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FINITE ELEMENT DISCRETIZATION ERROR ANALYSIS OF A SURFACE TENSION FORCE IN TWO-PHASE INCOMPRESSIBLE FLOWS*

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Abstract. We consider a standard model for a stationary two-phase incompressible flow with surface tension. In the variational formulation of the model a linear functional which describes the surface tension force occurs. This functional depends on the location and the curvature of the interface. In a finite element discretization method the functional has to be approximated. For an approximation method based on a Laplace-Beltrami representation of the curvature we derive sharp bounds for the approximation error. A new modified approximation method with a significantly smaller error is introduced.

Key words. two-phase flow, continuum surface force technique, interface, Laplace–Beltrami operator, finite elements

AMS subject classifications. 65M60, 65N15, 65N30, 76D45, 76T99

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1. Introduction.

Let $\Omega \subset \mathbb{R}^3$ be a polyhedral domain that contains a flow of two different immiscible incompressible newtonian phases (fluid-fluid or fluid-gas). At the interface between the two phases there are surface tension forces that are significant and cannot be neglected. An example is a (rising) liquid drop contained in a surrounding fluid. The standard model to describe such a flow problem consists of instationary Navier–Stokes equations with certain coupling conditions at the interface which describe the effect of surface tension. In this paper we analyze errors that are due to the discretization of the surface tension force that occurs in the continuous model. To simplify the presentation and the analysis we assume a *stationary* flow.

The domains which contain the phases are denoted by Ω_1 and Ω_2 with $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}$ and $\partial\Omega_1 \cap \partial\Omega = \emptyset$. The interface between the two phases ($\partial\Omega_1 \cap \partial\Omega_2$) is denoted by Γ. To model the forces at the interface we make the standard assumption that the surface tension balances the jump of the normal stress on the interface; i.e., we have an interface condition

$$[\boldsymbol{\sigma}\mathbf{n}]_\Gamma = \tau\mathcal{K}\mathbf{n},$$

with $\mathbf{n} = \mathbf{n}_\Gamma$ the unit normal at the interface (pointing from Ω_1 in Ω_2), τ the surface tension coefficient (material parameter), $\mathcal{K}$ the curvature of Γ, and $\boldsymbol{\sigma}$ the stress tensor, i.e.,

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mu\mathbf{D}(\mathbf{u}), \quad \mathbf{D}(\mathbf{u}) = \nabla\mathbf{u} + (\nabla\mathbf{u})^T,$$

with $p = p(x,t)$ the pressure, $\mathbf{u} = \mathbf{u}(x,t)$ the velocity vector, and μ the viscosity. We assume continuity of the velocity across the interface. In combination with the

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conservation laws of mass and momentum this yields the following standard model (cf., for example, [23, 22, 26, 25]):

$$
\begin{cases} -\operatorname{div}(\mu_i \mathbf{D}(\mathbf{u})) + \rho_i(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p = \rho_i \mathbf{g} & \text{in } \Omega_i \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega_i \end{cases} \quad \text{for } i = 1,2, \tag{1.1}
$$

$$
[\boldsymbol{\sigma}\mathbf{n}]_\Gamma = \tau\mathcal{K}\mathbf{n}, \quad [\mathbf{u}]_\Gamma = 0. \tag{1.2}
$$

The vector $\mathbf{g}$ is a known external force (gravity). In addition we need boundary conditions for $\mathbf{u}$ at $\partial\Omega$. For simplicity we take homogeneous Dirichlet boundary conditions. The *two* Navier–Stokes equations in (1.1) and the coupling conditions at the interface in (1.2) can be reformulated into *one* Navier–Stokes equation in the whole domain in which the effect of the surface tension is expressed in terms of a localized force at the interface; cf. the so-called continuum surface force (CSF) model [5, 6]. We consider this alternative formulation in a standard weak form (as in [12, 27, 28, 29, 30]) in the spaces

$$
\mathbf{V} := H_0^1(\Omega)^3, \quad Q := L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \mid \int_\Omega q\,dx = 0 \right\}.
$$

For the L^2 scalar product we use the notation $(f,g) := \int_\Omega fg\,dx$ (and similarly for vector functions). The standard norm in $\mathbf{V}$ is denoted by $\|\cdot\|_1$. The weak formulation is as follows: Determine $(\mathbf{u},p) \in \mathbf{V}\times Q$ such that

$$
\begin{aligned} \int_\Omega \frac{\mu}{2}\,\mathbf{D}(\mathbf{u}):\mathbf{D}(\mathbf{v})\,dx + (\rho\,\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{v}) + (\operatorname{div}\mathbf{v},p) &= (\rho\,\mathbf{g},\mathbf{v}) + f_\Gamma(\mathbf{v}) \quad \text{for all}\,\mathbf{v}\in\mathbf{V}, \\ (\operatorname{div}\mathbf{u},q) &= 0 \quad \text{for all}\,q\in Q, \end{aligned} \tag{1.3}
$$

with

$$
f_\Gamma(\mathbf{v}) = \tau \int_\Gamma \mathcal{K}\mathbf{n}_\Gamma\cdot\mathbf{v}\,ds, \tag{1.4}
$$

and $\mathbf{D}(\mathbf{u}):\mathbf{D}(\mathbf{v}) = \operatorname{tr}\big((\mathbf{D}(\mathbf{u})\mathbf{D}(\mathbf{v})\big)$. The functions μ and ρ are strictly positive and piecewise constant in $\Omega_i,\ i=1,2$. For Γ sufficiently smooth we have $\sup_{x\in\Gamma}|\mathcal{K}(x)| \leq c < \infty$, and thus

$$
|f_\Gamma(\mathbf{v})| \leq c\,\tau \int_\Gamma |\mathbf{n}_\Gamma\cdot\mathbf{v}|\,ds \leq c\,\|\mathbf{v}\|_{L^2(\Gamma)} \leq c\|\mathbf{v}\|_1 \quad \text{for all} \quad \mathbf{v}\in\mathbf{V}. \tag{1.5}
$$

Here and in the remainder we use the notation c for a generic constant. From (1.5) we see that $f_\Gamma \in \mathbf{V}'$, and thus under the usual assumptions (cf. [13]) the stationary Navier–Stokes equations (1.3) have a unique solution. We emphasize that the location of the interface is in general unknown and has to be determined (approximated) before the Navier–Stokes equations (1.3) can be solved. In this paper we assume that the unknown interface is captured using a level set technique. For a discussion of level set methods in incompressible two-phase flow problems we refer to the literature [6, 14, 21, 24]. We assume that the interface Γ is characterized as the zero level of the level set function d, which locally (close to the interface) is a signed distance function.

We now turn to the discretization of (1.3). We assume that $\mathcal{S}$ is a triangulation of Ω consisting of tetrahedra. With this triangulation we associate a mesh size parameter H. Let $\mathbf{V}_H \subset \mathbf{V}$, $Q_H \subset Q$ be standard polynomial finite element spaces corresponding to the triangulation $\mathcal{S}$, for example, the Hood–Taylor P_2-P_1 pair. In practice, the triangulation $\mathcal{S}$ is locally refined close to the interface Γ but *not* aligned with this interface; cf. Figures 2.1 and 6.1. The Galerkin Discretization is as follows: Determine $(\mathbf{u}_H, p_H) \in \mathbf{V}_H \times Q_H$ such that

$$\begin{aligned} &\int_\Omega \frac{\mu}{2}\mathbf{D}(\mathbf{u}_H) : \mathbf{D}(\mathbf{v}_H)\,dx + (\rho\,\mathbf{u}_H \cdot \nabla \mathbf{u}_H, \mathbf{v}_H) + (\operatorname{div}\mathbf{v}_H, p_H) \\ &= (\rho\,\mathbf{g}, \mathbf{v}_H) + f_\Gamma(\mathbf{v}_H), \quad \text{for all } \ \mathbf{v}_H \in \mathbf{V}_H, \\ &(\operatorname{div}\mathbf{u}_H, q_H) = 0 \quad \text{for all } \ q_H \in Q_H. \end{aligned} \tag{1.6}$$

For this discrete problem, many important theoretical issues are still unsolved. For example, regarding iterative solvers there is the issue of robustness w.r.t. large jumps in the density and viscosity coefficients (results for Stokes equations are given in [20, 19, 18]). A second example is the effect of errors in the approximation of $f_\Gamma(\mathbf{v}_H)$ on the accuracy of the flow variables. In this paper we treat the latter topic.

