_E := \{\, (x,y) \in \overline{\Omega} \mid x = 1 \,\} \\ u &= 0 \quad \text{on} \quad \partial\Omega \setminus \Gamma_E \end{aligned} \tag{1.2}$$

In this problem we have Neumann boundary conditions on the outflow boundary and Dirichlet boundary conditions on the remaining part of the boundary. Hence, the solution may have parabolic layers but exponential boundary layers at the outflow boundary do not occur. For this case an a priori regularity estimate of the form $\|u\|_{H^2} \leq c\,\varepsilon^{-1}\|f\|_{L^2}$ holds, whereas for the case with an exponential boundary layer one only has $\|u\|_{H^2} \leq c\,\varepsilon^{-\frac{3}{2}}\|f\|_{L^2}$.

Due to the Dirichlet boundary conditions a Fourier analysis is not applicable.

For the discretization we use conforming linear finite elements. As far as we know there is no multigrid convergence analysis for convection-dominated problems known in the literature that can be applied in a finite element setting with nonperiodic boundary conditions and yields robustness for the parameter range $0 \leq \varepsilon \leq h_k \leq 1$. In this paper we present an analysis which partly fills this gap. We use the streamline diffusion finite element method (SDFEM). The SDFEM ensures a higher order of accuracy than a first order upwind finite difference method (cf. [28, 38]). In SDFEM a mesh-dependent anisotropic diffusion, which acts only in streamline direction, is added to the discrete problem. Such anisotropy is important for the high order of convergence of this method and also plays a crucial role in our convergence analysis of the multigrid method. In this paper we only treat the case of a uniform triangulation which is taken such that the streamlines are aligned with gridlines. Whether our analysis can be generalized to the situation of an unstructured triangulation is an open question.

We briefly discuss the different components of the multigrid solver.
For the *prolongation and restriction* we use the canonical inter-grid transfer operators that are induced by the nesting of the finite element spaces.
The hierarchy of *coarse grid discretization operators* is constructed by applying the SDFEM on each grid level. Note that due to the level-dependent stabilization term we have level-dependent bilinear forms and the Galerkin property $A_{k-1} = r_k A_k p_k$ does not hold.
Related to the *smoother* we note the following. First we emphasize that due to a certain crosswind smearing effect in the finite element discretization the x-line Jacobi or Gauss-Seidel methods do *not* yield robust smoothers (i.e., they do not result in a direct solver in the limit case $\varepsilon = 0$, cf. [9]). This is explained in more detail in remark 6.1 in section 6. In the present paper we use a smoother of x-line-Jacobi type. These components are combined in a standard W-cycle algorithm.

The convergence analysis of the multigrid method is based on the framework of the smoothing- and approximation property as introduced by Hackbusch [9, 10]. However, the splitting of the two-grid iteration matrix that we use in our analysis is not the standard one. This splitting is given in (6.8). It turns out to be essential to keep the preconditioner corresponding to the smoother (W_k in (6.8)) as part of the approximation property. Moreover, in the analysis we have to distinguish between residuals which after presmoothing are zero close to the inflow boundary and those that are nonzero. This is done by using a cut-off operator (Φ_k in (6.8)). The main reason for this distinction is the following. As is usually done in the analysis of the approximation property we use finite element error bounds combined with regularity results. In the derivation of a L^2 bound for the finite element discretization error we use a duality argument. However, the formal dual problem has poor regularity properties, since the inflow boundary of the original problem is the outflow boundary of the dual problem. Thus Dirichlet outflow boundary conditions would appear and we obtain poor estimates due to the poor regularity. To avoid this, we consider a dual problem with Neumann outflow and Dirichlet inflow conditions. To be able to deal with the inconsistency caused by these "wrong" boundary conditions we assume the input residuals for the coarse grid correction to be zero near the inflow boundary. Numerical experiments from section 11 related to the approximation property show that such analysis is sharp.

In our estimates there are terms that grow logarithmically if the mesh size parameter h_k tends to zero. To compensate this the number of presmoothings has to be taken level dependent. This then results in a two-grid method with a contraction number $\|T_k\|_{A^TA} \leq c < 1$ and a complexity $\mathcal{O}(N_k(\ln N_k)^4)$, with $N_k = h_k^{-2}$. Using standard arguments we obtain a similar convergence result for the multigrid W-cycle.

The remainder of this paper is organized as follows. In section 2 we give the weak formulation of the problem (1.2) and describe the SDFEM. In section 3 some useful properties of the stiffnes matrix are derived. In section 4 we prove some a priori estimates for the continuous and the discrete solution. In section 5 we derive quantitative results concerning the upstream influence of a righthand side on the solution. These results are needed in the proof of the modified approximation property. *Section 6 contains the main results of this paper*. In this section we describe the multigrid algorithm and present the convergence analysis. In the sections 7–10 we give proofs of some important results that are used in the analysis in section 6. In section 11 we present results of a few numerical experiments.

2. The continuous problem and its discretization. For the weak formulation of the problem (1.2) we use the $\mathrm{L}^2(\Omega)$ scalar product which is denoted by $(\cdot,\cdot)$. For the corresponding norm we use the notation $\|\cdot\|$. With the Sobolev space $\mathbf{V} := \{\, v \in \mathrm{H}^1(\Omega) \mid v = 0 \ \text{ on } \ \partial\Omega \setminus \Gamma_E \,\}$ the weak formulation is as follows: find $u \in \mathbf{V}$ such that

$$a(u,v) := \varepsilon(u_x, v_x) + \varepsilon(u_y, v_y) + (u_x, v) = (f, v) \ \text{ for all } \ v \in \mathbf{V} \qquad (2.1)$$

¿From the Lax-Milgram lemma it follows that a unique solution of this problem exists. For the discretization we use linear finite elements on a uniform triangulation. For this we use a mesh size $h_k := 2^{-k}$ and grid points $x_{i,j} = (ih_k, jh_k),\ 0 \le i,j \le h_k^{-1}$. A uniform triangulation is obtained by inserting diagonals that are oriented from south-west to north-east. Let $\mathbb{V}_k \subset \mathbf{V}$ be the space of continuous functions that are piecewise linear on this triangulation and have zero values on $\partial\Omega \setminus \Gamma_E$. For the discretization of (2.1) we consider the streamline-diffusion finite element method: find $u_k \in \mathbb{V}_k$ satisfying

$$(\varepsilon + \delta_k h_k)((u_k)_x, v_x) + \varepsilon((u_k)_y, v_y) + ((u_k)_x, v) = (f, v + \delta_k h_k v_x) \ \text{ for all } \ v \in \mathbb{V}_k \quad (2.2)$$

with

$$\delta_k = \begin{cases} \bar{\delta} & \text{if } \ \frac{h_k}{2\,\varepsilon} \ge 1 \\ 0 & \text{otherwise} \end{cases} \qquad (2.3)$$

The stabilization parameter $\bar{\delta}$ is a given constant of order 1. For an analysis of the streamline diffusion finite element method we refer to [28, 15]. In this paper we assume

$$\bar{\delta} \in [\frac{1}{3}, 1] \ . \qquad (2.4)$$

The value $\frac{1}{3}$ for the lower bound is important for our analysis. The choice of 1 for the upper bound is made for technical reasons and this value is rather arbitrary.
The finite element formulation (2.2) gives rise to the (stabilized) bilinear form

$$a_k(u,v) := (\varepsilon + \delta_k h_k)(u_x, v_x) + \varepsilon(u_y, v_y) + (u_x, v), \quad u, v \in \mathbf{V} \qquad (2.5)$$

Note the following relation for the bilinear form $a_k(\cdot,\cdot)$:

$$a_k(v,v) = \varepsilon \|v_y\|^2 + (\varepsilon + \delta_k h_k)\|v_x\|^2 + \frac{1}{2}\int_{\Gamma_E} v^2 \, dy \quad \text{for } \ v \in \mathbf{V}. \qquad (2.6)$$

The main topic of this paper is a convergence analysis of a multigrid solver for the algebraic system of equations that corresponds to (2.2). In this convergence analysis the particular form of the righthand side in (2.2), which is essential for consistency in the streamline diffusion finite element method, does not play a role. Therefore for an arbitrary $f \in \mathrm{L}^2(\Omega)$ we will consider the problems:

$$u \in \mathbf{V} \quad \text{such that :} \quad a_k(u,v) = (f,v) \quad \text{for all} \quad v \in \mathbf{V} \tag{2.7}$$

$$u_k \in \mathbb{V}_k \quad \text{such that :} \quad a_k(u_k,v_k) = (f,v_k) \quad \text{for all} \quad v_k \in \mathbb{V}_k \tag{2.8}$$

Note that u and u_k depend on the stabilization term in the bilinear form and that these solutions differ from those in (2.1) and (2.2).

3. Representation of the stiffness matrix. We now derive a representation of the stiffness matrix corresponding to the bilinear form $a_k(\cdot,\cdot)$ that will be used in the analysis below. The standard nodal basis in $\mathbb{V}_k$ is denoted by $\{\phi_\ell\}_{1\le\ell\le N_k}$ with N_k the dimension of the finite element space, $N_k := h_k^{-1}(h_k^{-1}-1)$. Define the isomorphism:

$$P_k:\ X_k := \mathbb{R}^{N_k} \to \mathbb{V}_k, \quad P_k x = \sum_{i=1}^{N_k} x_i \phi_i .$$

On X_k we use a scaled Euclidean scalar product: $\langle x,y\rangle_k = h_k^2 \sum_{i=1}^{N_k} x_i y_i$ and corresponding norm denoted by $\|\cdot\|$ (note that this notation is also used to denote the $\mathrm{L}^2(\Omega)$ norm). The adjoint $P_k^*:\ \mathbb{V}_k \to X_k$ satisfies $(P_k x, v) = \langle x, P_k^* v\rangle_k$ for all $x \in X_k,\ v \in \mathbb{V}_k$. The following norm equivalence holds

$$C^{-1}\|x\| \le \|P_k x\| \le C\|x\| \quad \text{for all} \ x \in \mathrm{X}_k, \tag{3.1}$$

with a constant C independent of k. The stiffness matrix A_k on level k is defined by

$$\langle A_k x, y\rangle_k = a_k(P_k x, P_k y) \quad \text{for all} \ x,y \in \mathrm{X}_k. \tag{3.2}$$

In an interior grid point the discrete problem has the stencil

$$\frac{1}{h_k^2}\begin{bmatrix} 0 & -\varepsilon & 0 \\ -\varepsilon_k & 2(\varepsilon_k+\varepsilon) & -\varepsilon_k \\ 0 & -\varepsilon & 0 \end{bmatrix} + \frac{1}{h_k}\begin{bmatrix} 0 & -\frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & 0 \end{bmatrix}, \qquad \varepsilon_k := \varepsilon + \delta_k h_k \ . \tag{3.3}$$

For a matrix representation of the discrete operator we first introduce some notation and auxiliary matrices. Let $n_k := h_k^{-1}$ and

$$\hat{D}_x := \frac{1}{h_k}\mathrm{tridiag}(-1,1,0) \in \mathbb{R}^{n_k\times n_k} \ ,$$

$$\hat{A}_x := \hat{D}_x^T \hat{D}_x = \frac{1}{h_k^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix} \in \mathbb{R}^{n_k\times n_k} \ ,$$

$$\hat{A}_y := \frac{1}{h_k^2}\mathrm{tridiag}(-1,2,-1) \in \mathbb{R}^{(n_k-1)\times(n_k-1)} \ ,$$

$$\hat{J} := \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \frac{1}{2} \end{pmatrix} \in \mathbb{R}^{n_k\times n_k}, \quad \hat{T} := \mathrm{tridiag}(0,0,1) \in \mathbb{R}^{n_k\times n_k} \ .$$

Furthermore, let I_m be the $m \times m$ identity matrix. We finally introduce the following $N_k \times N_k$ matrices

$$D_x := I_{n_k-1} \otimes \hat{D}_x \ , \quad A_x := I_{n_k-1} \otimes \hat{A}_x = D_x^T D_x \ , \quad A_y := \hat{A}_y \otimes \hat{J}$$

and the $N_k \times N_k$ blocktridiagonal matrix

$$B := \text{blocktridiag}(I_{n_k}, 4I_{n_k}, \hat{T}) \ .$$

Using all this notation we consider the following representation for the stiffness matrix A_k in (3.2):

$$A_k = (\varepsilon + (\delta_k - \frac{1}{3})h_k)A_x + \varepsilon A_y + \frac{1}{6}BD_x \tag{3.4}$$

The latter decomposition can be written in stencil notation as

$$\frac{\bar{\varepsilon}_k}{h_k^2}\begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\varepsilon}{h_k^2}\begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{bmatrix} + \frac{1}{6h_k}\begin{bmatrix} 0 & -1 & 1 \\ -4 & 4 & 0 \\ -1 & 1 & 0 \end{bmatrix} \tag{3.5}$$

with $\bar{\varepsilon}_k = \varepsilon + (\delta_k - \frac{1}{3})h_k > 0$.

Some properties of the matrices used in the decomposition (3.4) are collected in the following lemma.

For $B, C \in \mathbb{R}^{n\times n}$ we write $B \geq C$ iff $x^T Bx \geq x^T Cx$ for all $x \in \mathbb{R}^n$.

LEMMA 3.1. *The following inequalities hold*

$$A_x D_x^{-1} \geq 0 \tag{3.6}$$

$$A_y D_x^{-1} \geq 0 \tag{3.7}$$

$$B \geq 2I \tag{3.8}$$

$$A_k D_x^{-1} \geq \frac{1}{3} I \tag{3.9}$$

$$\|D_x A_k^{-1}\| \leq 3 \tag{3.10}$$

Proof. To check (3.6) observe $A_x D_x^{-1} = D_x^T D_x D_x^{-1} = D_x^T$. Now note that $D_x^T + D_x$ is symmetric positive definite.