As mentioned above, the interface Γ has to be approximated. Furthermore, to evaluate the integral in (1.4) the curvature of Γ has to be approximated and a quadrature rule may be needed. Thus the term $f_\Gamma(\mathbf{v}_H)$ on the right-hand side in (1.6) will be replaced by an approximation $\tilde{f}(\mathbf{v}_H)$. For the effect of the surface tension force approximation error on the accuracy of the velocity and pressure variables, the quantity

$$\sup_{\mathbf{v}\in\mathbf{V}_H} \frac{f_\Gamma(\mathbf{v}_H) - \tilde{f}(\mathbf{v}_H)}{\|\mathbf{v}_H\|_1} \tag{1.7}$$

is crucial (Strang lemma). The two main ingredients in the approximation method that we use are the following. First, a Laplace–Beltrami characterization of the curvature is used. This technique has been applied in mean curvature flows (cf. [7]) and in flows with a free capillary surface (cf. [3, 4]). Application of this technique in two-phase incompressible flows can be found in [12, 11, 14, 17]. Second, the unknown interface Γ (zero level of d) is approximated as the zero level Γ_h of a finite element approximation d_h of d. The approximate interface Γ_h consists of triangular faces. The parameter h is the maximal diameter of these faces and is not necessarily of the same order of magnitude as H. For this approximation technique we derive a sharp bound for the quantity in (1.7). The main result of this paper is the $\mathcal{O}(\sqrt{h})$ bound given in Corollary 4.8. We do not know of any literature in which, for this technique or for any other technique for approximating $f_\Gamma(\mathbf{v}_H)$, rigorous bounds for the quantity in (1.7) are derived. A numerical experiment (given in section 6) indicates that the $\mathcal{O}(\sqrt{h})$ is sharp. Our analysis reveals how the approximation method can be improved. A modified new approach, resulting in an $\mathcal{O}(h)$ bound, is presented in section 5.

2. Approximation of the surface tension force $f_\Gamma(\mathbf{v}_H)$.

In this section we explain how the localized surface tension force term, $f_\Gamma(\mathbf{v}_H)$ in (1.6), is approximated. For this we first need some notions from differential geometry.

Let U be an open subset in $\mathbb{R}^3$ and Γ a connected C^2 compact hypersurface contained in U. For a sufficiently smooth function $g : U \to \mathbb{R}$ the tangential derivative (along Γ) is defined by projecting the derivative on the tangent space of Γ, i.e.,

$$\nabla_\Gamma g = \nabla g - \nabla g \cdot \mathbf{n}_\Gamma \, \mathbf{n}_\Gamma. \tag{2.1}$$

The *Laplace–Beltrami operator* of g on Γ is defined by

$$\Delta_\Gamma g := \nabla_\Gamma \cdot \nabla_\Gamma g.$$

It can be shown that $\nabla_\Gamma g$ and $\Delta_\Gamma g$ depend only on values of g on Γ. For vector valued functions $f,\, g : \Gamma \to \mathbb{R}^3$ we define

$$\Delta_\Gamma f := (\Delta_\Gamma f_1, \Delta_\Gamma f_2, \Delta_\Gamma f_3)^T, \qquad \nabla_\Gamma f \cdot \nabla_\Gamma g := \sum_{i=1}^{3} \nabla_\Gamma f_i \cdot \nabla_\Gamma g_i.$$

We recall the following basic result from differential geometry.

THEOREM 2.1. *Let $\mathrm{id}_\Gamma : \Gamma \to \mathbb{R}^3$ be the identity on Γ and $\mathcal{K} = \kappa_1 + \kappa_2$ the sum of the principal curvatures. For all sufficiently smooth vector functions $\mathbf{v}$ on Γ the following holds:*

$$\int_\Gamma \mathcal{K}\, \mathbf{n}_\Gamma \cdot \mathbf{v} \, ds = -\int_\Gamma (\Delta_\Gamma \, \mathrm{id}_\Gamma) \cdot \mathbf{v} \, ds = \int_\Gamma \nabla_\Gamma \, \mathrm{id}_\Gamma \cdot \nabla_\Gamma \mathbf{v} \, ds. \tag{2.2}$$

In a finite element setting (which is based on a weak formulation) it is natural to use the expression on the right-hand side of (2.2) as a starting point for the discretization. This idea is used in, for example, [10, 4, 12, 14]. In this discretization we use *an approximation Γ_h of Γ*.

For the formulation of assumptions on the approximate interface Γ_h it is convenient to introduce the signed distance function $d : U \to \mathbb{R}$, $|d(x)| := \mathrm{dist}(x, \Gamma)$ for all $x \in U$. Thus Γ is the zero level set of d. We assume $d < 0$ on the interior of Γ (that is, in Ω_1) and $d > 0$ on the exterior. Note that $\mathbf{n}_\Gamma = \nabla d$ on Γ. We define $\mathbf{n}(x) := \nabla d(x)$ for all $x \in U$. Thus $\mathbf{n} = \mathbf{n}_\Gamma$ on Γ and $\|\mathbf{n}(x)\| = 1$ for all $x \in U$. Here and in the remainder $\|\cdot\|$ denotes the Euclidean norm. The Hessian of d is denoted by $\mathbf{H}$:

$$\mathbf{H}(x) = D^2 d(x) \in \mathbb{R}^{3\times 3} \quad \text{for all} \quad x \in U. \tag{2.3}$$

The eigenvalues of $\mathbf{H}(x)$ are denoted by $\kappa_1(x), \kappa_2(x)$, and 0. For $x \in \Gamma$ the eigenvalues $\kappa_i(x)$, $i = 1, 2$, are the principal curvatures.

We will need the orthogonal projection

$$\mathbf{P}(x) = \mathbf{I} - \mathbf{n}(x)\mathbf{n}(x)^T \quad \text{for} \quad x \in U.$$

Note that the tangential derivative can be written as $\nabla_\Gamma g = \mathbf{P}\nabla g$.

Using the distance function d we now introduce assumptions on the approximate interface Γ_h. In Remark 2 below we indicate how in practice an approximate interface Γ_h can be constructed which satisfies these assumptions. Let $\{\Gamma_h\}_{h>0}$ be a family of polygonal approximations of Γ. Each Γ_h is contained in U and consists of a set $\mathcal{F}_h$ of triangular faces: $\Gamma_h = \cup_{T\in\mathcal{F}_h} T$. For $T_1, T_2 \in \mathcal{F}_h$ with $T_1 \neq T_2$ we assume that $T_1 \cap T_2$ is either empty or a common edge or a common vertex. The parameter h denotes the maximal diameter of the triangles in $\mathcal{F}_h$: $h = \max_{T\in\mathcal{F}_h} \mathrm{diam}(T)$. By $\mathbf{n}_h$ we denote the outward pointing unit normal on Γ_h. This normal is piecewise constant with possible discontinuities at the edges of the triangles in $\mathcal{F}_h$.

The approximation Γ_h is assumed to be close to Γ in the following sense:

$$|d(x)| \leq ch^2 \quad \text{for all} \quad x \in \Gamma_h, \tag{2.4}$$

$$\operatorname{ess\,inf}_{x\in\Gamma_h} \mathbf{n}(x)^T \mathbf{n}_h(x) \geq c > 0, \tag{2.5}$$

$$\operatorname{ess\,sup}_{x\in\Gamma_h} \|\mathbf{P}(x)\mathbf{n}_h(x)\| \leq ch. \tag{2.6}$$

Here c denotes a generic constant independent of h.

Remark 1. The conditions (2.5), (2.6) are satisfied if

$$\operatorname{ess\,sup}_{x\in\Gamma_h} \|\mathbf{n}(x) - \mathbf{n}_h(x)\| \leq \min\{c_0, ch\} \quad \text{with} \quad c_0 < \sqrt{2} \tag{2.7}$$

holds. This easily follows from

$$\|\mathbf{n}(x) - \mathbf{n}_h(x)\|^2 = 2\big(1 - \mathbf{n}(x)^T \mathbf{n}_h(x)\big)$$

and

$$\|\mathbf{P}(x)\mathbf{n}_h(x)\| = \|\mathbf{P}(x)\big(\mathbf{n}(x) - \mathbf{n}_h(x)\big)\| \leq \|\mathbf{n}(x) - \mathbf{n}_h(x)\|.$$

Remark 2. We briefly explain the approach that is used in [14] (cf. also [9]) for computing Γ_h. Let $\mathcal{S}$ be the (locally refined) triangulation of Ω, consisting of tetrahedra, that is used for the discretization of the flow variables with finite elements; cf. (1.6) (in our approach we use the Hood–Taylor P_2-P_1 pair). The level set equation for d is discretized with continuous piecewise quadratic finite elements on a triangulation $\mathcal{T}$. This triangulation is either equal to $\mathcal{S}$ or obtained from one or a few regular refinements of $\mathcal{S}$ (the subdivision of each tetrahedron in eight child tetrahedra). The piecewise *quadratic* finite element approximation of d on $\mathcal{T}$ is denoted by d_h. We now introduce one further regular refinement of $\mathcal{T}$, resulting in $\mathcal{T}'$. Let $I(d_h)$ be the continuous piecewise *linear* function on $\mathcal{T}'$ which interpolates d_h at all vertices of all tetrahedra in $\mathcal{T}'$. The approximation of the interface Γ is defined by

$$\Gamma_h := \{\, x \in \Omega \mid I(d_h)(x) = 0 \,\}$$

which consists of piecewise planar segments. The mesh size parameter h is the maximal diameter of these segments. This (maximal) diameter is approximately the (maximal) diameter of the tetrahedra in $\mathcal{T}'$ that contain the discrete interface; i.e., h is approximately the maximal diameter of the tetrahedra in $\mathcal{T}'$ *that are close to the interface*. In Figure 2.1 we illustrate this construction for the two-dimensional case.

Each of the planar segments of Γ_h is either a triangle or a quadrilateral. The quadrilaterals can (formally) be divided into two triangles. Thus Γ_h consists of a set $\mathcal{F}_h$ of triangular faces. For the example considered in section 6, in which Γ is a sphere, the resulting polygonal approximations Γ_h for $h = \frac{1}{5}$ and $h = \frac{1}{10}$, resp., are shown in Figure 2.2.

We note the following facts related to the assumptions (2.4)–(2.6). If we assume $|I(d_h)(x) - d(x)| \leq c\,h^2$ for all x in a neighborhood of Γ, which is reasonable for a smooth d and piecewise quadratic d_h, then for $x \in \Gamma_h$ we have $|d(x)| = |d(x) - I(d_h)(x)| \leq c\,h^2$, and thus (2.4) is satisfied. Instead of (2.5), (2.6) we consider the sufficient condition (2.7). We assume $\|\nabla d(x) - \nabla I(d_h)(x)\| \leq c\,h$ for all x in a neighborhood of Γ (x not on an edge), which again is reasonable for a smooth d and

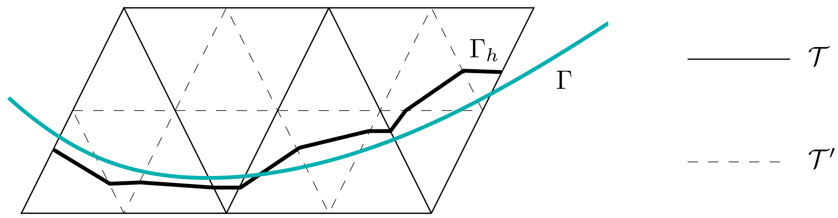

FIG. 2.1. *Construction of approximate interface for the two-dimensional case.*

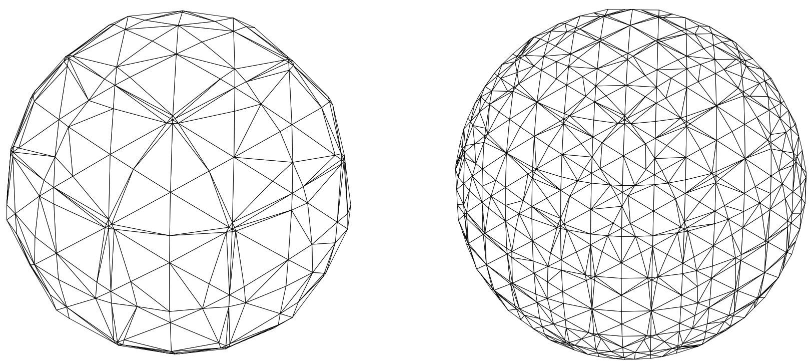

FIG. 2.2. *Approximate interface* Γ_h *for the example from section* 6 *on a coarse grid (left) and after one refinement (right).*

piecewise quadratic d_h. Due to $\|\nabla d\| = 1$ we then also have $\|\nabla I(d_h)(x)\| = 1 + \mathcal{O}(h)$ in a neighborhood of Γ. For $x \in \Gamma_h$ (not on an edge) we obtain

$$\|\mathbf{n}_h(x) - \mathbf{n}(x)\| = \Big\| \frac{\nabla I(d_h)(x)}{\|\nabla I(d_h)(x)\|} - \nabla d(x) \Big\|$$

$$\leq \Big| \frac{1}{\|\nabla I(d_h)(x)\|} - 1 \Big| \cdot \|\nabla I(d_h)(x)\| + \|\nabla I(d_h)(x) - \nabla d(x)\| \leq c\, h,$$

and thus (2.7) is satisfied (for h sufficiently small).

Given an approximate interface Γ_h, the localized force term $f_\Gamma(\mathbf{v}_H)$ is approximated by

$$\tilde{f}(\mathbf{v}_H) = f_{\Gamma_h}(\mathbf{v}_H) := \tau \int_{\Gamma_h} \nabla_{\Gamma_h} \mathrm{id}_{\Gamma_h} \cdot \nabla_{\Gamma_h} \mathbf{v}_H \, ds, \qquad \mathbf{v}_H \in \mathbf{V}_H. \tag{2.8}$$

Under the assumptions (2.4)–(2.6) on the family $\{\Gamma_h\}_{h>0}$ in section 4 we will derive a bound for the approximation error

$$\sup_{\mathbf{v}_H \in \mathbf{V}_H} \frac{f_\Gamma(\mathbf{v}_H) - f_{\Gamma_h}(\mathbf{v}_H)}{\|\mathbf{v}_H\|_1} \quad \text{with} \quad f_{\Gamma_h}(\mathbf{v}_H) \quad \text{as in (2.8)}. \tag{2.9}$$

Remark 3. From Theorem 2.1, the fact that $f_\Gamma(\mathbf{v}) = \tau \int_\Gamma \mathcal{K}\mathbf{v} \cdot \mathbf{n} \, ds$ is a bounded linear functional on $\mathbf{V}$, and a density argument, it follows that the linear functional

$$f_\Gamma : \ \mathbf{v} \to \tau \int_\Gamma \nabla_\Gamma \mathrm{id}_\Gamma \cdot \nabla_\Gamma \mathbf{v} \, ds \ , \qquad \mathbf{v} \in \big(C_0^\infty(\Omega)\big)^3, \tag{2.10}$$

has a unique bounded extension to $\mathbf{V}$. Therefore, for $f_\Gamma : \mathbf{V} \to \mathbb{R}$ we can use both the representation in (1.4) and the one in (2.10) (these are the same on a dense subset). This, however, is *not* the case for f_{Γ_h}. Because Γ_h is not sufficiently smooth, a partial integration result as in Theorem 2.1 does not hold. The linear functional

$$\mathbf{v} \to \tau \int_{\Gamma_h} \nabla_{\Gamma_h} \mathrm{id}_{\Gamma_h} \cdot \nabla_{\Gamma_h} \mathbf{v} \, ds$$

is *not* necessarily bounded on $\mathbf{V}$. For this reason the restriction to $\mathbf{v}_H$ from the finite element space $\mathbf{V}_H$ in (2.9) is essential.

Remark 4. At many places in this section, for example in (2.2), (2.3) and (implicitly) in (2.4), and also in the analysis presented in the next section, the assumption that Γ is a C^2 smooth interface plays a crucial role. We do not know of any literature in which smoothness properties of the interface are analyzed for a Navier–Stokes incompressible two-phase flow problem with surface tension. In [2] and [1] a two-phase Stokes flow problem without surface tension, in which the evolution is driven by the gravity force, is analyzed. In [2] it is proved that if the initial configuration has a C^2 smooth interface $\Gamma = \Gamma(0)$, then for arbitrary finite time $t > 0$ the interface $\Gamma(t)$ is a surface of class $C^{2-\varepsilon}$ for arbitrary $\varepsilon \in (0, 2]$. In [1] it is shown that if $\Gamma(0)$ is a $C^{2+\ell}$ smooth surface, with $\ell > 0$, then $\Gamma(t)$ is of class $C^{2+\ell}$, too, for all $t \in [0, T]$ and $T > 0$ sufficiently small.

3. Preliminaries. In this section we collect some results that will be used in the analysis in section 4. The techniques that we use come from the paper [8]. For proofs of certain results we will refer to that paper.

We introduce a locally (in a neighborhood of Γ) orthogonal coordinate system by using the projection $\mathbf{p} : U \to \Gamma$:

$$\mathbf{p}(x) = x - d(x)\mathbf{n}(x) \quad \text{for all} \quad x \in U.$$

We assume that the decomposition $x = \mathbf{p}(x) + d(x)\mathbf{n}(x)$ is unique for all $x \in U$. Note that

$$\mathbf{n}(x) = \mathbf{n}(\mathbf{p}(x)) \quad \text{for all} \quad x \in U.$$

We use an extension operator defined as follows. For a (scalar) function v defined on Γ we define

$$v_\Gamma^e(x) := v(x - d(x)\mathbf{n}(x)) = v(\mathbf{p}(x)) \quad \text{for all} \quad x \in U;$$

i.e., v is extended along normals on Γ. We will also need extensions of functions defined on Γ_h to U. This is done again by extending along normals $\mathbf{n}(x)$. For v defined on Γ_h we define, for $x \in \Gamma_h$,

$$v_{\Gamma_h}^e(x + \alpha\mathbf{n}(x)) := v(x) \quad \text{for all} \quad \alpha \in \mathbb{R} \quad \text{with} \quad x + \alpha\mathbf{n}(x) \in U. \tag{3.1}$$

The projection $\mathbf{p}$ and the extensions v_Γ^e, $v_{\Gamma_h}^e$ are illustrated in Figure 3.1.