To prove (3.7) it suffices to show that $D_x^T A_y \geq 0$ holds. We have

$$K := D_x^T A_y = (I_{n_k-1} \otimes \hat{D}_x^T)(\hat{A}_y \otimes \hat{J}) = \hat{A}_y \otimes \tilde{D}_x^T \ ,$$

with the matrix

$$\tilde{D}_x^T = \frac{1}{h_k}\begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -\frac{1}{2} \\ & & & \frac{1}{2} \end{pmatrix}$$

Hence in the matrix $K + K^T = \hat{A}_y \otimes (\tilde{D}_x^T + \tilde{D}_x)$ both factors $\hat{A}_y$ and $\tilde{D}_x^T + \tilde{D}_x$ are symmetric positive definite. From this the result follows.

To prove (3.8) we define $R := B - 4I$ and note that $\|R\|^2 \leq \|R\|_\infty \|R\|_1 \leq 4$. Using this we get

$$\langle Bx, x\rangle_k = 4\|x\|^2 + \langle Rx, x\rangle_k \geq 4\|x\|^2 - \|R\|\|x\|^2 \geq 2\|x\|^2$$

which proves the desired result. Inequality (3.9) follows immediately from the representation of A_k in (3.4) and inequalities (3.6)–(3.8). ¿From the result in (3.9) it follows that $D_x^T A_k \geq \frac{1}{3} D_x^T D_x$. This implies $\|D_x x\|^2 \leq 3\langle A_k x, D_x x\rangle_k \leq 3\|A_k x\|\|D_x x\|$ for all $x \in \mathrm{X}_k$ and thus estimate (3.10) is also proved. □

4. A priori estimates. In this paper we study the convergence of a multigrid method for solving the system of equations

$$A_k x_k = b \ , \tag{4.1}$$

with A_k the stiffness matrix from the previous section. As already noted in the introduction our analysis relies on smoothing and approximation properties. For establishing a suitable approximation property we will use regularity results and a priori estimates for solutions of the continuous and the discrete problems. Such results are collected in this section. In the remainder of the paper we restrict ourselves to the convection-dominated case:

ASSUMPTION 4.1. *We only consider values of k and ε such that $\varepsilon \leq \frac{1}{2}\, h_k$.*

If instead of the factor $\frac{1}{2}$ in this assumption we take another constant C, our analysis can still be applied but some technical modifications are needed (to distinguish between $\delta_k = \bar{\delta}$ and $\delta_k = 0$) which make the presentation less transparent.
We consider this convection-dominated case to be the most interesting one. Many results that will be presented also hold for the case of an arbitrary positive ε but the proofs for the diffusion-dominated case often differ from those for the convection-dominated case. In view of the presentation we decided to treat only the convection-dominated case. Note that then

$$\delta_k = \bar{\delta} \in [\frac{1}{3}, 1] \ \text{ and } \ \frac{1}{3} h_k \leq \varepsilon_k = \varepsilon + \bar{\delta} h_k \leq \frac{3}{2} h_k \ . \tag{4.2}$$

For the inflow boundary we use the notation $\Gamma_W := \{\, (x,y) \in \overline{\Omega} \mid x = 0 \,\}$. For the continuous solution u the following a-priori estimates hold

THEOREM 4.1. *For $f \in L_2(\Omega)$ let u be the solution of (2.7). There is a constant c independent of k and ε such that:*

$$\|u\| + \|u_x\| \leq c\|f\|, \tag{4.3}$$

$$\sqrt{\varepsilon}\|u_y\| \leq c\|f\|, \tag{4.4}$$

$$h_k\|u_{xx}\| + \sqrt{\varepsilon h_k}\|u_{xy}\| + \varepsilon\|u_{yy}\| \leq c\|f\|, \tag{4.5}$$

$$\int_{\Gamma_E} u^2\, dy + h_k \int_{\Gamma_W} u_x^2\, dy + \varepsilon \int_{\Gamma_E} u_y^2\, dy \leq c\|f\|^2. \tag{4.6}$$

Proof. Since $f \in L_2(\Omega)$, the regularity theory from [8] ensures that the solution u of (2.7) belongs to $\mathrm{H}^2(\Omega)$. Hence we can consider the strong formulation of (2.7)

$$-\varepsilon u_{yy} - \varepsilon_k u_{xx} + u_x = f \tag{4.7}$$

with boundary conditions as in (1.2). Now we multiply (4.7) with u_x and integrate by parts. Taking boundary conditions into account, we get the following terms:

$$\begin{aligned} -\varepsilon(u_{yy}, u_x) &= \frac{\varepsilon}{2}((u_y^2)_x, 1) = \frac{\varepsilon}{2}\int_{\Gamma_E} u_y^2\, dy, \\ -\varepsilon_k(u_{xx}, u_x) &= -\frac{\varepsilon_k}{2}((u_x^2)_x, 1) = \frac{\varepsilon_k}{2}\int_{\Gamma_W} u_x^2\, dy \geq c\, h_k \int_{\Gamma_W} u_x^2\, dy, \quad \text{(we use (4.2))} \\ (u_x, u_x) &= \|u_x\|^2 \geq \|u\|^2, \\ (f, u_x) &\leq \frac{1}{2}\|f\|^2 + \frac{1}{2}\|u_x\|^2. \end{aligned}$$

From these relations the results (4.3) and (4.6), except the bound for $\int_{\Gamma_E} u^2\,dy$, easily follow. Next we multiply (4.7) with u and integrate by parts to obtain

$$\varepsilon\|u_y\|^2+\varepsilon_k\|u_x\|^2+\frac{1}{2}\int_{\Gamma_E} u^2\,dy=(f,u)\le\|f\|\|u\|\le c\,\|f\|^2. \quad \text{(we use (4.3))}$$

Estimate (4.4) and the remainder of (4.6) now follow. To prove (4.5) we introduce $F=f-u_x$. Due to (4.3) we have $\|F\|\le c\,\|f\|$. Moreover $-\varepsilon u_{yy}-\varepsilon_k u_{xx}=F$ holds. If we square both sides of this equality and integrate over Ω we obtain

$$\varepsilon^2\|u_{yy}\|^2+2\varepsilon\varepsilon_k(u_{yy},u_{xx})+\varepsilon_k^2\|u_{xx}\|^2=\|F\|^2\le c\,\|f\|^2. \tag{4.8}$$

Further note that for any sufficiently smooth function v, satisfying the boundary conditions in (1.2), the relations

$$v_{xx}(x,0)=v_{xx}(x,1)=0,\ x\in(0,1),\quad v_y(0,y)=v_{xy}(1,y)=0,\ y\in(0,1),$$

hold, and thus

$$(v_{yy},v_{xx})=-(v_y,v_{xxy})=(v_{xy},v_{xy}).$$

Using a standard density argument we conclude that for the solution $u\in \mathrm{H}^2(\Omega)$ of (2.7) the relation $(u_{yy},u_{xx})=(u_{xy},u_{xy})$ holds. Now (4.8) gives

$$\varepsilon^2\|u_{yy}\|^2+2\varepsilon\varepsilon_k\|u_{xy}\|^2+\varepsilon_k^2\|u_{xx}\|^2\le c\,\|f\|^2.$$

In combination with (4.2) this yields (4.5). □

The next lemma states that the x-derivative of the *discrete* solution is also uniformly bounded if the righthand side is from $\mathbb{V}_k$.

LEMMA 4.2. *For $f_k\in\mathbb{V}_k$ let $u_k\in\mathbb{V}_k$ be a solution to (2.8), then*

$$\|(u_k)_x\|\le c\,\|f_k\|. \tag{4.9}$$

Proof. The result in (4.9) follows from the estimate (3.10) in lemma 3.1. To show this we need some technical considerations.

First we show how the size of the x-derivative of a finite element function $v\in\mathbb{V}_k$ can be determined from its corresponding coefficent vector $P_k^{-1}v\in X_k$. Let $\mathcal{I}$ be the index set $\{(i,j)\mid 0\le i\le n_k-1,\quad 1\le j\le n_k-1\}$. For $(i,j)\in\mathcal{I}$ let $T^l_{(i,j)}$ and $T^u_{(i,j)}$ be the two triangles in the triangulation which have the line between the grid points $x_{i,j}$ and $x_{i+1,j}$ as a common edge. Let $v\in\mathbb{V}_k$ be given. For $1\le j\le n_k-1$ we introduce the vector $\mathbf{v}_j=(v(x_{1,j}),\ldots,v(x_{n_k,j}))^T$. We then obtain

$$\begin{aligned}\|v_x\|^2&=\sum_{(i,j)\in\mathcal{I}}\Big(\int_{T^l_{(i,j)}} v_x^2\,dxdy+\int_{T^u_{(i,j)}} v_x^2\,dxdy\Big)\\&=\sum_{(i,j)\in\mathcal{I}}\Big(\frac{v(x_{i+1,j})-v(x_{i,j})}{h_k}\Big)^2 h_k^2=h_k^2\sum_{1\le j\le n_k-1}\big(D_x\mathbf{v}_j\big)^T(D_x\mathbf{v}_j)\\&=h_k^2\big(D_xP_k^{-1}v\big)^T\big(D_xP_k^{-1}v\big)=\|D_xP_k^{-1}v\|^2.\end{aligned}$$

Therefore

$$\|v_x\| = \|D_x P_k^{-1} v\| \quad \text{for any} \;\; v \in \mathbb{V}_k. \tag{4.10}$$

For the discrete solution of (2.8) with $f = f_k$ we have the representation $u_k = P_k A_k^{-1} P_k^* f_k$. Now from (3.10) and (4.10) it follows that

$$\|(u_k)_x\| = \|D_x A_k^{-1} P_k^* f_k\| \leq 3\,\|P_k^* f_k\| \leq c\,\|f_k\|$$

with a constant c independent of k and ε. ☐

The next lemma gives some bounds on the difference between discrete and continuous solutions

LEMMA 4.3. *Define the error $e_k = u - u_k$, where u and u_k are solutions of the problems (2.7) and (2.8) with righthand side $f = f_k \in \mathbb{V}_k$. Then the following estimates hold*

$$\|(e_k)_x\| \leq c\|f_k\| \tag{4.11}$$

$$\varepsilon\|(e_k)_y\|^2 + \frac{1}{2}\int_{\Gamma_E} e_k^2\, dy \leq \; c\frac{h_k^2}{\varepsilon}\|f_k\|^2. \tag{4.12}$$

Proof. Estimate (4.11) directly follows from (4.3) and (4.9) by a triangle inequality. The proof of (4.12) is based on standard arguments: the Galerkin orthogonality, approximation properties of $\mathbb{V}_k$ and a priori estimates from (4.5). Indeed

$$\begin{aligned}
&\varepsilon\|(e_k)_y\|^2 + (\varepsilon + \bar{\delta} h_k)\|(e_k)_x\|^2 + \frac{1}{2}\int_{\Gamma_E} e_k^2\, dy = a_k(e_k, e_k) = \inf_{v_k \in \mathbb{V}_k} a_k(e_k, u - v_k)\\
&\leq \inf_{v_k \in \mathbb{V}_k} \left(\varepsilon\|(e_k)_y\|\|(u - v_k)_y\| + (\varepsilon + \bar{\delta} h_k)\|(e_k)_x\|\|(u - v_k)_x\| + \|(e_k)_x\|\|u - v_k\|\right)\\
&\leq c\left(\varepsilon\, h_k\|(e_k)_y\|\|u\|_{H^2} + h_k^2\|(e_k)_x\|\|u\|_{H^2}\right)\\
&\leq c\left(h_k\|(e_k)_y\|\|f_k\| + \frac{h_k^2}{\varepsilon}\|f_k\|^2\right) \leq \frac{\varepsilon}{2}\|(e_k)_y\|^2 + c\frac{h_k^2}{\varepsilon}\|f_k\|^2.
\end{aligned}$$

The estimate (4.12) follows. ☐

5. Upstream influence of the streamline diffusion method. Consider the continuous problem (2.7). The goal of this section is to estimate the upstream influence of the righthand side function f on the solution u. The same will be done for the corresponding discrete problem. In the literature results of such type are known for the problem with Dirichlet boundary conditions and typically formulated in the form of estimates on the (discrete) Greens function (see, e.g., [31, 20, 16]). A typical result is that the value of the solution at a point x is essentially determined by the values of the righthand side in a "small" strip that contains x. This strip has a crosswind width of size $O(\varepsilon^*|\ln h|)$, where $\varepsilon^* = \max\{\varepsilon, h^{\frac{3}{2}}\}$, and in the streamline direction it ranges from the inflow boundary to a $\mathcal{O}(h|\ln h|)$ upstream distance from x. In our analysis we need precise quantitative results for the case with Neumann outflow boundary conditions. In the literature we did not find such results. Hence we present proofs of the results that are needed for the multigrid convergence analysis further on. Our analysis uses the known technique of cut-off functions (e.g., [7, 16]), it avoids the use of an adjoint problem and is based on the following lemma.