We define a discrete analogue of the orthogonal projection $\mathbf{P}$:

$$\mathbf{P}_h(x) := \mathbf{I} - \mathbf{n}_h(x)\mathbf{n}_h(x)^T \quad \text{for} \quad x \in \Gamma_h, \ x \text{ not on an edge.}$$

The tangential derivative along Γ_h can be written as $\nabla_{\Gamma_h} g = \mathbf{P}_h \nabla g$. In the analysis a further technical assumption is used, namely that the neighborhood U of Γ is

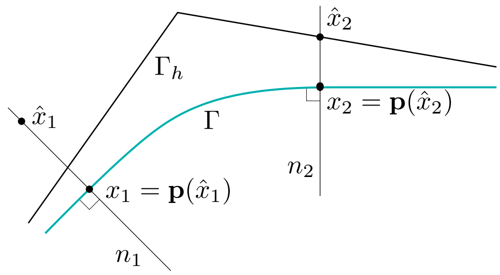

FIG. 3.1. *Example for projection* $\mathbf{p}$ *and construction of extension operators.* n_1 *and* n_2 *are straight lines perpendicular to* Γ*. For* v *defined on* Γ *we have* $v_\Gamma^e \equiv v(x_1)$ *on* n_1*. For* v_h *defined on* Γ_h *we have* $v_{\Gamma_h}^e \equiv v_h(\hat{x}_2)$ *on* n_2*.*

sufficiently small in the following sense. We assume that U is a strip of width $\delta > 0$ with

$$\delta^{-1} > \max_{i=1,2} \|\kappa_i(x)\|_{L^\infty(\Gamma)}. \tag{3.2}$$

Assumption 1. In the remainder of the paper we assume that (2.4), (2.5), (2.6), and (3.2) hold.

We present two lemmas from [8]. Proofs are elementary and can be found in [8].

LEMMA 3.1. *For the projection operator* $\mathbf{P}$ *and the Hessian* $\mathbf{H}$ *the relation*

$$\mathbf{P}(x)\mathbf{H}(x) = \mathbf{H}(x)\mathbf{P}(x) = \mathbf{H}(x) \quad \textit{for all} \quad x \in U$$

holds. For v *defined on* Γ *and sufficiently smooth the following holds:*

$$\nabla_{\Gamma_h} v_\Gamma^e(x) = \mathbf{P}_h(x)\big(\mathbf{I} - d(x)\mathbf{H}(x)\big)\mathbf{P}(x)\nabla_\Gamma v(\mathbf{p}(x)) \quad \textit{a.e. on} \quad \Gamma_h. \tag{3.3}$$

Proof. The proof is given in section 2.3 in [8]. ∎

In (3.3) (and also below) we have the result "a.e. on Γ_h" because quantities (derivatives, $\mathbf{P}_h$, etc.) are not well defined on the edges of the triangulation Γ_h.

LEMMA 3.2. *For* $x \in \Gamma_h$ *(not on an edge) define*

$$\mu(x) = \big[\Pi_{i=1}^2 (1 - d(x)\kappa_i(x))\big]\mathbf{n}(x)^T\mathbf{n}_h(x), \tag{3.4}$$

$$\mathbf{A}(x) = \frac{1}{\mu(x)}\mathbf{P}(x)\big[\mathbf{I} - d(x)\mathbf{H}(x)\big]\mathbf{P}_h(x)\big[\mathbf{I} - d(x)\mathbf{H}(x)\big]\mathbf{P}(x). \tag{3.5}$$

Let $\mathbf{A}_{\Gamma_h}^e$ *be the extension of* $\mathbf{A}$ *as in* (3.1)*. The following identity holds for functions* v *and* ψ *that are defined on* Γ_h *and sufficiently smooth:*

$$\int_{\Gamma_h} \nabla_{\Gamma_h} v \cdot \nabla_{\Gamma_h} \psi \, ds = \int_\Gamma \mathbf{A}_{\Gamma_h}^e \nabla_\Gamma v_{\Gamma_h}^e \cdot \nabla_\Gamma \psi_{\Gamma_h}^e \, ds. \tag{3.6}$$

Proof. The proof is given in section 2.3 in [8]. ∎

Due to the assumptions in (2.5) and (3.2) we have $\operatorname{ess\,inf}_{x\in\Gamma_h} \mu(x) > 0$, and thus $\mathbf{A}(x)$ is well defined.

We now derive two further results that are needed in the analysis in section 4.

LEMMA 3.3. *There exists a constant* c *independent of* h *such that*

$$\|\nabla_\Gamma v_{\Gamma_h}^e\|_{L^2(\Gamma)} \le c\,\|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} \quad \textit{for all} \quad v \in H^1(\Gamma_h) \cap C(\Gamma_h).$$

Proof. Due to Lemma 3.2 we have

$$\|\nabla_{\Gamma_h} v\|^2_{L^2(\Gamma_h)} = \int_\Gamma \mathbf{A}^e_{\Gamma_h}(y)\nabla_\Gamma v^e_{\Gamma_h}(y)\cdot\nabla_\Gamma v^e_{\Gamma_h}(y)\,ds(y)$$

with $ds(y)$ the surface measure on Γ. Take $x\in\Gamma_h$ with $\mathbf{p}(x)=y$. If x does not lie on an edge, we have $\mathbf{A}^e_{\Gamma_h}(y)=\mathbf{A}(x)$ with $\mathbf{A}(x)$ as in (3.5). We drop the symbol x in the notation and write $\mathbf{A}(x)=\mathbf{A}=\frac{1}{\mu}\mathbf{P}(\mathbf{I}-d\mathbf{H})\mathbf{P}_h(\mathbf{I}-d\mathbf{H})\mathbf{P}$. Decompose $\mathbf{n}_h$ as $\mathbf{n}_h=\alpha\mathbf{n}+\beta\mathbf{n}^\perp$ with $\|\mathbf{n}^\perp\|=1$ and $\mathbf{n}^T\mathbf{n}^\perp=0$. From (2.5) is follows that $\alpha\geq c>0$ and thus $\beta^2\leq 1-c^2<1$. Take $z\in\mathrm{range}(\mathbf{P})$ with $\|z\|=1$. We then have $\|\mathbf{P}_h z\|\geq\|z\|-|z^T\mathbf{n}_h|=\|z\|-|\beta||z^T\mathbf{n}^\perp|\geq(1-|\beta|)\|z\|$. Hence, there is a constant $c>0$ such that

$$\|\mathbf{P}_h\mathbf{P}w\|\geq c\,\|\mathbf{P}w\|\quad\text{for all}\;\; w\in\mathbb{R}^3.$$

Using (3.2) it follows that there is a constant $c>0$ such that $\|(\mathbf{I}-d\mathbf{H})w\|\geq c\,\|w\|$ for all $w\in\mathbb{R}^3$. Note that $\mu=\mu(x)\geq c>0$ holds. From these results we obtain, using $\mathbf{PH}=\mathbf{HP}$ (Lemma 3.1),

$$\begin{aligned} w^T\mathbf{A}w &= \frac{1}{\mu}w^T\mathbf{P}(\mathbf{I}-d\mathbf{H})\mathbf{P}_h(\mathbf{I}-d\mathbf{H})\mathbf{P}w \\ &= \frac{1}{\mu}\|\mathbf{P}_h\mathbf{P}(\mathbf{I}-d\mathbf{H})w\|^2\geq c\,\|\mathbf{P}w\|^2\quad\text{for all}\;\; w\in\mathbb{R}^3, \end{aligned}$$

with a constant $c>0$. This yields, using $\mathbf{n}(x)=\mathbf{n}(\mathbf{p}(x))=\mathbf{n}(y)$,

$$\mathbf{A}^e_{\Gamma_h}(y)w\cdot w=w^T\mathbf{A}(x)w\geq c\,\|\mathbf{P}(x)w\|^2=\|\mathbf{P}(y)w\|^2,$$

with $c>0$. For $w=\nabla_\Gamma v^e_{\Gamma_h}(y)$ we have $\mathbf{P}(y)w=\mathbf{P}(y)\nabla_\Gamma v^e_{\Gamma_h}(y)=\nabla_\Gamma v^e_{\Gamma_h}(y)$, and thus we get

$$\begin{aligned} \|\nabla_{\Gamma_h} v\|^2_{L^2(\Gamma_h)} &= \int_\Gamma \mathbf{A}^e_{\Gamma_h}(y)\nabla_\Gamma v^e_{\Gamma_h}(y)\cdot\nabla_\Gamma v^e_{\Gamma_h}(y)\,ds(y) \\ &\geq c\int_\Gamma \nabla_\Gamma v^e_{\Gamma_h}(y)\cdot\nabla_\Gamma v^e_{\Gamma_h}(y)\,ds(y)=c\,\|\nabla_\Gamma v^e_{\Gamma_h}\|^2_{L^2(\Gamma)}, \end{aligned}$$

with a constant $c>0$. ∎

Lemma 3.4. *The following holds:*

$$\text{ess sup}_{y\in\Gamma}\|\big(\mathbf{A}^e_{\Gamma_h}(y)-\mathbf{I}\big)\mathbf{P}(y)\|\leq ch^2.$$

Proof. Take $y\in\Gamma$ and a corresponding $x\in\Gamma_h$ such that $\mathbf{p}(x)=y$. Assume that x does not lie on an edge of the triangulation Γ_h, which is true for almost all $y\in\Gamma$. Then we have

$$(\mathbf{A}^e_{\Gamma_h}(y)-\mathbf{I})\mathbf{P}(y)=(\mathbf{A}(x)-\mathbf{I})\mathbf{P}(x).$$

We drop the symbol x in the notation and write $\mathbf{A}(x)=\mathbf{A}=\frac{1}{\mu}\mathbf{P}(\mathbf{I}-d\mathbf{H})\mathbf{P}_h(\mathbf{I}-d\mathbf{H})\mathbf{P}$. Note that $|\mu|=\mu(x)\geq c>0$ holds. Decompose $\mathbf{n}_h$ as $\mathbf{n}_h=\alpha\mathbf{n}+\beta\mathbf{n}^\perp$ with $\|\mathbf{n}^\perp\|=1$ and $\mathbf{n}^T\mathbf{n}^\perp=0$. Due to (2.5) we have $\alpha=\mathbf{n}^T\mathbf{n}_h\geq c>0$. From (2.6) we get $\|\mathbf{P}\mathbf{n}_h\|=|\beta|\leq ch$. Hence,

$$|\mathbf{n}^T\mathbf{n}_h-1|=1-\alpha=\frac{1-\alpha^2}{1+\alpha}\leq 1-\alpha^2=\beta^2\leq c\,h^2. \tag{3.7}$$

Using this and $|d(x)| \leq ch^2$, $|\kappa_i(x)| \leq c$, we obtain $|\mu - 1| \leq c\,h^2$. Thus

$$\Big|\frac{1}{\mu} - 1\Big| = \frac{|\mu - 1|}{\mu} \leq c\,h^2 \tag{3.8}$$

holds. We have

$$\begin{aligned}(\mathbf{A} - \mathbf{I})\mathbf{P} &= \frac{1}{\mu}\mathbf{P}(\mathbf{I} - d\mathbf{H})\mathbf{P}_h(\mathbf{I} - d\mathbf{H})\mathbf{P} - \mathbf{P} \\ &= \left[\left(\frac{1}{\mu} - 1\right)\mathbf{P}(\mathbf{I} - d\mathbf{H})\mathbf{P}_h(\mathbf{I} - d\mathbf{H})\mathbf{P}\right] + \Big[\mathbf{P}(\mathbf{I} - d\mathbf{H})\mathbf{P}_h(\mathbf{I} - d\mathbf{H})\mathbf{P} - \mathbf{P}\Big]\end{aligned}$$

and consider the two terms on the right-hand side separately. For the first term we get, using (3.8),

$$\left\|\left(\frac{1}{\mu} - 1\right)\mathbf{P}(\mathbf{I} - d\mathbf{H})\mathbf{P}_h(\mathbf{I} - d\mathbf{H})\mathbf{P}\right\| \leq \left|\frac{1}{\mu} - 1\right|(1 + c\,h^2)(1 + c\,h^2) \leq c\,h^2.$$

For the second term we obtain, using (2.6),

$$\begin{aligned}&\|\mathbf{P}(\mathbf{I} - d\mathbf{H})\mathbf{P}_h(\mathbf{I} - d\mathbf{H})\mathbf{P} - \mathbf{P}\| \leq \|\mathbf{P}\mathbf{P}_h\mathbf{P} - \mathbf{P}\| + c\,h^2 \\ &= \|\mathbf{P}\mathbf{n}_h\mathbf{n}_h^T\mathbf{P}\| + c\,h^2 = \|\mathbf{P}\mathbf{n}_h\|^2 + c\,h^2 \leq c\,h^2.\end{aligned}$$

Combination of these bounds completes the proof. ∎

4. Approximation error analysis. We are interested in the difference between the terms

$$\tau\int_\Gamma \nabla_\Gamma \operatorname{id}_\Gamma \cdot \nabla_\Gamma \mathbf{v}_H\,ds \quad \text{and} \quad \tau\int_{\Gamma_h} \nabla_{\Gamma_h} \operatorname{id}_{\Gamma_h} \cdot \nabla_{\Gamma_h} \mathbf{v}_H\,ds \quad \text{for} \quad \mathbf{v}_H \in \mathbf{V}_H.$$

Since $\nabla_\Gamma \operatorname{id}_\Gamma \cdot \nabla_\Gamma \mathbf{v}_H = \sum_{i=1}^3 \nabla_\Gamma (\operatorname{id}_\Gamma)_i \cdot \nabla_\Gamma (\mathbf{v}_H)_i$ we consider only one term in this sum, say the ith. We write id_Γ and v for the *scalar* functions $(\operatorname{id}_\Gamma)_i$ and $(\mathbf{v}_H)_i$, respectively. We write $\operatorname{id}_{\Gamma_h}$ for $(\operatorname{id}_{\Gamma_h})_i$. Note that

$$\nabla_\Gamma \operatorname{id}_\Gamma = \mathbf{P}\nabla \operatorname{id}_\Gamma = \mathbf{P}e_i, \quad \nabla_{\Gamma_h} \operatorname{id}_{\Gamma_h} = \mathbf{P}_h \nabla \operatorname{id}_{\Gamma_h} = \mathbf{P}_h e_i,$$

with e_i the ith basis vector in $\mathbb{R}^3$. We introduce scalar versions of the functionals f_Γ and f_{Γ_h} defined in (2.10) and (2.8) (without loss of generality we can take $\tau := 1$):

$$g(v) := \int_\Gamma \nabla_\Gamma \operatorname{id}_\Gamma \cdot \nabla_\Gamma v\,ds, \quad g_h(v) := \int_{\Gamma_h} \nabla_{\Gamma_h} \operatorname{id}_{\Gamma_h} \cdot \nabla_{\Gamma_h} v\,ds.$$

As noted in Remark 3, g is a bounded linear functional on $H^1(U)$. To guarantee that g_h and the extension operator in (3.1) are well defined we assume $v \in H^1(\Gamma_h) \cap C(\Gamma_h)$. Therefore, in the analysis in this section we use the subspace W of $H^1(U)$ consisting of functions whose restriction to Γ_h belongs to $H^1(\Gamma_h) \cap C(\Gamma_h)$.

Remark 5. If we use a Hood–Taylor pair $\mathbf{V}_H \times Q_H$ in the discretization of the Navier–Stokes equations, then the ith component $v \in V_H$ of $\mathbf{v}_H \in \mathbf{V}_H = (V_H)^3$ is continuous and piecewise polynomial (on the tetrahedral triangulation $\mathcal{S}$). Thus $v \in W$ holds.

In this section we first derive, for $v \in W$, a bound for $|g(v) - g_h(v)|$ in terms of $\|v\|_{1,U} := \|v\|_{H^1(U)}$ and $\|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)}$. This bound is given in Corollary 4.4. Using this bound we then derive a bound for

$$\sup_{v \in V_H} \frac{g(v) - g_h(v)}{\|v\|_1};$$

cf. Theorem 4.7. This immediately implies a bound for the approximation error as in (2.9); cf. Corollary 4.8.

The analysis is based on the following splitting:

(4.1)

$$\begin{aligned}
&g(v) - g_h(v) \\
&= \int_\Gamma \nabla_\Gamma \operatorname{id}_\Gamma \cdot \nabla_\Gamma v \, ds - \int_{\Gamma_h} \nabla_{\Gamma_h} \operatorname{id}_\Gamma^e \cdot \nabla_{\Gamma_h} v \, ds + \int_{\Gamma_h} \nabla_{\Gamma_h} (\operatorname{id}_\Gamma^e - \operatorname{id}_{\Gamma_h}) \cdot \nabla_{\Gamma_h} v \, ds \\
&\overset{(3.6)}{=} \int_\Gamma \nabla_\Gamma \operatorname{id}_\Gamma \cdot \nabla_\Gamma v \, ds - \int_\Gamma \mathbf{A}_{\Gamma_h}^e \nabla_\Gamma \operatorname{id}_\Gamma \cdot \nabla_\Gamma v_{\Gamma_h}^e \, ds + \int_{\Gamma_h} \nabla_{\Gamma_h} (\operatorname{id}_\Gamma^e - \operatorname{id}_{\Gamma_h}) \cdot \nabla_{\Gamma_h} v \, ds \\
&= \int_\Gamma \nabla_\Gamma \operatorname{id}_\Gamma \cdot \nabla_\Gamma (v - v_{\Gamma_h}^e) \, ds + \int_\Gamma (\mathbf{I} - \mathbf{A}_{\Gamma_h}^e) \nabla_\Gamma \operatorname{id}_\Gamma \cdot \nabla_\Gamma v_{\Gamma_h}^e \, ds \\
&\quad + \int_{\Gamma_h} \nabla_{\Gamma_h} (\operatorname{id}_\Gamma^e - \operatorname{id}_{\Gamma_h}) \cdot \nabla_{\Gamma_h} v \, ds.
\end{aligned}$$

In the lemmas below we derive bounds for the three terms in (4.1). Note that the first two terms do *not* involve $\operatorname{id}_{\Gamma_h}$.