LEMMA 5.1. *For $\varepsilon_k = \varepsilon + \bar{\delta} h_k$ assume a function $\phi \in H^1_\infty(0,1)$, such that $0 \leq -\varepsilon_k \phi_x \leq \phi$. Denote by $\|\cdot\|_\phi$ a semi-norm induced by the scalar product $(\phi\cdot, \cdot)$.*

Then the solution u of (2.7) satisfies

$$\|u_x\|_\phi \le 2\|f\|_\phi \tag{5.1}$$

$$\varepsilon_k\ \phi(0)\int_{\Gamma_W} u_x^2\,dy \le \|f\|_\phi^2 \tag{5.2}$$

$$\frac{1}{4}\|u\|^2_{-\phi_x} + \varepsilon\|u_y\|^2_\phi \le (\phi\, f, u). \tag{5.3}$$

Proof. We consider the strong formulation (4.7) and multiply it with ϕu_x and integrate by parts. We then get the following terms

$$\begin{aligned}
-\varepsilon(u_{yy},\phi u_x) &= \frac{\varepsilon}{2}\|u_y\|^2_{-\phi_x} + \frac{\varepsilon}{2}\phi(1)\int_{\Gamma_E} u_y^2\,dy \ge 0\ ,\\
-\varepsilon_k(u_{xx},\phi u_x) &= -\frac{\varepsilon_k}{2}\|u_x\|^2_{-\phi_x} + \frac{\varepsilon_k}{2}\phi(0)\int_{\Gamma_W} u_x^2\,dy \ge -\frac{1}{2}\|u_x\|^2_\phi + \frac{\varepsilon_k}{2}\phi(0)\int_{\Gamma_W} u_x^2\,dy\ ,\\
(u_x,\phi u_x) &= \|u_x\|^2_\phi\ ,\\
(f,\phi u_x) &\le \|f\|_\phi\|u_x\|_\phi \le \|f\|^2_\phi + \frac{1}{4}\|u_x\|^2_\phi.
\end{aligned}$$

Now (5.1) and (5.2) immediately follow. To obtain the estimate (5.3) we multiply (4.7) with $\phi\, u$ and integrate by parts. We get the following terms:

$$\begin{aligned}
-\varepsilon(u_{yy},\phi\, u) &= \varepsilon\|u_y\|^2_\phi,\\
-\varepsilon_k(u_{xx},\phi\, u) &= \varepsilon_k\|u_x\|^2_\phi + \varepsilon_k(u_x,\phi_x\, u)\\
&\ge \varepsilon_k\|u_x\|^2_\phi - \varepsilon_k^2\|u_x\|^2_{-\phi_x} - \frac{1}{4}\|u\|^2_{-\phi_x} \ge -\frac{1}{4}\|u\|^2_{-\phi_x}.\\
(u_x,\phi\, u) &= \frac{1}{2}\|u\|^2_{-\phi_x} + \frac{\phi(1)}{2}\int_{\Gamma_E} u^2\,dy.
\end{aligned}$$

Thus (5.3) follows. ☐

For arbitrary $\xi \in [0,1]$ consider the function

$$\phi_\xi(x) = \begin{cases} 1 & \text{for } x \in [0,\xi],\\ \exp\left(-\frac{x-\xi}{\varepsilon_k}\right) & \text{for } x \in (\xi,1].\end{cases}$$

For any ξ the function $\phi_\xi(x)$ satisfies the assumptions of lemma 5.1. For $0 < \xi < \eta < 1$ we define the domains

$$\Omega_\xi = \{(x,y)\in\Omega :\ x < \xi\}\,,\quad \Omega_\eta = \{(x,y)\in\Omega :\ x > \eta\}\ .$$

Direct application of lemma 5.1 with $\phi = \phi_\xi$ gives the following corollary.

COROLLARY 5.2. *Consider $f \in L_2(\Omega)$ such that* $\mathrm{supp}(f) \in \Omega_\eta$ *and let u be the corresponding solution of problem (2.7). Assume $\eta - \xi \ge 2\,\varepsilon_k\, p\,|\ln h_k|,\ p > 0$. Then we have*

$$\|u_x\|_{L_2(\Omega_\xi)} \le h_k^p\|f\|, \tag{5.4}$$

$$\varepsilon_k\int_{\Gamma_W} u_x^2\,dy \le h_k^{2p}\|f\|^2, \tag{5.5}$$

$$\sqrt{\varepsilon}\|u_y\|_{L_2(\Omega_\xi)} \le \sqrt{\varepsilon_k}\, h_k^p\|f\|. \tag{5.6}$$

Proof. The estimate $\|f\|_\phi^2 = (\phi f, f)_{\Omega_\eta} \le \phi(\eta)\|f\|^2_{\Omega_\eta} = h_k^{2p}\|f\|^2$ and (5.1), (5.2) imply the results (5.4) and (5.5). We also have

$$(\phi f, u) = (\phi f, u)_{\Omega_\eta} \le \varepsilon_k \|f\|_\phi^2 + \frac{1}{4\varepsilon_k}(\phi u, u)_{\Omega_\eta} = \varepsilon_k\|f\|_\phi^2 + \frac{1}{4}(-\phi_x u, u)_{\Omega_\eta}$$
$$\le \varepsilon_k \|f\|_\phi^2 + \frac{1}{4}\|u\|^2_{-\phi_x}.$$

Together with (5.3) this yields (5.6). □

We need an analogue of estimate (5.1) for the finite element solution u_k of (2.8). To this end consider a vector $\phi = (\phi_0, \dots, \phi_{n_k})$, such that $\phi_i > 0$ for all i and

$$0 \le -\varepsilon_k \frac{\phi_i - \phi_{i-1}}{h_k} \le c_0 \phi_i, \quad i = 1, \dots, n_k \tag{5.7}$$

with a constant $c_0 \in (0, \frac{4}{9})$ and $\varepsilon_k = \varepsilon + \bar{\delta} h_k$.

Define $\hat{\Phi}_k := \mathrm{diag}(\phi_i)_{1 \le i \le n_k}$, $\Phi_k := I_{n_k - 1} \otimes \hat{\Phi}_k$ with ϕ_i satisfying (5.7). Let $\langle \cdot, \cdot \rangle_\Phi = \langle \Phi_k \cdot, \cdot \rangle_k$.

LEMMA 5.3. *There exists a constant $c > 0$ independent of k and ε such that*

$$\langle A_k x, D_x x\rangle_\Phi \ge c\, \|D_x x\|^2_\Phi \quad \textit{for all} \;\; x \in X_k$$

Proof. We use similar arguments as in the proof of (3.10). We use the representation (3.4) of the stiffness matrix: $A_k = \bar{\varepsilon}_k A_x + \varepsilon A_y + \frac{1}{6} B D_x$. Note that

$$D_x^T \Phi_k A_y = (I_{n_k-1} \otimes \hat{D}_x^T)(I_{n_k - 1} \otimes \hat{\Phi}_k)(\hat{A}_y \otimes \hat{J}) = \hat{A}_y \otimes \hat{D}_x^T \hat{\Phi}_k \hat{J}.$$

The matrix $\hat{A}_y$ is symmetric positive definite. Using $\phi_i \le \phi_{i-1}$ and a Gershgorin theorem it follows that $\hat{D}_x^T \hat{\Phi}_k \hat{J} + \hat{J}\hat{\Phi}_k \hat{D}_x$ is symmetric positive definite, too. Hence, $D_x^T \Phi_k A_y \ge 0$ holds, i.e.,

$$\langle A_y x, D_x x\rangle_\Phi \ge 0 \quad \text{for all} \;\; x \in X_k. \tag{5.8}$$

From the assumption on ϕ it follows that $\phi_{i-1} \le (1 + \frac{c_0 h_k}{\varepsilon_k})\phi_i$ for all i. Using this and the relation

$$\frac{1}{2}(\hat{\Phi}_k^{\frac{1}{2}} \hat{D}_x^T \hat{\Phi}_k^{-\frac{1}{2}} + \hat{\Phi}_k^{-\frac{1}{2}} \hat{D}_x \hat{\Phi}_k^{\frac{1}{2}}) = \frac{1}{2h_k}\mathrm{tridiag}\Big(\sqrt{\frac{\phi_{i-1}}{\phi_i}}\,, 2\,, \sqrt{\frac{\phi_i}{\phi_{i+1}}}\Big)$$

it follows that

$$\Phi_k^{\frac{1}{2}} D_x^T \Phi_k^{-\frac{1}{2}} \ge \frac{1}{2h_k}\Big(2 - 2\sqrt{1 + \frac{c_0 h_k}{\varepsilon_k}}\Big)\, I \;\ge\; -\frac{c_0}{2\varepsilon_k} I \;\ge\; -\frac{c_0}{2\bar{\varepsilon}_k} I$$

holds. And thus

$$\bar{\varepsilon}_k \langle A_x x, D_x x\rangle_\Phi = \bar{\varepsilon}_k \langle \Phi_k D_x^T D_x x, D_x x\rangle \ge -\frac{1}{2} c_0 \langle D_x x, D_x x\rangle_\Phi \quad \text{for all} \;\; x \in X_k. \tag{5.9}$$

We decompose B as $B = 4I - R$. A simple computation yields

$$\|\Phi_k^{\frac{1}{2}} R \Phi_k^{-\frac{1}{2}}\|_1 \le 1 + \sqrt{1 + \frac{c_0 h_k}{\varepsilon_k}} \le 1 + \sqrt{1 + 3c_0} \le 2 + \frac{3}{2} c_0.$$

Similarly we get $\|\Phi_k^{\frac{1}{2}} R \Phi_k^{-\frac{1}{2}}\|_\infty \leq 2 + \frac{3}{2}c_0$ and thus $\|\Phi_k^{\frac{1}{2}} R \Phi_k^{-\frac{1}{2}}\| \leq 2 + \frac{3}{2}c_0$. Hence

$$\Phi_k^{\frac{1}{2}} B \Phi_k^{-\frac{1}{2}} \geq \big(4 - (2 + \frac{3}{2}c_0)\big) I = (2 - \frac{3}{2}c_0) I$$

and thus

$$\frac{1}{6}\langle B D_x x, D_x x\rangle_\Phi \geq (\frac{1}{3} - \frac{1}{4}c_0)\langle D_x x, D_x x\rangle_\Phi \quad \text{for all} \;\; x \in X_k \tag{5.10}$$

Combination of the results in (5.8), (5.9) and (5.10) yields

$$\langle A_k x, D_x x\rangle_\Phi \geq (\frac{1}{3} - \frac{3}{4}c_0)\langle D_x x, D_x x\rangle_\Phi \geq c\langle D_x x, D_x x\rangle_\Phi \quad \text{for all} \;\; x \in X_k$$

with a constant $c > 0$ (use that $c_0 \in (0, \frac{4}{9})$). ☐

LEMMA 5.4. *For $f = f_k \in \mathbb{V}_k$ let u_k be the solution of the problem (2.8). Then*

$$\sum_{i=1}^{n_k} \sum_{j=1}^{n_k - 1} h_k^2 \phi_i \left(\frac{u_{i,j} - u_{i-1,j}}{h_k}\right)^2 \leq C \sum_{i=1}^{n_k} \sum_{j=1}^{n_k - 1} h_k^2 \phi_i (M_k \hat{f})_{i,j}^2 \tag{5.11}$$

holds. Here u_{ij} is the nodal value of u_k at the grid point $x_{i,j}$, $\hat{f}$ is the vector of nodal values of f_k, M_k is the mass matrix and ϕ_i satisfies (5.7).

Proof. Let $\hat{u}_k = P_k^{-1} u_k \in X_k$ be the vector of nodal values of u_k, then

$$A_k \hat{u}_k = M_k \hat{f} =: \hat{b}_k. \tag{5.12}$$

The diagonal matrices Φ_k and $\hat{\Phi}_k$ are as in lemma 5.3. The statement of the lemma is equivalent to $\langle \Phi_k D_x \hat{u}_k, D_x \hat{u}_k\rangle_k \leq c \langle \Phi_k \hat{b}_k, \hat{b}_k\rangle_k$, with a constant c that is independent of $\hat{b}_k$. This is the same as

$$\|D_x A_k^{-1}\|_\Phi \leq c. \tag{5.13}$$

Note that (5.13) is a generalization of the result in (3.10). From lemma 5.3 we obtain

$$\|D_x x\|_{\hat{\Phi}}^2 < \frac{1}{c}\langle A_k x, D_x x\rangle_{\hat{\Phi}} \leq \frac{1}{c}\|A_k x\|_\Phi \|D_x x\|_\Phi \quad \text{for all} \;\; x \in X_k,$$

and thus $\|D_x x\|_\Phi \leq \tilde{c}\|A_k x\|_\Phi$ for all x. Hence we have proved the result in (5.13). ☐

For the discrete case we consider

$$\phi_i^\xi = \begin{cases} 1 & \text{for } ih_k \in [0, \xi], \\ \exp\left(-\frac{ih_k - \xi}{4h_k}\right) & \text{for } ih_k > \xi. \end{cases} \tag{5.14}$$

It is straightforward to check that $-(\phi_i^\xi - \phi_{i-1}^\xi) = (\exp(\frac{1}{4}) - 1)\,\phi_i^\xi$ if $ih_k > \xi$. Therefore, using $\varepsilon_k \leq \frac{3}{2}h_k$,

$$0 \leq -\varepsilon_k \frac{\phi_i^\xi - \phi_{i-1}^\xi}{h_k} \leq \frac{3}{2}(\exp(\frac{1}{4}) - 1)\phi_i^\xi, \quad i = 1, 2, \ldots \tag{5.15}$$

For any ξ the vector ϕ_i^ξ, $1 \leq i \leq n_k$, satisfies the condition (5.7) with $c_0 = \frac{3}{2}(\exp(\frac{1}{4}) - 1)$. This constant is less than $\frac{4}{9}$. As a consequence of lemma 5.4 we obtain discrete versions of the results in corollary 5.2:

Corollary 5.5. *Consider $f_k \in \mathbb{V}_k$ such that* $\mathrm{supp}(f_k) \in \Omega_\eta$ *and let u_k be a the corresponding solution of problem (2.8). Assume $\eta - \xi \geq 8\, h_k\, p\, |\ln h_k|,\ p > 0$, then*

$$\|(u_k)_x\|_{L_2(\Omega_\xi)} \leq c\, h_k^p \|f_k\|, \tag{5.16}$$

$$\|(u_k)_y\|_{L_2(\Omega_\xi)} \leq c\, \xi\, h_k^{p-1} \|f_k\|. \tag{5.17}$$

Proof. Estimate (5.16) is a consequence of (5.11). Indeed, observe the following inequalities:

$$\begin{aligned}
\|(u_k)_x\|_{L_2(\Omega_\xi)} &\leq c \sum_{i:\, ih\leq\xi} \sum_{j=1}^{n_k-1} h_k^2 \left(\frac{u_{i,j}-u_{i-1,j}}{h_k}\right)^2 \\
&= c \sum_{i:\, ih\leq\xi} \sum_{j=1}^{n_k-1} h_k^2 \phi_i \left(\frac{u_{i,j}-u_{i-1,j}}{h_k}\right)^2 \leq c \sum_{i=1}^{n_k} \sum_{j=1}^{n_k-1} h_k^2 \phi_i (M_k \hat{f})_{i,j}^2 \\
&\leq c\, (\max_{ih\geq\eta} \phi_i) \sum_{i=1}^{n_k} \sum_{j=1}^{n_k-1} h_k^2 (M_h \hat{f})_{i,j}^2 \leq c\, (\max_{ih\geq\eta} \phi_i) \|f_k\|^2 \leq c\, h_k^{2p} \|f_k\|^2.
\end{aligned}$$

Estimate (5.17) follows from an inverse inequality, the Friedrichs inequality and (5.16):

$$\|(u_k)_y\|_{L_2(\Omega_\xi)} \leq c\, h_k^{-1} \|u_k\|_{L_2(\Omega_\xi)} \leq c\, \xi\, h_k^{-1} \|(u_k)_x\|_{L_2(\Omega_\xi)} \leq c\, \xi h_k^{p-1} \|f\|.$$

□

Corollary 5.6. *Consider $f_k \in \mathbb{V}_k$ such that* $\mathrm{supp}(f_k) \in \Omega_\eta$. *Let u and u_k be the solutions (2.7) and (2.8), respectively. Assume $\eta - \xi \geq 8\, h_k\, p\, |\ln h_k|,\ p > 0$. Then for $e_k = u - u_k$ we have*

$$\begin{aligned}
&\|(e_k)_x\|_{L_2(\Omega_\xi)} \leq c\, h_k^p \|f_k\|, \\
&\|(e_k)_y\|_{L_2(\Omega_\xi)} \leq c\, \max\{\sqrt{\frac{\varepsilon_k}{\varepsilon}};\ \frac{\xi}{h_k}\}\, h_k^p \|f_k\|.
\end{aligned}$$

Proof. Direct superposition of estimates in the corollaries 5.2 and 5.5. □

The result in corollary 5.6 shows that the (H^1-norm of) errors close to the inflow boundary can be made arbitrarily small if the righthand side is zero on a sufficiently large subdomain ($\Omega \setminus \Omega_\eta$) that is adjacent to this inflow boundary. In the proof of the approximation property in section 10 we will need these estimates for the case $\xi = h_k$ and $p = \frac{1}{2}$. Hence we take $\eta = 4h_k|\ln h_k| + h_k$. Note that for the results in the previous corollaries to be applicable we need righthand side functions f_k which are zero in $\Omega \setminus \Omega_\eta$. For technical reasons we take Ω_η such that the right boundary of the domain $\Omega \setminus \Omega_\eta$ coincides with a grid line. We use $|\ln h_k| = k \ln 2$ and thus $4h_k|\ln h_k| + h_k \leq (3k+1)h_k$ and introduce the following auxiliary domains for each grid level

$$\Omega_k^{in} := \{\, (x,y) \in \Omega \mid x < (3k+1)h_k \,\}. \tag{5.18}$$

As a direct consequence of the previous corollary we then obtain

Corollary 5.7. *Consider $f_k \in \mathbb{V}_k$ such that f_k is zero on the subdomain Ω_k^{in}. Let u and u_k be the solutions of (2.7) and (2.8), respectively. Then for $e_k = u - u_k$ we have*

$$\|(e_k)_x\|_{L_2(\Omega_{h_k})} \leq c\, h_k^{\frac{1}{2}} \|f_k\|, \tag{5.19}$$

$$\|(e_k)_y\|_{L_2(\Omega_{h_k})} \leq c\, \frac{h_k}{\sqrt{\varepsilon}} \|f_k\|. \tag{5.20}$$

6. Multigrid method and convergence analysis. In this section we describe the multigrid method for solving a problem of the form $A_k x = \hat{b}$ with the stiffness matrix A_k from section 2 and present a convergence analysis.

For the prolongation and restriction in the multigrid algorithm we use the canonical choice:

$$p_k : \mathrm{X}_{k-1} \to \mathrm{X}_k,\ p_k = P_k^{-1} P_{k-1}, \quad r_k = \frac{1}{4} p_k^T \tag{6.1}$$

Let $W_k : \mathrm{X}_k \to \mathrm{X}_k$ be a nonsingular matrix. We consider a smoother of the form

$$x^{\text{new}} = \mathcal{S}_k(x^{\text{old}}, \hat{b}) = x^{\text{old}} - \omega_k W_k^{-1}(A_k x^{\text{old}} - \hat{b}), \quad \text{for } x^{\text{old}}, \hat{b} \in \mathrm{X}_k, \tag{6.2}$$

with corresponding iteration matrix denoted by

$$S_k = I - \omega_k W_k^{-1} A_k. \tag{6.3}$$

The preconditioner W_k we use is of line-Jacobi type:

$$W_k = \frac{4\varepsilon}{h_k^2} I + D_x\ . \tag{6.4}$$

Note that W_k is a blockdiagonal matrix with diagonal blocks that are $n_k \times n_k$ bidiagonal matrices. A suitable choice for the parameter ω_k follows from the analysis below.

Remark 6.1. In the literature it is often recommended to apply a so-called *robust smoother* for solving singularly perturbed elliptic problem using multigrid. Such a smoother should have the property that it becomes a direct solver if the singular perturbation parameter tends to zero (cf. [9], chapter 10). In the formulation (6.2) one then must have a splitting such that $A_k - W_k = \mathcal{O}(\varepsilon)$ (the constant in $\mathcal{O}$ may depend on k). Such robust smoothers are well-known for some anisotropic problems. For anisotropic problems in which the anisotropy is *aligned with the gridlines* one can use a line (Jacobi or Gauss-Seidel) method or an ILU factorization as a robust smoother. Theoretical analyses of these methods can be found in [29, 30, 34].
If the convection-diffusion problem (1.2) is discretized using standard finite differences it is easy to see that an appropriate line solver yields a robust smoother. However, in the finite element setting such line methods do *not yield a robust smoother*. This is clear from the stencil in (3.3). For $\varepsilon \to 0$ the diffusion part yields an x-line difference operator which can be represented exactly by an x-line smoother, but in the convection stencil the $[0\ \ -\frac{1}{6}\ \ \frac{1}{6}]$ and $[-\frac{1}{6}\ \ \frac{1}{6}\ \ 0]$ parts of the difference operator are not captured by such a smoother. It is not clear to us how for the finite element discretization, with a stencil as in (3.3), a robust smoother can be constructed.

In multigrid analyses for reaction-diffusion or anisotropic diffusion problems one usually observes a ε^{-1} dependence in the standard approximation property that is then compensated by an ε factor from the smoothing property (cf. [21, 22, 29, 30, 34]). However, we can not apply a similar technique, due to the fact that for our problem class a robust smoother is not available. Instead, we use another splitting of the iteration matrix of the two-grid method, leading to modified (ε-independent) smoothing and approximation properties. $\square$

We consider a standard multigrid method with pre- and postsmoothers of the form as in (6.2), (6.4). In the analysis we will need different damping parameters for the pre- and postsmoother. Thus we introduce

$$S_{k,pr} := I - \omega_{k,pr} W_k^{-1} A_k, \quad S_{k,po} := I - \omega_{k,po} W_k^{-1} A_k$$

We also define the transformed iteration matrices

$$\tilde{S}_{k,pr} := A_k S_{k,pr} A_k^{-1}, \quad \tilde{S}_{k,po} := A_k S_{k,po} A_k^{-1}$$

We will analyze a standard two-grid method with iteration matrix

$$T_k = S_{k,po}^{\nu_k} \big(I - p_k A_{k-1}^{-1} r_k A_k \big) S_{k,pr}^{\mu_k} \tag{6.5}$$

For the corresponding multigrid W-cycle the iteration matrix (cf. [10]) is given by

$$M_0^{\mathrm{mgm}} := 0, \quad M_k^{\mathrm{mgm}} = T_k + S_{k,po}^{\nu_k} p_k (M_{k-1}^{\mathrm{mgm}})^2 A_{k-1}^{-1} r_k A_k S_{k,pr}^{\mu_k} \quad , k > 1 \tag{6.6}$$

In the convergence analysis of this method the auxiliary inflow domain Ω_k^{in} defined in (5.18) plays a crucial role. As in the analysis of the upstream influence in section 5 we will use a cut-off function in x-direction. We define diagonal matrices $\hat{\Phi}_k$, Φ_k as follows:

$$\xi := (3k+1)h_k, \quad \hat{\Phi}_k := \mathrm{diag}(\phi_1^\xi, \ldots, \phi_{n_k}^\xi), \quad \Phi_k := I_{n_k-1} \otimes \hat{\Phi}_k \ , \tag{6.7}$$

here ϕ_i^ξ is the cut-off function defined in (5.14) with $\xi = (3k+1)h_k$. For notational simplicity we drop the superscript ξ in ϕ_i^ξ in the remainder. Note that the diagonal matrix Φ_k is positive definite.
For any symmetric positive definite matrix $C \in \mathbb{R}^{m \times m}$ we define

$$\langle x, y \rangle_C := x^T C y \ , \quad \|x\|_C^2 := \langle x, x \rangle_C \ , \quad \|B\|_C := \|C^{\frac{1}{2}} B C^{-\frac{1}{2}}\|$$

with $x, y \in \mathbb{R}^m$, $B \in \mathbb{R}^{m \times m}$. Note that if $C = E^T E$ for some nonsingular matrix E then $\|B\|_C = \|EBE^{-1}\|$.

The convergence analysis is based on the following splitting, with $A := A_k$:

$$\begin{aligned}
\|T_k\|_{A^TA} &= \|S_{k,po}^{\nu_k}(I - p_k A_{k-1}^{-1} r_k A_k) S_{k,pr}^{\mu_k}\|_{A^TA} \\
&= \|S_{k,po}^{\nu_k}(A_k^{-1} - p_k A_{k-1}^{-1} r_k)\big((I - \Phi_k^{\frac{1}{2}}) + \Phi_k^{\frac{1}{2}}\big) A_k S_{k,pr}^{\mu_k}\|_{A^TA} \\
&\le \|S_{k,po}^{\nu_k}(A_k^{-1} - p_k A_{k-1}^{-1} r_k)(I - \Phi_k^{\frac{1}{2}}) A_k S_{k,pr}^{\mu_k}\|_{A^TA} \\
&\quad + \|S_{k,po}^{\nu_k}(A_k^{-1} - p_k A_{k-1}^{-1} r_k)\Phi_k^{\frac{1}{2}} A_k S_{k,pr}^{\mu_k}\|_{A^TA} \\
&\le \|\tilde{S}_{k,po}^{\nu_k} A_k W_k^{-1}\| \|W_k(A_k^{-1} - p_k A_{k-1}^{-1} r_k)(I - \Phi_k^{\frac{1}{2}})\| \|\tilde{S}_{k,pr}^{\mu_k}\| \\
&\quad + \|\tilde{S}_{k,po}^{\nu_k}\| \|I - A_k p_k A_{k-1}^{-1} r_k\| \|\Phi_k^{\frac{1}{2}} \tilde{S}_{k,pr}^{\mu_k}\|
\end{aligned} \tag{6.8}$$

REMARK 6.2. Note that the splitting in (6.8) differs from the usual splitting that is used in the theory based on the smoothing and approximation property introduced by Hackbusch (cf. [10]). In this theory the approximation property of the form $\|A_k^{-1} - p_k A_{k-1}^{-1} r_k\| \leq C_A\, g(h_k, \varepsilon)$ is combined with a smoothing property of the form $\|A_k S_{k,po}^{\mu_k}\| \leq \eta(\mu_k)\, g(h_k,\varepsilon)^{-1}$ with some $\eta(\mu_k)$ such that $\eta(\mu_k) \to 0,\ \mu_k \to \infty$ uniformly with respect to h_k and ε. In numerical experiments we observed that bounds of this type are not likely to be valid. Due to the fact that the smoother is not an exact solver for $\varepsilon \downarrow 0$ (cf. remark 6.1), it is essential to have the preconditioner W_k as part of the approximation property. Furthermore it turns out that for obtaining an appropriate bound for $\|W_k(A_k^{-1} - p_k A_{k-1}^{-1} r_k) f_k\|$ the righthand side function f_k must vanish near the inflow boundary. We illustrate this by numerical experiments in section 11. This motivates the introduction of the "cut-off" matrix Φ_k in the decomposition.

We now formulate the main results on which the convergence analysis will be based. The proofs of these results will be given further on.