Lemma 4.1. *The following holds:*

$$\left| \int_\Gamma (\mathbf{I} - \mathbf{A}_{\Gamma_h}^e) \nabla_\Gamma \operatorname{id}_\Gamma \cdot \nabla_\Gamma v_{\Gamma_h}^e \, ds \right| \le c\, h^2 \, \|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} \quad \textit{for all} \ \ v \in W.$$

Proof. Using the Cauchy–Schwarz inequality and the results in Lemmas 3.3 and 3.4 we obtain

$$\begin{aligned}
&\left| \int_\Gamma (\mathbf{I} - \mathbf{A}_{\Gamma_h}^e) \nabla_\Gamma \operatorname{id}_\Gamma \cdot \nabla_\Gamma v_{\Gamma_h}^e \, ds \right| = \left| \int_\Gamma (\mathbf{I} - \mathbf{A}_{\Gamma_h}^e) \mathbf{P} \nabla_\Gamma \operatorname{id}_\Gamma \cdot \nabla_\Gamma v_{\Gamma_h}^e \, ds \right| \\
&\le \operatorname{ess\,sup}_{y \in \Gamma} \|(\mathbf{I} - \mathbf{A}_{\Gamma_h}^e(y)) \mathbf{P}(y)\| \, \|\nabla_\Gamma \operatorname{id}_\Gamma\|_{L^2(\Gamma)} \, \|\nabla_\Gamma v_{\Gamma_h}^e\|_{L^2(\Gamma)} \\
&\le c\, h^2 \, \|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)},
\end{aligned}$$

and thus the result holds. ◻

Lemma 4.2. *The following holds:*

$$\left| \int_{\Gamma_h} \nabla_{\Gamma_h} (\operatorname{id}_\Gamma^e - \operatorname{id}_{\Gamma_h}) \cdot \nabla_{\Gamma_h} v \, ds \right| \le c\, h \|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} \quad \textit{for all} \ \ v \in W.$$

Proof. From Lemma 3.1 we get for $x \in \Gamma_h$ (not on an edge),

$$\begin{aligned}
\nabla_{\Gamma_h} \operatorname{id}_\Gamma^e(x) &= \mathbf{P}_h(x) \big(\mathbf{I} - d(x) \mathbf{H}(x)\big) \mathbf{P}(x) \nabla_\Gamma \operatorname{id}_\Gamma(\mathbf{p}(x)) \\
&= \mathbf{P}_h(x) \big(\mathbf{I} - d(x) \mathbf{H}(x)\big) \mathbf{P}(x) e_i.
\end{aligned}$$

We also have $\nabla_{\Gamma_h} \mathrm{id}_{\Gamma_h} = \mathbf{P}_h \nabla \mathrm{id}_{\Gamma_h} = \mathbf{P}_h e_i$. Hence,

$$
\begin{aligned}
(4.2)\qquad & \left| \int_{\Gamma_h} \nabla_{\Gamma_h} (\mathrm{id}_\Gamma^e - \mathrm{id}_{\Gamma_h}) \cdot \nabla_{\Gamma_h} v \, ds \right| \\
& = \left| \int_{\Gamma_h} \big(\mathbf{P}_h (\mathbf{I} - d\mathbf{H}) \mathbf{P} e_i - \mathbf{P}_h e_i \big) \cdot \nabla_{\Gamma_h} v \, ds \right| \\
& \le c \operatorname{ess\,sup}_{x \in \Gamma_h} \| \mathbf{P}_h(x) \big(\mathbf{I} - d(x) \mathbf{H}(x) \big) \mathbf{P}(x) - \mathbf{P}_h(x) \| \, \| \nabla_{\Gamma_h} v \|_{L^2(\Gamma_h)} \\
(4.3)\qquad & \le c \operatorname{ess\,sup}_{x \in \Gamma_h} \big(\| \mathbf{P}_h(x) \mathbf{P}(x) - \mathbf{P}_h(x) \| \\
(4.4)\qquad & \qquad + |d(x)| \| \mathbf{P}_h(x) \mathbf{H}(x) \mathbf{P}(x) \| \big) \| \nabla_{\Gamma_h} v \|_{L^2(\Gamma_h)}.
\end{aligned}
$$

Note that $|d(x)| \le c\, h^2$ for $x \in \Gamma_h$, and

$$
\operatorname{ess\,sup}_{x \in \Gamma_h} \| \mathbf{P}_h(x) \mathbf{H}(x) \mathbf{P}(x) \| \le \operatorname{ess\,sup}_{x \in \Gamma_h} \| \mathbf{H}(x) \| \le c.
$$

For the term in (4.3) we have (we drop x in the notation)

$$
\| \mathbf{P}_h \mathbf{P} - \mathbf{P}_h \| = \| \mathbf{P}_h \mathbf{n} \mathbf{n}^T \| \le \| \mathbf{P}_h \mathbf{n} \| \le \| \mathbf{P}_h \mathbf{n} + \mathbf{P} \mathbf{n}_h \| + \| \mathbf{P} \mathbf{n}_h \|.
$$

For the first term we get, using (3.7),

$$
\| \mathbf{P}_h \mathbf{n} + \mathbf{P} \mathbf{n}_h \| = \big\| (1 - \mathbf{n}^T \mathbf{n}_h)(\mathbf{n} + \mathbf{n}_h) \big\| \le 2 |1 - \mathbf{n}^T \mathbf{n}_h| \le c\, h^2.
$$

From (2.6) we get $\| \mathbf{P} \mathbf{n}_h \| \le c\, h$ (a.e. on Γ_h). Thus $\| \mathbf{P}_h(x) \mathbf{P}(x) - \mathbf{P}_h(x) \| \le c\, h$ holds a.e. on Γ_h. As an upper bound for (4.2) we obtain $ch \, \| \nabla_{\Gamma_h} v \|_{L^2(\Gamma_h)}$. ∎

LEMMA 4.3. *The following holds:*

$$
\left| \int_\Gamma \nabla_\Gamma \mathrm{id}_\Gamma \cdot \nabla_\Gamma (v - v_{\Gamma_h}^e) \, ds \right| \le ch \|v\|_{1,U} \quad \text{for all} \quad v \in W.
$$

Proof. We take $v \in C^1(U)$. For $y \in \Gamma$ we have $v_{\Gamma_h}^e(y) = v(y \pm \delta(y) \mathbf{n}(y))$ with a unique $\delta(y) \ge 0$ such that $y \pm \delta(y) \mathbf{n}(y) \in \Gamma_h$. Note that $\delta(y) \le c\, h^2$ holds. Let $U_m \subset U$ be a strip around Γ that contains Γ_h and has width $m \le c\, h^2$. We now have

$$
\begin{aligned}
\left| \int_\Gamma \nabla_\Gamma \mathrm{id}_\Gamma \cdot \nabla_\Gamma (v - v_{\Gamma_h}^e) \, ds(y) \right| & = \left| \int_\Gamma \Delta_\Gamma \mathrm{id}_\Gamma \, \big(v(y) - v(y \pm \delta(y) \mathbf{n}(y)) \big) \, ds(y) \right| \\
& \le \int_\Gamma |\Delta_\Gamma \mathrm{id}_\Gamma| \left| \int_0^{\delta(y)} \frac{\partial v}{\partial t} (y \pm t \mathbf{n}(y)) \, dt \right| \, ds(y) \\
& \le c \int_\Gamma \int_0^{\delta(y)} \left| \frac{\partial v}{\partial t} (y \pm t \mathbf{n}(y)) \right| \, dt \, ds(y).
\end{aligned}
$$

For $x = y \pm t \mathbf{n}(y)$ with $0 \le t \le \delta(y)$ we use $\mathbf{n}(x) = \mathbf{n}(\mathbf{p}(x)) = \mathbf{n}(y)$ and obtain

$$
\begin{aligned}
\left| \int_\Gamma \nabla_\Gamma \mathrm{id}_\Gamma \cdot \nabla_\Gamma (v - v_{\Gamma_h}^e) \, ds(y) \right| & \le c \int_{U_m} |\mathbf{n}(x) \cdot \nabla v(x)| \, dx \\
& \le c \left(\int_{U_m} 1 \, dx \right)^{\frac{1}{2}} \left(\int_{U_m} (\nabla v)^2 \, dx \right)^{\frac{1}{2}} \le c\, h \, \|v\|_{1,U}.
\end{aligned}
$$

A density argument yields the same bound for all $v \in W$. □

A direct consequence of the previous three lemmas is the following corollary.

Corollary 4.4. *The three terms in* (4.1) *can be bounded by*

$$\left| \int_\Gamma \nabla_\Gamma \, \mathrm{id}_\Gamma \cdot \nabla_\Gamma (v - v^e_{\Gamma_h}) \, ds \right| \le c\, h\, \|v\|_{1,U}, \tag{4.5}$$

$$\left| \int_\Gamma (\mathbf{I} - \mathbf{A}^e_{\Gamma_h}) \nabla_\Gamma \, \mathrm{id}_\Gamma \cdot \nabla_\Gamma v^e_{\Gamma_h} \, ds \right| \le c\, h^2\, \|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)}, \tag{4.6}$$

$$\left| \int_{\Gamma_h} \nabla_{\Gamma_h} (\mathrm{id}^e_\Gamma - \mathrm{id}_{\Gamma_h}) \cdot \nabla_{\Gamma_h} v \, ds \right| \le c\, h\, \|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)}, \tag{4.7}$$

and thus

$$|g(v) - g_h(v)| \le c\, h\, \|v\|_{1,U} + c\, h^2\, \|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} + c\, h\, \|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} \quad \textit{for all} \;\; v \in W$$

holds.

In view of Corollary 4.4 and the error measure in (2.9), we want to derive a bound for $\|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)}$ in terms of $\|v\|_1$ for v from the scalar finite element space V_H. An obvious approach is to apply an inverse inequality combined with a trace theorem, resulting in

$$\|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} \le c\, h_{\min}^{-1} \|v\|_{L^2(\Gamma_h)} \le c\, h_{\min}^{-1} \|v\|_1 \quad \text{for all} \;\; v \in V_H. \tag{4.8}$$

This, however, is too crude (cf. the bound in Corollary 4.4). To derive a better bound than the one in (4.8) we have to introduce some further assumptions related to the family of triangulations $\{\Gamma_h\}_{h>0}$. We assume that to each triangulation $\Gamma_h = \cup_{T \in \mathcal{F}_h} T$ there can be associated a set of tetrahedra $\mathcal{S}_h$ with the following properties:

(4.9) For each $T \in \mathcal{F}_h$ there is a corresponding $S_T \in \mathcal{S}_h$ with $T \subset S_T$.

(4.10) For $T_1, T_2 \in \mathcal{F}_h$ with $T_1 \neq T_2$ we have $\mathrm{meas}_3(S_{T_1} \cap S_{T_2}) = 0$.

(4.11) The family $\{\mathcal{S}_h\}_{h>0}$ is shape-regular.

(4.12) $c_0 h \le \mathrm{diam}(S_T) \le ch$ for all $T \in \mathcal{F}_h$ with $c_0 > 0$ (quasi-uniformity).

(4.13) For each $S_T \in \mathcal{S}_h$ there is a tetrahedron $S \in \mathcal{S}$ such that $S_T \subset S$.

Recall that $\mathcal{S}$ is the (fixed) tetrahedral triangulation that is used in the finite element discretization of the Navier–Stokes problem in (1.6). Note that the set of tetrahedra $\mathcal{S}_h$ has to be defined only close to the approximate interface Γ_h and that this set does not necessarily form a regular tetrahedral triangulation of Ω. Furthermore, it is *not* assumed that the family $\{\Gamma_h\}_{h>0}$is shape-regular or quasi-uniform.

Remark 6. Consider the construction of $\{\Gamma_h\}_{h>0}$ as in Remark 2. The approximate interface Γ_h is the zero level of the function $I(d_h)$, which is continuous piecewise linear on the tetrahedral triangulation $\mathcal{T}'$:

$$\Gamma_h = \cup_T T.$$

Each T is a triangle or a quadrilateral. To each T there can be associated a tetrahedron $S_T \in \mathcal{T}'$ such that $T \subset S_T$. If T is a quadrilateral, then we can subdivide T and

S_T in two disjoint triangles T_1, T_2 and two disjoint tetrahedra S_{T_1}, S_{T_2}, respectively, such that $T_i \subset S_{T_i} \subset S_T$ for $i = 1, 2$. One can check that this construction results in a family $\{\mathcal{S}_h\}_{h>0}$ that satisfies conditions (4.9)–(4.13).

In the following lemma we consider a standard affine mapping between a tetrahedron $S_T \in \mathcal{S}_h$ and the reference unit tetrahedron and apply it to the triangle $T \subset S_T$.

LEMMA 4.5. *Assume that the family $\{\Gamma_h\}_{h>0}$ is such that for the associated family of sets of tetrahedra $\{\mathcal{S}_h\}_{h>0}$ conditions* (4.9)*–*(4.13) *are satisfied. Take $T \in \mathcal{F}_h$ and the corresponding $S_T \in \mathcal{S}_h$. Let $\hat{S}$ be the reference unit tetrahedron and $F(x) = \mathbf{J}x + \mathbf{b}$ an affine mapping such that $F(\hat{S}) = S_T$. Define $\hat{T} := F^{-1}(T)$. The following holds:*

$$\|\mathbf{J}\|^2 \frac{\mathrm{meas}_3(\hat{S})}{\mathrm{meas}_3(S_T)} \leq c\, h^{-1}, \tag{4.14}$$

$$\|\mathbf{J}^{-1}\|^2 \frac{\mathrm{meas}_2(T)}{\mathrm{meas}_2(\hat{T})} \leq c, \tag{4.15}$$

with constants c independent of T and h.

Proof. Let $\rho(S_T)$ be the diameter of the maximal ball contained in S_T and similarly for $\rho(\hat{S})$. From standard finite element theory we have

$$\|\mathbf{J}\| \leq \frac{\mathrm{diam}(S_T)}{\rho(\hat{S})}, \quad \|\mathbf{J}^{-1}\| \leq \frac{\mathrm{diam}(\hat{S})}{\rho(S_T)}.$$

Using (4.11) and (4.12) we then get

$$\|\mathbf{J}\|^2 \frac{\mathrm{meas}_3(\hat{S})}{\mathrm{meas}_3(S_T)} \leq c\, \frac{\mathrm{diam}(S_T)^2}{\mathrm{meas}_3(S_T)} \leq c\, \mathrm{diam}(S_T)^{-1} \leq c\, h^{-1},$$

and thus the result in (4.14) holds.

The vertices of $\hat{T} = F^{-1}(T)$ are denoted by $\hat{V}_i$, $i = 1, 2, 3$. Let $\hat{V}_1\hat{V}_2$ be the longest edge of $\hat{T}$ and $\hat{M}$ the point on this edge such that $\hat{M}\hat{V}_3$ is perpendicular to $\hat{V}_1\hat{V}_2$. Define $V_i := F(\hat{V}_i)$, $i = 1, 2, 3$, and $M := F(\hat{M})$. Then V_i, $i = 1, 2, 3$, are the vertices of T and M lies on the edge V_1V_2. We then have

$$\begin{aligned}\mathrm{meas}_2(\hat{T}) &= \frac{1}{2}\|\hat{V}_1 - \hat{V}_2\|\|\hat{V}_3 - \hat{M}\| = \frac{1}{2}\|\mathbf{J}^{-1}(V_1 - V_2)\|\|\mathbf{J}^{-1}(V_3 - M)\| \\ &\geq \frac{1}{2}\|\mathbf{J}\|^{-2}\|V_1 - V_2\|\|V_3 - M\| \geq c\, \frac{\rho(\hat{S})^2}{\mathrm{diam}(S_T)^2}\, \mathrm{meas}_2(T),\end{aligned}$$

with a constant $c > 0$. Thus we obtain

$$\|\mathbf{J}^{-1}\|^2 \frac{\mathrm{meas}_2(T)}{\mathrm{meas}_2(\hat{T})} \leq c\, \frac{\mathrm{diam}(\hat{S})^2}{\rho(S_T)^2} \frac{\mathrm{diam}(S_T)^2}{\rho(\hat{S})^2} \leq c,$$

which completes the proof. $\square$

THEOREM 4.6. *Assume that the family $\{\Gamma_h\}_{h>0}$ is such that for the associated family of sets of tetrahedra $\{\mathcal{S}_h\}_{h>0}$ conditions* (4.9)*–*(4.13) *are satisfied. The following holds:*

$$\|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} \leq c\, h^{-\frac{1}{2}} \|v\|_1 \quad \textit{for all} \;\; v \in V_H.$$

Proof. Note that

$$\|\nabla_{\Gamma_h} v\|^2_{L^2(\Gamma_h)} = \sum_{T\in\mathcal{F}_h} \|\nabla_T v\|^2_{L^2(T)}.$$