THEOREM 6.1. *The following holds:*

$$W_k A_k^{-1} \geq \frac{1}{8} I \quad \text{for} \quad k = 1, 2, \ldots \tag{6.9}$$

Proof. Given in section 7. □

LEMMA 6.2. *From (6.9) it follows that*

$$\|I - \omega A_k W_k^{-1}\| \leq 1 \quad \text{for all} \ \ \omega \in [0, \frac{1}{4}] \ .$$

Proof. Elementary. □

ASSUMPTION 6.1. *In the postsmoother $S_{k,po}$ we take $\omega_{k,po} := \frac{1}{8}$.*

We note that the analysis below applies for any fixed $\omega_{k,po} \in (0, \frac{1}{8}]$. We obtain the following smoothing property:

COROLLARY 6.1. *There exists a constant c_1 independent of k and ε such that*

$$\|\tilde{S}_{k,po}^{\nu_k} A_k W_k^{-1}\| \leq \frac{c_1}{\sqrt{\nu_k}} \ . \tag{6.10}$$

Proof. Follows from lemma 6.2 and theorem 10.6.8 in [10] (or results in [12, 24]). The result holds with $c_1 = \frac{32}{\sqrt{2\pi}}$. □

We now turn to the presmoother:

THEOREM 6.3. *There exist constants $d_1 > 0, d_2 > 0$ independent of k and ε such that*

$$\|\Phi_k^{\frac{1}{2}} (I - \frac{d_1}{k^2} A_k W_k^{-1}) \Phi_k^{-\frac{1}{2}}\| \leq 1 - \frac{d_2}{k^4} \tag{6.11}$$

Proof. Given in section 8. □

ASSUMPTION 6.2. *In the presmoother $S_{k,pr}$ we take $\omega_{k,pr} := \min\{\, \frac{1}{4}\, , \frac{d_1}{k^2}\, \}$.*

REMARK 6.3. The result in (6.11) can be written as $\|I - \frac{d_1}{k^2} A_k W_k^{-1}\|_{\Phi_k} \leq 1 - \frac{d_2}{k^4}$. Hence, we have a contraction result in the almost degenerated norm $\|\cdot\|_{\Phi_k}$. This norm, however, coincides with the Euclidean one for the vectors that have a support only in Ω_k^{in}. Hence the result in (6.11) indicates that the presmoother is a fast solver *near the inflow boundary.* (cf. section 11).

Concerning the approximation property the following result holds:

THEOREM 6.4. *There exists a constant c_2 independent of k and ε such that*

$$\|W_k(A_k^{-1} - p_k A_{k-1}^{-1} r_k)(I - \Phi_k^{\frac{1}{2}})\| \leq c_2 \quad \textit{for} \quad k = 2, 3, \ldots \tag{6.12}$$

Proof. Given in section 10. □

Finally we present two results related to stability of the coarse grid correction. It is well-known that for the canonical restriction operator the inequality

$$\|r_k\| \leq c_r$$

holds with a constant c_r independent of k. The second stability result is:

THEOREM 6.5. *There exists a constant c_3 independent of k and ε such that*

$$\|A_k p_k A_{k-1}^{-1}\| \leq c_3 \quad \textit{for} \quad k = 2, 3, \ldots \tag{6.13}$$

Proof. Given in section 9. □

We now obtain a two-grid convergence result:

THEOREM 6.6. *For the two-grid method we then have*

$$\|T_k\|_{A^TA} \leq \frac{c_1 c_2}{\sqrt{\nu_k}} + (1 + c_r c_3)(1 - \frac{d_2}{k^4})^{\mu_k}$$

Proof. The proof is based on results from (6.9), (6.11), (6.12) and (6.13). We use the splitting in (6.8). From the assumptions 6.1 and 6.2 and lemma 6.2 it follows that $\|\tilde{S}_{k,pr}\| \leq 1$ and $\|\tilde{S}_{k,po}\| \leq 1$. From assumption 6.2, theorem 6.3 and $\|\Phi_k\| \leq 1$ we obtain

$$\|\Phi_k^{\frac{1}{2}} \tilde{S}_{k,pr}^{\mu_k}\| \leq \|(\Phi_k^{\frac{1}{2}} \tilde{S}_{k,pr} \Phi_k^{-\frac{1}{2}})^{\mu_k}\| \|\Phi_k^{\frac{1}{2}}\| \leq (1 - \frac{d_2}{k^4})^{\mu_k}$$

Combine these bounds with the results in corollary 6.1, theorem 6.4, theorem 6.5. □

Using the two-grid result of theorem 6.6 we derive a multigrid W-cycle convergence result based on standard arguments:

THEOREM 6.7. *In addition to the assumptions of theorem 6.6 we assume that the number of smoothing steps on every grid level is sufficiently large:*

$$\nu_k \geq c_{po}, \quad \mu_k \geq c_{pr}\, k^4$$

with suitable constants c_{po}, c_{pr}. Then for the contraction number of the multigrid W-cycle the inequality

$$\|M_k^{\mathrm{mgm}}\|_{A^TA} \leq \xi^* \tag{6.14}$$

holds, with a constant $\xi^* < 1$ independent of k and ε.

Proof. Define $\xi_k := \|M_k^{\rm mgm}\|_{A_k^TA_k}$. Using the recursion relation (6.6) for $M_k^{\rm mgm}$ it follows that

$$\begin{aligned}\xi_k &\le \|T_k\|_{A_k^TA_k} + \|\tilde S_{k,po}\|^{\nu_k}\|A_kp_kA_{k-1}^{-1}\|\xi_{k-1}^2\|r_k\|\|\tilde S_{k,pr}\|^{\mu_k}\\ &\le \|T_k\|_{A_k^TA_k} + c_3c_r\xi_{k-1}^2\end{aligned}$$

Now use the two-grid bound given in theorem 6.6 and a fixed point argument. ∎

Remark 6.4. We briefly discuss the arithmetic work needed in one W-cycle iteration. The arithmetic work for a matrix vector multiplication on level k is of order $\mathcal{O}(N_k) = \mathcal{O}(n_k^2)$. The work needed in one smoothing iteration is of order $\mathcal{O}(N_k)$. The number of smoothings behaves like $\nu_k+\mu_k \sim k^4$. Using a standard recursive argument it follows that for a multigrid W-cycle iteration the arithmetic complexity is of the order $N_k(\ln N_k)^4$. Hence this multigrid method has suboptimal complexity.

7. Proof of theorem 6.1. We recall the representation of the stiffness matrix in (3.4)

$$A_k = (\varepsilon + (\bar\delta - \frac{1}{3})h_k)A_x + \varepsilon A_y + \frac{1}{6}BD_x.$$

We will need the following lemma

Lemma 7.1. *The inequality $BD_x \ge 0$ holds.*

Proof. The matrix $\frac{1}{6}BD_x - \frac{1}{3}h_kA_x$ is the stiffness matrix corresponding to the bilinear form $(u,v) \to \int_\Omega u_xv\ dxdy$. For any $x \in X_k$ we get

$$\frac{1}{6}\langle BD_xx,x\rangle_k - \frac{1}{3}\langle h_kA_xx,x\rangle_k = \int_\Omega (P_kx)_x(P_kx)\ dxdy = \frac{1}{2}\int_{\Gamma_E}(P_kx)^2\ dxdy \ge 0$$

Since the matrix A_x is symmetric positive definite the result now follows. ∎

We now consider the preconditioner $W_k = \frac{4\varepsilon}{h_k^2}I + D_x$, as in (6.4).

Theorem 7.2 (=theorem 6.1). *The inequality $W_kA_k^{-1} \ge \frac{1}{8}I$ holds.*

Proof. First note that

$$h_k\hat D_x\hat D_x^T = \hat D_x + \hat D_x^T - \frac{1}{h_k}(1,0,\dots,0)^T(1,0,\dots,0) \le \hat D_x + \hat D_x^T$$

and thus $h_k\hat D_x^T\hat D_x\hat D_x^T\hat D_x \le \hat D_x^T(\hat D_x + \hat D_x^T)\hat D_x$ holds. Using $\hat A_x = \hat D_x^T\hat D_x$ this results in $h_k\hat A_x^2 \le 2\hat D_x^T\hat A_x$ and thus:

$$\frac{1}{2}h_kA_x^2 \le D_x^TA_x \tag{7.1}$$

Note that the following inequality holds for any $a,b,c \in \mathbb{R}$ and $\sigma_1,\sigma_2,\sigma_3 > 0$:

$$(a+b+c)^2 \le (1+\sigma_2+\sigma_3^{-1})a^2 + (1+\sigma_3+\sigma_1^{-1})b^2 + (1+\sigma_1+\sigma_2^{-1})c^2.$$

We apply this inequality with $\sigma_2 = 2, \sigma_1 = \sigma_3 = 1$. Also using $\|A_y\| \le 4h_k^{-2}$ and $\|B\| \le 6$ we get for any $x \in \mathrm{X}_k$

$$\begin{aligned}\|A_kx\|^2 &\le 4\varepsilon^2\|A_yx\|^2 + 3\bar\varepsilon_k^2\|A_xx\|^2 + \frac{5}{2}\|\frac{1}{6}BD_xx\|^2\\ &\le 16\big(\frac{\varepsilon}{h_k}\big)^2\langle A_yx,x\rangle_k + 3\bar\varepsilon_k^2\|A_xx\|^2 + \frac{5}{2}\|D_xx\|^2\end{aligned} \tag{7.2}$$

We recall that $\bar{\varepsilon}_k = \varepsilon_k - \bar{\delta} h_k \leq \frac{7}{6} h_k$. Now apply the result (7.1) and the estimates in lemma 3.1, lemma 7.1 to obtain

$$\begin{aligned}
\langle W_k x, A_k x\rangle_k &= \langle \frac{4\varepsilon}{h_k^2} x + D_x x\, ,\, \varepsilon A_y x + \bar{\varepsilon}_k A_x x + \frac{1}{6} B D_x x\rangle_k \\
&\geq 4\big(\frac{\varepsilon}{h_k}\big)^2 \langle A_y x, x\rangle_k + \bar{\varepsilon}_k \langle D_x x, A_x x\rangle_k + \langle D_x x, \frac{1}{6} B D_x x\rangle_k \\
&\geq 4\big(\frac{\varepsilon}{h_k}\big)^2 \langle A_y x, x\rangle_k + \frac{3}{7}\bar{\varepsilon}_k^2 \|A_x x\|^2 + \frac{1}{3}\|D_x x\|^2 \\
&= \frac{1}{8}\big(32\big(\frac{\varepsilon}{h_k}\big)^2 \langle A_y x, x\rangle_k + \frac{24}{7}\bar{\varepsilon}_k^2 \|A_x x\|^2 + \frac{8}{3}\|D_x x\|^2\big) \\
&\geq \frac{1}{8}\big(16\big(\frac{\varepsilon}{h_k}\big)^2 \langle A_y x, x\rangle_k + 3\bar{\varepsilon}_k^2 \|A_x x\|^2 + \frac{5}{2}\|D_x x\|^2\big)
\end{aligned}$$

Combination of this with the inequality in (7.2) proves the theorem. ∎

8. Proof of theorem 6.3. We start with an elementary known result on the convergence of basic iterative methods:

LEMMA 8.1. *Assume $C, A, W \in \mathbb{R}^{n\times n}$ with C symmetric positive definite. If there are constants $c_0 > 0,\ c_1$ such that*

$$c_0 \langle Ay, Ay\rangle_C \leq \langle Wy, Wy\rangle_C \leq c_1 \langle Wy, Ay\rangle_C \quad \textit{for all} \ \ y \in \mathbb{R}^n \tag{8.1}$$

then for arbitrary $d \in [0,1]$ we have

$$\|I - \alpha \frac{c_0}{c_1} A W^{-1}\|_C \leq \sqrt{1 - d\frac{c_0}{c_1^2}} \qquad \textit{if} \quad 1 - \sqrt{1-d} \leq \alpha \leq 1 + \sqrt{1-d}$$

Proof. Let $D := AW^{-1}$. From (8.1) we get

$$\langle Dy, y\rangle_C \geq c_1^{-1} \langle y, y\rangle_C \ , \quad \langle Dy, Dy\rangle_C \leq c_0^{-1}\langle y, y\rangle_C \quad \text{for all} \ \ y$$

Note that

$$\begin{aligned}
\|(I - \alpha\frac{c_0}{c_1} A W^{-1})y\|_C^2 &= \langle y, y\rangle_C - 2\alpha\frac{c_0}{c_1}\langle Dy, y\rangle_C + \alpha^2 \frac{c_0^2}{c_1^2}\langle Dy, Dy\rangle_C \\
&\leq \big(1 - 2\alpha\frac{c_0}{c_1^2} + \alpha^2\frac{c_0}{c_1^2}\big)\|y\|_C^2 = \big(1 - (2\alpha - \alpha^2)\frac{c_0}{c_1^2}\big)\|y\|_C^2
\end{aligned}$$

and $2\alpha - \alpha^2 \geq d$ if $1 - \sqrt{1-d} \leq \alpha \leq 1 + \sqrt{1-d}$. ∎

Below we use the scalar product $\langle \cdot, \cdot\rangle_\Phi := \langle \Phi_k \cdot, \cdot\rangle_k$ with Φ_k defined in (6.7). We recall the result proved in lemma 5.3

$$\langle A_k x, D_x x\rangle_\Phi \geq c\,\|D_x x\|_\Phi^2 \quad \text{for all} \ \ x \in X_k \tag{8.2}$$

with $c > 0$ independent of k and of ε.
We introduce the diagonal projection matrix $J_k := I_{n_k - 1} \otimes \hat{J}_k$ with $\hat{J}_k$ the $n_k \times n_k$ diagonal matrix with $(\hat{J}_k)_{i,i} = 1$ if $(\hat{\Phi}_k)_{i,i} = 1$ and $(\hat{J}_k)_{i,i} = 0$ otherwise.

LEMMA 8.2. *There exist a constant $c > 0$ independent of k and ε such that*

$$\|W_k x\|_\Phi^2 \leq c k^2 \big(\frac{\varepsilon}{h_k^3}\|(I - J_k)x\|_\Phi^2 + \|D_x x\|_\Phi^2\big) \quad \textit{for all} \ \ x \in X_k$$

Proof. Note that

$$\begin{aligned}\|J_k x\|_\Phi &= \|J_k D_x^{-1} J_k D_x x\|_\Phi \le \|J_k D_x^{-1} J_k\|_\Phi \|D_x x\|_\Phi \\ &= \|J_k D_x^{-1} J_k\| \|D_x x\|_\Phi \le (3k+1) h_k \|D_x x\|_\Phi\end{aligned}$$

And thus, using $\varepsilon \le \frac{1}{2} h_k$ we get

$$\begin{aligned}\|W_k x\|_\Phi = \|\frac{4\varepsilon}{h_k^2} x + D_x x\|_\Phi &\le \frac{4\varepsilon}{h_k^2}\|(I - J_k)x\|_\Phi + \frac{4\varepsilon}{h_k^2}\|J_k x\|_\Phi + \|D_x x\|_\Phi \\ &\le \frac{4\varepsilon}{h_k^2}\|(I - J_k)x\|_\Phi + ck\|D_x x\|_\Phi \le ck\big(\frac{4\varepsilon}{h_k^2}\|(I - J_k)x\|_\Phi + \|D_x x\|_\Phi\big)\end{aligned}$$

Squaring this result and using $(\frac{\varepsilon}{h_k^2})^2 \le \frac{1}{2}\frac{\varepsilon}{h_k^3}$ completes the proof. □

We define $\hat{\Phi}_x := \frac{1}{h_k}\mathrm{diag}(\phi_i - \phi_{i+1})_{1 \le i \le n_k}$ with $\phi_i = \phi_i^\xi$ as in (6.7). Consider the diagonal matrix $\Phi_x := I_{n_k - 1} \otimes \hat{\Phi}_x$. Note that $\Phi_x \ge 0$.

LEMMA 8.3. *The following estimate holds:*

$$\langle A_k x, x\rangle_\Phi \ge \frac{1}{30}\|\Phi_x^{\frac{1}{2}} x\|^2 \quad \textit{for all} \quad x \in X_k$$

Proof. Recall

$$A_k = \bar{\varepsilon}_k A_x + \varepsilon A_y + \frac{1}{6} B D_x \tag{8.3}$$

Note that

$$\Phi_k A_y = (I_{n_k - 1} \otimes \hat{\Phi}_k)(\hat{A}_y \otimes \hat{J}) = \hat{A}_y \otimes \hat{\Phi}_k \hat{J} \ge 0 \tag{8.4}$$

We consider the term $\bar{\varepsilon}_k \Phi_k A_x = \bar{\varepsilon}_k (I_{n_k - 1} \otimes \hat{\Phi}_k \hat{A}_x)$. Note that $\hat{\Phi}_k \hat{A}_x = \hat{\Phi}_k \hat{D}_x^T \hat{D}_x$. A simple computation yields $\hat{\Phi}_k \hat{D}_x^T - \hat{D}_x^T \hat{\Phi}_k = -\hat{\Phi}_x \hat{T}$, with $\hat{T} := \mathrm{tridiag}(0, 0, 1)$, and thus

$$\bar{\varepsilon}_k \hat{\Phi}_k \hat{A}_x = \bar{\varepsilon}_k \hat{D}_x^T \hat{\Phi}_k \hat{D}_x - \bar{\varepsilon}_k \hat{\Phi}_x \hat{T} \hat{D}_x \tag{8.5}$$

From the Cauchy-Schwarz inequality it follows that

$$\bar{\varepsilon}_k \langle \hat{\Phi}_x \hat{T} \hat{D}_x y, y\rangle \le \bar{\varepsilon}_k^2 \frac{9}{4}\|\hat{\Phi}_x^{\frac{1}{2}} \hat{T} \hat{D}_x y\|^2 + \frac{1}{9}\|\hat{\Phi}_x^{\frac{1}{2}} y\|^2 \quad \text{for all} \quad y \in \mathbb{R}^{n_k} \tag{8.6}$$

Using the property (5.15) we get

$$\hat{T}^T \hat{\Phi}_x \hat{T} \le \bar{\varepsilon}_k^{-1} c_0 \hat{\Phi}_k \tag{8.7}$$

Combination of the results in (8.5), (8.6), (8.7) and using $c_0 \le \frac{4}{9}$ yields

$$\begin{aligned}\bar{\varepsilon}_k \langle \hat{\Phi}_k \hat{A}_x y, y\rangle &\ge \bar{\varepsilon}_k \|\hat{D}_x y\|_{\hat{\Phi}_k}^2 - \bar{\varepsilon}_k \frac{9}{4} c_0 \|\hat{D}_x y\|_{\hat{\Phi}_k}^2 - \frac{1}{9}\|\hat{\Phi}_x^{\frac{1}{2}} y\|^2 \\ &\ge -\frac{1}{9}\|\hat{\Phi}_x^{\frac{1}{2}} y\|^2 \quad \text{for all} \quad y \in \mathbb{R}^{n_k}\end{aligned}$$

And thus

$$\bar{\varepsilon}_k \Phi_k A_x \geq -\frac{1}{9}\Phi_x \tag{8.8}$$

holds. Finally we consider the term $\frac{1}{6}\langle BD_x x, x\rangle_\Phi$. First we note

$$BD_x = \text{blocktridiag}(\hat{D}_x, 4\hat{D}_x, \hat{S}_x), \quad \hat{S}_x := \frac{1}{h_k}\begin{pmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{R}^{n_k \times n_k}$$

and thus $K := \frac{1}{6}\Phi_k BD_x = \frac{1}{6}\text{blocktridiag}(\hat{\Phi}_k\hat{D}_x, 4\hat{\Phi}_k\hat{D}_x, \hat{\Phi}_k\hat{S}_x)$. Hence

$$\frac{1}{2}(K+K^T) = \frac{1}{12}\text{blocktridiag}\big(\hat{\Phi}_k\hat{D}_x + \hat{S}_x^T\hat{\Phi}_k, 4(\hat{\Phi}_k\hat{D}_x + \hat{D}_x^T\hat{\Phi}_k), \hat{\Phi}_k\hat{S}_x + \hat{D}_x^T\hat{\Phi}_k\big)$$

A simple computation yields

$$\hat{\Phi}_k\hat{D}_x + \hat{D}_x^T\hat{\Phi}_k = \hat{\Phi}_x + \frac{1}{h_k}\text{tridiag}(-\phi_i, \phi_i + \phi_{i+1}, -\phi_{i+1})_{1\leq i\leq n_k} =: \hat{\Phi}_x + R \tag{8.9}$$

and $\hat{\Phi}_k\hat{S}_x + \hat{D}_x^T\hat{\Phi}_k = \hat{\Phi}_x\hat{T} + \frac{1}{h_k}\phi_n e_n e_n^T$, with $n := n_k$ and e_n the n-th basis vector in $\mathbb{R}^n$. Thus we obtain

$$\begin{aligned} \frac{1}{2}(K+K^T) &= \frac{1}{12}\text{blocktridiag}\big(\hat{T}^T\hat{\Phi}_x, 4\hat{\Phi}_x, \hat{\Phi}_x\hat{T}\big) \\ &\quad + \frac{1}{12}\text{blocktridiag}\big(\frac{1}{h_k}\phi_n e_n e_n^T, 4R, \frac{1}{h_k}\phi_n e_n e_n^T\big) \\ &\geq \frac{1}{12}\text{blocktridiag}\big(\hat{T}^T\hat{\Phi}_x, 4\hat{\Phi}_x, \hat{\Phi}_x\hat{T}\big) \end{aligned}$$

By $\hat{\Phi}_x^{-1}$ (Φ_x^{-1}) we denote the pseudo inverse of $\hat{\Phi}_x$ (Φ_x). We then have

$$\frac{1}{2}\Phi_x^{-\frac{1}{2}}(K+K^T)\Phi_x^{-\frac{1}{2}} \geq \frac{1}{12}\text{blocktridiag}\big(\hat{\Phi}_x^{-\frac{1}{2}}\hat{T}^T\hat{\Phi}_x^{\frac{1}{2}}, 4I, \hat{\Phi}_x^{\frac{1}{2}}\hat{T}\hat{\Phi}_x^{-\frac{1}{2}}\big)$$

Note that

$$\|\hat{\Phi}_x^{-\frac{1}{2}}\hat{T}^T\hat{\Phi}_x^{\frac{1}{2}}\|_\infty = \|\hat{\Phi}_x^{\frac{1}{2}}\hat{T}\hat{\Phi}_x^{-\frac{1}{2}}\|_\infty = \max_{i\geq 3k+2}\left(\frac{\phi_{i-1}-\phi_i}{\phi_i - \phi_{i+1}}\right)^{\frac{1}{2}} = e^{\frac{1}{8}}$$

And thus we get $\frac{1}{2}\Phi_x^{-\frac{1}{2}}(K+K^T)\Phi_x^{-\frac{1}{2}} \geq \frac{1}{12}(4-2e^{\frac{1}{8}})I$. Hence

$$\frac{1}{6}\Phi_k BD_x = K \geq \frac{1}{6}(2-e^{\frac{1}{8}})\Phi_x \tag{8.10}$$

Combination of the results in (8.3), (8.4), (8.8) and (8.10) yields

$$\Phi_k A_k \geq \left(-\frac{1}{9} + \frac{1}{6}(2-e^{\frac{1}{8}})\right)\Phi_x > \frac{1}{30}\Phi_x$$

□

Using the previous two lemmas we can show a result as in the second inequality in (8.1):

THEOREM 8.4. *There exists a constant c_1 independent of k and ε such that*

$$\langle W_k x, W_k x\rangle_\Phi \leq c_1 k^2 \langle W_k x, A_k x\rangle_\Phi \quad \textit{for all} \;\; x \in X_k$$

Proof. From lemma 8.3 and (8.2) we get

$$\begin{aligned} \langle W_k x, A_k x\rangle_\Phi &= \frac{4\varepsilon}{h_k^2}\langle x, A_k x\rangle_\Phi + \langle D_x x, A_k x\rangle_\Phi \\ &\geq c\big(\frac{\varepsilon}{h_k^2}\langle \Phi_x x, x\rangle_k + \|D_x x\|_\Phi^2\big) \end{aligned} \tag{8.11}$$

with $c > 0$ independent of k and ε. Using $\phi_i - \phi_{i+1} = (1 - e^{-\frac{1}{4}})\phi_i \geq \frac{1}{5}\phi_i$ for $i \geq 3k+1$ we get

$$\langle \Phi_x x, x\rangle_k \geq \frac{1}{5} h_k^{-1}\langle (I - J_k)\Phi_k x, x\rangle_k = \frac{1}{5} h_k^{-1}\|(I - J_k)x\|_\Phi^2 \tag{8.12}$$

From (8.11) and (8.12) we obtain

$$\langle W_k x, A_k x\rangle_\Phi \geq c\big(\frac{\varepsilon}{h_k^3}\|(I - J_k)x\|_\Phi^2 + \|D_x x\|_\Phi^2\big)$$

Now combine this with the result in lemma 8.2. ◻

We now consider the first inequality in (8.1):

THEOREM 8.5. *There exists a constant $c_0 > 0$ independent of k and ε such that*

$$c_0\langle A_k x, A_k x\rangle_\Phi \leq \langle W_k x, W_k x\rangle_\Phi \quad \textit{for all} \;\; x \in X_k$$

Proof. The constants c that appear in the proof are all strictly positive and independent of k and ε. First note that $\|A_k x\|_\Phi \leq \bar{\varepsilon}_k\|A_x x\|_\Phi + \varepsilon\|A_y x\|_\Phi + \frac{1}{6}\|BD_x x\|_\Phi$. We have

$$\|A_y\|_\Phi = \|(I_{n_k-1}\otimes \hat{\Phi}_k^{\frac{1}{2}})(\hat{A}_y \otimes \hat{J})(I_{n_k-1}\otimes \hat{\Phi}_k^{-\frac{1}{2}})\| = \|\hat{A}_y \otimes \hat{J}\| \leq \frac{4}{h_k^2}$$

Note that $|\phi_i\phi_{i+1}^{-1}| \leq e^{\frac{1}{4}}$ and thus $\|\hat{\Phi}_k^{\frac{1}{2}}\hat{D}_x^T\hat{\Phi}_k^{-\frac{1}{2}}\| \leq ch_k^{-1}$ holds. From this it follows that $\|D_x^T\|_\Phi \leq ch_k^{-1}$ holds. With a similar argument we get $\|B\|_\Phi \leq c$. Thus we obtain, using $\bar{\varepsilon}_k \leq \frac{3}{2}h_k$:

$$\begin{aligned} \|A_k x\|_\Phi &\leq \bar{\varepsilon}_k\|D_x^T\|_\Phi\|D_x x\|_\Phi + \frac{4\varepsilon}{h_k^2}\|x\|_\Phi + c\|D_x x\|_\Phi \\ &\leq c\big(\frac{\varepsilon}{h_k^2}\|x\|_\Phi + \|D_x x\|_\Phi\big) \end{aligned} \tag{8.13}$$

From (8.9) it follows that $\langle D_x x, x\rangle_\Phi \geq 0$ holds. Using this we get

$$\begin{aligned} \|W_k x\|_\Phi^2 &= \frac{16\varepsilon^2}{h_k^4}\|x\|_\Phi^2 + \frac{16\varepsilon}{h_k^2}\langle D_x x, x\rangle_\Phi + \|D_x x\|_\Phi^2 \\ &\geq c\big(\frac{\varepsilon^2}{h_k^4}\|x\|_\Phi^2 + \|D_x x\|_\Phi^2\big) \end{aligned} \tag{8.14}$$

Now combine (8.13) with (8.14). ◻

Combination of the results of theorem 8.4, theorem 8.5 with the second result in lemma 8.1 shows that theorem 6.3 holds.

9. Proof of theorem 6.5. Let $\mathrm{g}_{k-1} \in X_{k-1}$ be given and define $g_{k-1} := (P^*_{k-1})^{-1}\mathrm{g}_{k-1} \in \mathbb{V}_{k-1}$. Let $u_{k-1} \in \mathbb{V}_{k-1}$ be such that

$$a_{k-1}(u_{k-1}, v_{k-1}) = (g_{k-1}, v_{k-1}) \quad \text{for all} \;\; v_{k-1} \in \mathbb{V}_{k-1}.$$

Then $A^{-1}_{k-1}\mathrm{g}_{k-1} = P^{-1}_{k-1}u_{k-1}$ holds. The corresponding continuous solution $u \in \mathbf{V}$ satisfies $a_{k-1}(u, v) = (g_{k-1}, v)$ for all $v \in \mathbf{V}$. Now note that

$$\begin{aligned} \|A_k p_k A^{-1}_{k-1}\mathrm{g}_{k-1}\| &= \max_{y\in X_k} \frac{\langle A_k p_k P^{-1}_{k-1}u_{k-1}, y\rangle_k}{\|y\|} \leq c \max_{v_k\in\mathbb{V}_k} \frac{a_k(u_{k-1}, v_k)}{\|v_k\|} \\ &\leq c \max_{v_k\in\mathbb{V}_k} \frac{a_{k-1}(u_{k-1}, v_k)}{\|v_k\|} + c \max_{v_k\in\mathbb{V}_k} \frac{a_k(u_{k-1}, v_k) - a_{k-1}(u_{k-1}, v_k)}{\|v_k\|} \end{aligned} \tag{9.1}$$

Define $e_{k-1} := u - u_{k-1}$. For the first term in (9.1) we get, using the results of lemma 4.3 :

$$\begin{aligned} a_{k-1}(u_{k-1}, v_k) &\leq |a_{k-1}(e_{k-1}, v_k)| + |a_{k-1}(u, v_k)| \\ &\leq ch_k\|(e_{k-1})_x\|\|(v_k)_x\| + \varepsilon\|(e_{k-1})_y\|\|(v_k)_y\| + \|(e_{k-1})_x\|\|v_k\| + |(g_{k-1}, v_k)| \\ &\leq c\big(\|(e_{k-1})_x\| + \frac{\varepsilon}{h_k}\|(e_{k-1})_y\|\big)\|v_k\| + \|g_{k-1}\|\|v_k\| \\ &\leq c\|g_{k-1}\|\|v_k\| \leq c\|\mathrm{g}_{k-1}\|\|v_k\| \end{aligned} \tag{9.2}$$

For the second term in (9.1) we have, using lemma 4.2 :

$$\begin{aligned} |a_k(u_{k-1}, v_k) - a_{k-1}(u_{k-1}, v_k)| &= \bar{\delta}h_k|((u_{k-1})_x, (v_k)_x)| \\ &\leq c\|(u_{k-1})_x\|\|v_k\| \\ &\leq c\|g_{k-1}\|\|v_k\| \leq c\|\mathrm{g}_{k-1}\|\|v_k\| \end{aligned} \tag{9.3}$$

Combination of the results in (9.1), (9.2) and (9.3) yields $\|A_k p_k A^{-1}_{k-1}\mathrm{g}_{k-1}\| \leq c\|\mathrm{g}_{k-1}\|$ and thus the result in theorem 6.5 holds. ☐

10. Proof of theorem 6.4. We briefly comment on the idea of the proof. As usual to prove an estimate for the error in the L^2-norm we use a duality argument. However, the formal dual problem has poor regularity properties, since in this dual problem Γ_E is the "inflow" boundary and Γ_W is the "outflow" boundary. Thus Dirichlet outflow boundary conditions would appear and we obtain poor estimates due to the poor regularity. To avoid this, we consider a dual problem with Neumann outflow and Dirichlet inflow conditions. To be able to deal with the inconsistency caused by these "wrong" boundary conditions we *assume* the righthand side is zero near the boundary Γ_W. In order to satisfy this assumption we use the cut-off operator with matrix Φ_k.
A further problem we have to deal with is the fact that due to the level dependent stabilization term we have to treat k-dependent bilinear forms.

We introduce the space

$$\mathbb{V}^0_k := \{v_k \in \mathbb{V}_k \mid v_k(x) = 0 \quad \text{for all} \;\; x \in \Omega^{in}_k\}$$

Let $\hat{b}_k \in X_k$ be given. In view of theorem 6.4 we must prove an estimate $\|W_k(A^{-1}_k - p_k A^{-1}_{k-1} r_k)(I - \Phi_k)\hat{b}_k\| \leq c\|\hat{b}_k\|$ with a constant c that is independent of k, ε and

$\hat{b}_k$. Note that $(P_k^*)^{-1}(I-\Phi_k^{\frac{1}{2}})\hat{b}_k =: f_k \in \mathbb{V}_k^0$ holds. For this $f_k \in \mathbb{V}_k^0$ we define corresponding discrete solutions and continuous solutions as follows:

$$\begin{array}{rlll} u_k \in \mathbb{V}_k: & a_k(u_k, v_k) = (f_k, v_k) & \text{for all} & v_k \in \mathbb{V}_k \\ u \in \mathbf{V}: & a_k(u, v) = (f_k, v) & \text{for all} & v \in \mathbf{V} \\ u_{k-1} \in \mathbb{V}_{k-1}: & a_{k-1}(u_{k-1}, v_{k-1}) = (f_k, v_{k-1}) & \text{for all} & v_{k-1} \in \mathbb{V}_{k-1} \\ \tilde{u} \in \mathbf{V}: & a_{k-1}(\tilde{u}, v) = (f_k, v) & \text{for all} & v \in \mathbf{V} \end{array} \tag{10.1}$$

In the proof of lemma 4.2 we showed that $\|v_x\| = \|D_x P_k^{-1} v\|$ holds for all $v \in \mathbb{V}_k$. We use that $W_k = \frac{4\varepsilon}{h_k^2} I + D_x$ and obtain

$$\begin{aligned} &\|W_k(A_k^{-1} - p_k A_{k-1}^{-1} r_k)(I-\Phi_k^{\frac{1}{2}})\hat{b}_k\| \\ &\le \frac{4\varepsilon}{h_k^2}\|(A_k^{-1} - p_k A_{k-1}^{-1} r_k)(I-\Phi_k^{\frac{1}{2}})\hat{b}_k\| + \|D_x A_k^{-1}(I-\Phi_k^{\frac{1}{2}})\hat{b}_k\| + \|D_x p_k A_{k-1}^{-1} r_k (I-\Phi_k^{\frac{1}{2}})\hat{b}_k\| \\ &\le c\big(\frac{\varepsilon}{h_k^2}\|u_k - u_{k-1}\| + \|(u_k)_x\| + \|(u_{k-1})_x\|\big) \\ &\le c\Big(\frac{\varepsilon}{h_k^2}\big(\|u-u_k\| + \|\tilde{u} - u_{k-1}\| + \|u-\tilde{u}\|\big) + \|(u_k)_x\| + \|(u_{k-1})_x\|\Big) \end{aligned} \tag{10.2}$$

From lemma 4.2 we get

$$\|(u_k)_x\| + \|(u_{k-1})_x\| \le c\|f_k\| \tag{10.3}$$

From the result in theorem 10.1 below it follows that

$$\|u_k - u\| + \|u_{k-1} - \tilde{u}\| \le c\,\frac{h_k^2}{\varepsilon}\,\|f_k\| \tag{10.4}$$

Finally, from theorem 10.4 we have

$$\|u - \tilde{u}\| \le c\, h_k \|f_k\| \tag{10.5}$$

If we insert the results (10.3),(10.4) and (10.5) in (10.2) we get

$$\|W_k(A_k^{-1} - p_k A_{k-1}^{-1} r_k)(I-\Phi_k^{\frac{1}{2}})\hat{b}_k\| \le c\|f_k\| \le c\|(P_k^*)^{-1}\|\|I-\Phi_k^{\frac{1}{2}}\|\|\hat{b}_k\| \le c\|\hat{b}_k\|$$

and thus the result of theorem 6.4 is proved.
It remains to prove the results in the theorems 10.1 and 10.4.

THEOREM 10.1. *For $f_k \in \mathbb{V}_k^0$ let u and u_k be as defined in (10.1). Then*

$$\|u - u_k\| \le c\,\frac{h_k^2}{\varepsilon}\,\|f_k\| \tag{10.6}$$

holds.

Proof. Define $e_k := u - u_k$. Let $w \in H^2(\Omega)$ be such that

$$-\varepsilon w_{yy} - \varepsilon_k w_{xx} - w_x = e_k \tag{10.7}$$

with

$$w_x = 0 \text{ on } \Gamma_W\ , \quad w = 0 \text{ on } \Gamma \setminus \Gamma_W. \tag{10.8}$$

Note that for this problem Γ_E is the "inflow" boundary and Γ_W is the "outflow" boundary. We multiply (10.7) with e_k and integrate by parts to get

$$\begin{aligned}\|e_k\|^2 &= \varepsilon((e_k)_y, w_y) + \varepsilon_k((e_k)_x, w_x) - \varepsilon_k \int_{\Gamma_E} w_x e_k\, dy + ((e_k)_x, w)\\ &= a_k(e_k, w) - \varepsilon_k \int_{\Gamma_E} w_x e_k\, dy\end{aligned}$$

We use (4.6) with w and e_k instead of u and f, respectively, and (4.12) to estimate

$$\left|\varepsilon_k \int_{\Gamma_E} w_x e_k\, dy\right| \le \varepsilon_k^{\frac{1}{2}} \left(\varepsilon_k \int_{\Gamma_E} w_x^2\, dy\right)^{\frac{1}{2}} \left(\int_{\Gamma_E} e_k^2\, dy\right)^{\frac{1}{2}} \le c\, h_k^{\frac{1}{2}} \|e_k\| \frac{h_k}{\sqrt{\varepsilon}} \|f_k\|. \quad (10.9)$$

From this estimate and the Galerkin orthogonality for the error it follows that for any $v_k \in \mathbb{V}_k$

$$\begin{aligned}\|e_k\|^2 \le\; &\varepsilon\,((e_k)_y, (w - v_k)_y) + \varepsilon_k\,((e_k)_x, (w - v_k)_x)\\ &+ ((e_k)_x, w - v_k) + c\,\|e_k\| \frac{h_k^{\frac{3}{2}}}{\sqrt{\varepsilon}} \|f_k\|.\end{aligned} \quad (10.10)$$

Let $\Omega_h := \Omega_{h_k}$ be as defined in (5.4), i.e., Ω_h is the set of triangles with at least one vertex on Γ_W. In the remainder of the domain, $\omega = \Omega\backslash\Omega_h$, we take v_k as a nodal interpolant to w and we put $v_k = 0$ on Γ_W to ensure $v_k \in \mathbb{V}_k$. Note that v_k is a proper interpolant of w everywhere in Ω except in Ω_h. Therefore we will estimate scalar products in (10.10) over ω and Ω_h, separately. We continue (10.10) with:

$$\begin{aligned}\|e_k\|^2 &\le c\,\varepsilon\, h_k \|(e_k)_y\|_\omega \|w\|_{H^2(\omega)} + c\,\varepsilon_k\, h_k \|(e_k)_x\|_\omega \|w\|_{H^2(\omega)}\\ &\quad + c\, h_k^2 \|(e_k)_x\|_\omega \|w\|_{H^2(\omega)} + c\,\|e_k\| \frac{h_k^{\frac{3}{2}}}{\sqrt{\varepsilon}} \|f_k\| + \mathrm{I}_{\Omega_h}\\ &\le c\, h_k^2 \|f_k\| \frac{1}{\varepsilon} \|e_k\| + \mathrm{I}_{\Omega_h}.\end{aligned} \quad (10.11)$$

The term I_{Ω_h} collects integrals over Ω_h:

$$\mathrm{I}_{\Omega_h} = \varepsilon\,((e_k)_y, (w - v_k)_y)_{\Omega_h} + \varepsilon_k\,((e_k)_x, (w - v_k)_x)_{\Omega_h} + ((e_k)_x, w - v_k)_{\Omega_h}$$

To estimate I_{Ω_h} we use corollary 5.7 and the following auxiliary estimate for the interpolant $v_k \in \mathbb{V}_k$ of w, with $\omega_h = \{(x, y) \in \Omega :\ x \in (h_k, 2h_k)\}$:

$$\begin{aligned}\|v_k\|_{\Omega_h} &\le c\|v_k\|_{\omega_h} \le c(\|w\|_{\omega_h} + \|v_k - w\|_\omega)\\ &= c\Big(\Big(\int_0^1 \int_{h_k}^{2h_k} \Big[w(0, y) + \int_0^x w_\eta(\eta, y)\, d\eta\Big]^2 dx\, dy\Big)^{\frac{1}{2}} + \|v_k - w\|_\omega\Big)\\ &\le c\big(h_k^{\frac{1}{2}} (\int_{\Gamma_W} w^2\, dy)^{\frac{1}{2}} + h_k \|w_x\| + h_k^2 \|w\|_{H^2(\omega)}\big) \le c\big(h_k^{\frac{1}{2}} + \frac{h_k^2}{\varepsilon}\big)\|e_k\|.\end{aligned}$$

We proceed estimating terms from I_{Ω_h}, where we use the previous result:

$$
\begin{aligned}
\varepsilon\left((e_k)_y,(w-v_k)_y\right)_{\Omega_h} &\le \varepsilon\|(e_k)_y\|_{\Omega_h}\left(\|w_y\|+\|(v_k)_y\|_{\Omega_h}\right)\\
&\le c\,\varepsilon^{\frac12}h_k\|f_k\|\;(\varepsilon^{-\frac12}\|e_k\|+h_k^{-1}\|v_k\|_{\Omega_h})\\
&\le c\,\varepsilon^{\frac12}h_k\|f_k\|(\varepsilon^{-\frac12}+h_k^{-\frac12}+\frac{h_k}{\varepsilon})\|e_k\|\le c\,(h_k+\frac{h_k^2}{\sqrt{\varepsilon}})\|f_k\|\|e_k\|,\\
\varepsilon_k\left((e_k)_x,(w-v_k)_x\right)_{\Omega_h} &\le \varepsilon_k\|(e_k)_x\|_{\Omega_h}\left(\|w_x\|+\|(v_k)_x\|_{\Omega_h}\right)\\
&\le c\,h_k^{\frac12}\varepsilon_k\|f_k\|\;(\|e_k\|+h_k^{-1}\|v_k\|_{\Omega_h})\le c\,(h_k+\frac{h_k^{\frac52}}{\varepsilon})\|f_k\|\|e_k\|,\\
((e_k)_x,w-v_k)_{\Omega_h} &\le \|(e_k)_x\|_{\Omega_h}(\|w\|_{\Omega_h}+\|v_k\|_{\Omega_h})\\
&\le c\,h_k^{\frac12}\|f_k\|\;\left(h_k^{\frac12}(\int_{\Gamma_W}w^2\,dy)^{\frac12}+h_k\|w_x\|_{\Omega_h}+\|v_k\|_{\Omega_h}\right)\\
&\le c\,(h_k+\frac{h_k^{\frac52}}{\varepsilon})\|f_k\|\|e_k\|.
\end{aligned}
$$

Inserting this estimates into (10.11) and using $\varepsilon\le\frac12 h_k$ we obtain

$$
\|e_k\|^2\le c\,\frac{h_k^2}{\varepsilon}\|f_k\|\|e_k\|+c\,(h_k+\frac{h_k^2}{\sqrt{\varepsilon}}+\frac{h_k^{\frac52}}{\varepsilon})\|f_k\|\|e_k\|\le c\,\frac{h_k^2}{\varepsilon}\|f_k\|\|e_k\|.
$$

and thus the theorem is proved.
□

For the proof of theorem 10.4 we first formulate two lemmas.

LEMMA 10.2. *Consider a function $g\in\mathrm{H}^1(\Omega)$. The solution of*

$$
-\varepsilon_k u_{xx}-\varepsilon u_{yy}+u_x=g_x \tag{10.12}
$$

with boundary conditions as in (1.2) satisfies

$$
\int_{\Gamma_E}u^2\,dy\le c\left(h_k^{-1}\|g\|^2+\int_{\Gamma_E}g^2\,dy+h_k\,|g_x|^2\right). \tag{10.13}
$$

Proof. We multiply (10.12) with u and integrate by parts to get

$$
\varepsilon_k\|u_x\|^2+\varepsilon\|u_y\|^2+\frac12\int_{\Gamma_E}u^2\,dy=-(g,u_x)+\int_{\Gamma_E}g\,u\,dy. \tag{10.14}
$$

For the righthand side in (10.14) we have

$$
|(g,u_x)|\le\|g\|\|u_x\|\le c\,\|g\|\|g_x\|\le c\,\left(h_k^{-1}\,\|g\|^2+h_k\|g_x\|^2\right)
$$

and

$$
\int_{\Gamma_E}g\,u\,dy\le\int_{\Gamma_E}g^2\,dy+\frac14\int_{\Gamma_E}u^2\,dy.
$$

Combining these estimates and (10.14) the lemma is proved. □

LEMMA 10.3. *Assume $g \in H^1$ and $g|_{\Gamma_E} = 0$, let u be the corresponding solution of (10.12). Then the following holds:*

$$\|u\| \le c\Big(\|g\| + h_k\|g_x\| + \big(\int_{\Gamma_W} g^2\,dy\big)^{\frac{1}{2}} + h_k\big(\int_{\Gamma_W} u_x^2\,dy\big)^{\frac{1}{2}}\Big). \qquad (10.15)$$

(Note that the standard a priori estimates would give only $\|u\| \le c\,\|g_x\|$.)

Proof. Consider the auxiliary function $v(x,y) := \int_0^x u(\xi,y)\,d\xi$. It satisfies

$$-\varepsilon_k v_{xx} - \varepsilon v_{yy} + v_x = g + \varepsilon_k\,u_{in} + g_{in}, \qquad (10.16)$$

with $u_{in}(x,y) = u_x(0,y)$ and $g_{in} = g(0,y)$. The corresponding boundary conditions are

$$v_x = u(1,y) \text{ on } \Gamma_E\ , \qquad v = 0 \text{ on } \partial\Omega\setminus\Gamma_E. \qquad (10.17)$$

Then the estimate (10.15) is equivalent to

$$\|v_x\| \le c\Big(\|g\| + h_k\|g_x\| + \big(\int_{\Gamma_W} g^2\,dy\big)^{\frac{1}{2}} + h_k\left(\int_{\Gamma_W} u_x^2\,dy\right)^{\frac{1}{2}}\Big). \qquad (10.18)$$

The estimate (10.18) is proved by the following arguments. We multiply (10.16) with v_x and integrate by parts to obtain

$$\begin{aligned}\|v_x\|^2 + \frac{\varepsilon}{2}\int_{\Gamma_E}(v_y)^2\,dy + \frac{\varepsilon_k}{2}\int_{\Gamma_W}(v_x)^2\,dy \\ = (g,v_x) + \varepsilon_k(u_{in},v_x) + (g_{in},v_x) + \frac{\varepsilon_k}{2}\int_{\Gamma_E}(v_x)^2\,dy. \qquad (10.19)\end{aligned}$$

Since $g|_{\Gamma_E} = 0$ the estimate (10.13) yields

$$\int_{\Gamma_E}(v_x)^2\,dy = \int_{\Gamma_E} u^2\,dy \le c\,\left(h_k^{-1}\,\|g\|^2 + h_k\,\|g_x\|^2\right). \qquad (10.20)$$

Now (10.18) follows from (10.19) by applying the Cauchy inequality and estimate (10.20). ∎

Using these lemmas we can prove the final result we need.

THEOREM 10.4. *For $f \in \mathbb{V}_k^0$ let u and $\tilde{u}$ be the continuous solutions defined in (10.1). Then the following holds*

$$\|u - \tilde{u}\| \le c\,h_k\|f_k\|. \qquad (10.21)$$

Proof. The difference $e := u - \tilde{u}$ solves the equation

$$-\varepsilon_k e_{xx} - \varepsilon e_{yy} + e_x = g_x\ , \qquad (10.22)$$

with $g = -\bar{\delta}h_k\tilde{u}_x$ and boundary conditions as in (1.2). Now the result of lemma 10.3 can be applied. We obtain

$$\begin{aligned}\|e\| &\le c\Big(\|g\| + h_k\|g_x\| + \big(\int_{\Gamma_W} g^2\,dy\big)^{\frac{1}{2}} + h_k\big(\int_{\Gamma_W} e_x^2\,dy\big)^{\frac{1}{2}}\Big) \\ &\le c\,h_k\Big(\|\tilde{u}_x\| + h_k\|\tilde{u}_{xx}\| + \big(\int_{\Gamma_W} u_x^2\,dy\big)^{\frac{1}{2}} + \big(\int_{\Gamma_W} \tilde{u}_x^2\,dy\big)^{\frac{1}{2}}\Big).\end{aligned}$$

To estimate the norms $\|\tilde{u}_x\|$ and $\|\tilde{u}_{xx}\|$ we use a priori bounds from theorem 4.1. Further we use the fact that $f_k = 0$ in Ω_k^{in}. Due to the choice of Ω_k^{in} (cf. (5.18)) we can apply corollary 5.2 with $\xi = h_k$, $\eta = \varepsilon_k |\ln h_k| + h_k$ and $p = \frac{1}{2}$. Using (5.5) and $\varepsilon_k \geq \frac{1}{3} h_k$ we get $\int_{\Gamma_W} u_x^2 \, dy \leq c \, \|f_k\|^2$. The same estimate holds for $\int_{\Gamma_W} \tilde{u}_x^2 \, dy$. Thus we obtain $\|e\| \leq c \, h_k \|f_k\|$. $\square$

11. Numerical experiments. In this section we present results of a few numerical experiments to illustrate that in a certain sense our analysis is sharp. In particular it will be shown that the nonstandard splitting in (6.8) which forms the basis of our convergence analysis reflects some important phenomena.

In the experiments we use the following parameters. For $\bar{\delta}$ in (2.4) we take $\bar{\delta} = \frac{1}{2}$. The pre- and postsmoother are as in (6.2), (6.4) with $\omega_k = 1$. We take a random righthand side vector and a starting vector equal to zero. For the stopping criterion we take a reduction of the relative residual by a factor 10^9 . Thus in the tables below convergence is measured in the norm $\| \cdot \|_{A^T A}$. We use the notation $Pe_h := \frac{h}{2\varepsilon}$.

First we present results for a standard V-cycle with $\mu_k = \nu_k = 2$. In table 11.1 we give the number of iterations needed to satisfy the stopping criterion and (between brackets) the average residual reduction per iteration. These results clearly show robustness of the multigrid solver.

TABLE 11.1
Multigrid convergence: V-cycle with $\nu_k = \nu_k = 2$

	h			
Pe_h	1/8	1/32	1/128	1/512
1	**8**(0.06)	**10**(0.12)	**11**(0.13)	**11**(0.13)
10	**7**(0.04)	**8**(0.07)	**8**(0.07)	**8**(0.07)
1e+3	**8**(0.05)	**11**(0.14)	**11**(0.14)	**11**(0.14)
1e+5	**7**(0.04)	**11**(0.14)	**11**(0.14)	**11**(0.14)

Number of iterations and average reduction factor

If we only consider the smoother and do not use a coarse grid correction, then for $\varepsilon \approx h$ this method has an h-dependent convergence rate. This is illustrated in table 11.2.

TABLE 11.2
h-dependence of convergence of the smoothing iterations

	h			
Pe_h	1/8	1/32	1/128	1/512
1	**119**(0.83)	**244**(0.91)	**533**(0.94)	**1495**(0.986)
10	**26**(0.44)	**51**(0.61)	**66**(0.72)	**173**(0.88)

Number of iterations and average reduction factor

We consider the standard splitting in the convergence analysis based on the smoothing and approximation property. For $\varepsilon = h^2$ some results are presented in table 11.3. The estimates that are given in this table result from the computation of

$$\frac{\|(A_h^{-1} - pA_{2h}^{-1} r)\hat{f}\|}{\|\hat{f}\|} \quad \text{and} \quad \frac{\|(A_h S_h^2)\hat{f}\|}{\|\hat{f}\|}$$

with $\hat{f} \in \mathbb{V}_h$ a discrete point-source in the grid point $(\frac{1}{2}, \frac{1}{2})$. These results indicate $\mathcal{O}(h^{-1})$ behavior for the smoothing property (as expected) and $\mathcal{O}(\sqrt{h})$ behavior for the approximation property. Hence this splitting is not satisfactory for proving a robustness result.

Table 11.3

Standard splitting for approximation and smoothing properties.

	h			
Estimates for	1/8	1/32	1/128	1/512
$\|A_h^{-1} - pA_{2h}^{-1}r\|$	8.4e-2	5.0e-2	2.7e-2	1.4e-2
$\|A_h S_h^2\|$	1.25	4.48	17.7	70.8

The proof of the modified approximation property is based on the result in theorem 10.1. In that theorem a $\frac{h_k^2}{\varepsilon}$ bound is proved *provided the righthand side function f_k is zero close to the inflow boundary.* We performed an experiment with a function f_k which has values equal to one in all grid points (h_k, jh_k), $j = 1, \ldots, n_k$, and zero elsewhere. Results are given in table 11.4. We observe an $h_k^{-\frac{1}{2}}$ effect. This justifies the splitting using the cut-off operator Φ_k.

Table 11.4

Approximation property if f_k has support near inflow

	h			
Pe_h	1/8	1/32	1/128	1/512
1	0.31	0.60	1.23	2.53
10	0.07	0.17	0.23	0.46

Values of $\frac{h^2}{\varepsilon}\|(A_h^{-1} - pA_{2h}^{-1}r)f\|/\|f\|$.

Finally we performed a numerical experiment related to the result in theorem 6.3. For the smoother we computed residual reduction factors in the almost degenerated norm $\|\Phi_k^{\frac{1}{2}} \cdot \|$ with $\Phi_k := I_{n_{k-1}} \otimes \mathrm{diag}(\phi)$ and

$$\phi_i = \begin{cases} 1 & \text{for } 1 \le i < 5, \\ \exp(4 - i) & \text{for } 5 \le i \le n_k \end{cases}$$

For the relaxation parameter ω in the smoother we take the value $\omega = 1.2$. The results in table 11.5 show h-independent and "fast" convergence of the smoother in this norm.

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Table 11.5
Residual reduction of the smoother in the $\|\Phi^{\frac{1}{2}} \cdot \|$-norm.

	h			
Pe_h	1/8	1/32	1/128	1/512
1	**93**(0.8)	**131**(0.85)	**133**(0.85)	**133**(0.85)
10	**23**(0.40)	**28**(0.47)	**28**(0.47)	**28**(0.47)

Number of iterations and average reduction factor

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