Take $T \in \mathcal{F}_h$ and let S_T be the associated tetrahedron as explained above. Let $\hat{S}$ be the reference unit tetrahedron and $F : \hat{S} \to S_T$ as in Lemma 4.5. Define $\hat{v} := v \circ F$. Using standard transformation rules and Lemma 4.5 we get

$$\begin{aligned}
\|\nabla_T v\|^2_{L^2(T)} &= \|\mathbf{P}_h \nabla v\|^2_{L^2(T)} \le \|\nabla v\|^2_{L^2(T)} = \sum_{|\alpha|=1} \|\partial^\alpha v\|^2_{L^2(T)} \\
&\le c\,\|\mathbf{J}^{-1}\|^2 \sum_{|\alpha|=1} \|(\partial^\alpha \hat{v}) \circ F^{-1}\|^2_{L^2(T)} \\
&\le c\,\|\mathbf{J}^{-1}\|^2 \frac{\mathrm{meas}_2(T)}{\mathrm{meas}_2(\hat{T})} \sum_{|\alpha|=1} \|\partial^\alpha \hat{v}\|^2_{L^2(\hat{T})} \le c \sum_{|\alpha|=1} \|\partial^\alpha \hat{v}\|^2_{L^2(\hat{T})} \\
&\le c \sum_{|\alpha|=1} \max_{x\in\hat{T}} \left|\partial^\alpha \hat{v}(x)\right|^2 \le c \sum_{|\alpha|=1} \max_{x\in\hat{S}} \left|\partial^\alpha \hat{v}(x)\right|^2,
\end{aligned}$$

with a constant c independent of T. From (4.13) it follows that $\hat{v}$ is a polynomial on $\hat{S}$ of maximal degree k, where k depends only on the choice of the finite element space $\mathbf{V}_H$. On $P_k^* := \{\, p \in P_k \mid p(0) = 0 \,\}$ we have, due to equivalence of norms,

$$\sum_{|\alpha|=1} \max_{x\in\hat{S}} \left|\partial^\alpha \hat{v}(x)\right|^2 \le c \sum_{|\alpha|=1} \|\partial^\alpha \hat{v}\|^2_{L^2(\hat{S})} \quad \text{for all} \;\; \hat{v} \in P_k^*.$$

Because $\partial^\alpha \hat{v}$ is independent of $\hat{v}(0)$ for $\hat{v} \in P_k$ and $|\alpha| = 1$, the same inequality holds for all $\hat{v} \in P_k$. Thus we get

$$\begin{aligned}
\|\nabla_T v\|^2_{L^2(T)} &\le c \sum_{|\alpha|=1} \|\partial^\alpha \hat{v}\|^2_{L^2(\hat{S})} \le c\,\|\mathbf{J}\|^2 \sum_{|\alpha|=1} \|(\partial^\alpha v) \circ F\|^2_{L^2(\hat{S})} \\
&= c\,\|\mathbf{J}\|^2 \frac{\mathrm{meas}_3(\hat{S})}{\mathrm{meas}_3(S_T)} \sum_{|\alpha|=1} \|\partial^\alpha v\|^2_{L^2(S_T)} \le c\,h^{-1} \|\nabla v\|^2_{L^2(S_T)}\,,
\end{aligned}$$

with a constant c independent of T and h. Using (4.10) we finally obtain

$$\begin{aligned}
\|\nabla_{\Gamma_h} v\|^2_{L^2(\Gamma_h)} &\le c\,h^{-1} \sum_{T\in\mathcal{F}_h} \|\nabla v\|^2_{L^2(S_T)} \\
&\le c\,h^{-1} \int_\Omega (\nabla v)^2\, dx \le c\,h^{-1} \|v\|_1^2\,,
\end{aligned}$$

which proves the result. ∎

We now present the main result of this paper.

THEOREM 4.7. *Let the assumptions be as in Theorem* 4.6. *The following holds:*

$$\sup_{v\in V_H} \frac{g(v) - g_h(v)}{\|v\|_1} \le c\sqrt{h}.$$

Proof. Combine the result in Corollary 4.4 with that in Theorem 4.6. ∎

As a direct consequence we obtain the following.

COROLLARY 4.8. *Let the assumptions be as in Theorem* 4.6. *For* f_Γ *and* f_{Γ_h} *as defined in section* 2 *the following holds:*

$$\sup_{\mathbf{v}\in\mathbf{V}_H} \frac{f_\Gamma(\mathbf{v}_H) - f_{\Gamma_h}(\mathbf{v}_H)}{\|\mathbf{v}_H\|_1} \leq \tau c\sqrt{h}.$$

Proof. Note that

$$\begin{aligned} &f_\Gamma(\mathbf{v}_H) - f_{\Gamma_h}(\mathbf{v}_H) \\ &\quad = \tau \sum_{i=1}^{3} \left(\int_\Gamma \nabla_\Gamma(\mathrm{id}_\Gamma)_i \cdot \nabla_\Gamma(\mathbf{v}_H)_i \, ds - \int_{\Gamma_h} \nabla_{\Gamma_h}(\mathrm{id}_{\Gamma_h})_i \cdot \nabla_{\Gamma_h}(\mathbf{v}_H)_i \, ds \right), \end{aligned}$$

and use the result in Theorem 4.7. ∎

An upper bound $\mathcal{O}(\sqrt{h})$ as in Corollary 4.8 for the error in the approximation of the localized force term may seem rather pessimistic, because Γ_h is an $\mathcal{O}(h^2)$ accurate approximation of Γ. Numerical experiments in section 6, however, indicate that the bound is sharp.

5. Improved approximation of the localized force term $f_\Gamma(\mathbf{v}_h)$.

In this section we show how the approximation of the localized force term can be improved, resulting in an improved error bound of the form $\mathcal{O}(h)$ (instead of $\mathcal{O}(\sqrt{h})$).
From Corollary 4.4 and Theorem 4.6 we see that the $\sqrt{h}$ behavior is caused by the estimate in (4.7):

$$\left| \int_{\Gamma_h} \nabla_{\Gamma_h}(\mathrm{id}_\Gamma^e - \mathrm{id}_{\Gamma_h}) \cdot \nabla_{\Gamma_h} v \, ds \right| \leq c\, h\, \|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)}. \tag{5.1}$$

The term $\nabla_{\Gamma_h} \mathrm{id}_{\Gamma_h}$ that is used in $g_h(v)$ occurs in (5.1) but not in the other two terms of the splitting; cf. (4.5), (4.6). We consider

$$\tilde{g}_h(v) = \int_{\Gamma_h} m_h \cdot \nabla_{\Gamma_h} v \, ds$$

and try to find a function $m_h = m_h(x)$ such that $\tilde{g}_h(v)$ remains easily computable and the bound in (5.1) is improved if we use m_h instead of $\nabla_{\Gamma_h} \mathrm{id}_{\Gamma_h}$. The latter condition is trivially satisfied for $m_h = \nabla_{\Gamma_h} \mathrm{id}_\Gamma^e$ (leading to a bound 0 in (5.1)). This choice, however, does not satisfy the first condition, because Γ is not known. We now discuss another possibility that is used in the experiments in section 6.

Due to $|d(x)| \leq ch^2$ we get from Lemma 3.1, for $x \in \Gamma_h$:

$$\nabla_{\Gamma_h} \mathrm{id}_\Gamma^e(x) = \mathbf{P}_h(x)\mathbf{P}(x)\nabla_\Gamma \mathrm{id}_\Gamma(\mathbf{p}(x)) + \mathcal{O}(h^2) = \mathbf{P}_h(x)\mathbf{P}(x)e_i + \mathcal{O}(h^2).$$

In the construction of the interface Γ_h (cf. Remark 2), we have available a piecewise *quadratic* function $d_h \approx d$. Define

$$\tilde{\mathbf{n}}_h(x) := \frac{\nabla d_h(x)}{\|\nabla d_h(x)\|}, \qquad \tilde{\mathbf{P}}_h(x); = \mathbf{I} - \tilde{\mathbf{n}}_h(x)\tilde{\mathbf{n}}_h(x)^T, \quad x \in \Gamma_h.$$

Thus an obvious modification is based on the choice $m_h(x) = \mathbf{P}_h(x)\tilde{\mathbf{P}}_h(x)e_i$, i.e.,

$$\tilde{g}_h(v) := \int_{\Gamma_h} \mathbf{P}_h(x)\tilde{\mathbf{P}}_h(x)e_i \cdot \nabla_{\Gamma_h} v \, ds = \int_{\Gamma_h} \tilde{\mathbf{P}}_h(x)e_i \cdot \nabla_{\Gamma_h} v \, ds. \tag{5.2}$$

In this approach the approximate interface Γ_h is *not* changed (piecewise planar). For piecewise quadratics d_h and v, the function $\nabla_{\Gamma_h} v = \mathbf{P}_h \nabla v$ is piecewise linear and $\tilde{\mathbf{P}}_h e_i$ is piecewise (very) smooth on the segments of Γ_h. Hence, the functional in (5.2) can be evaluated easily.

Under reasonable assumptions the modified functional indeed yields a better error bound.

LEMMA 5.1. *Assume that there exists $p > 0$ such that*

$$\|\nabla d_h(x) - \nabla d(x)\| \leq c\, h^p \quad \textit{for} \ \ x \in \Gamma_h. \tag{5.3}$$

Then the following holds:

$$\left| \int_{\Gamma_h} \left(\nabla_{\Gamma_h} \mathrm{id}_\Gamma^e - \mathbf{P}_h \tilde{\mathbf{P}}_h e_i \right) \cdot \nabla_{\Gamma_h} v \, ds \right| \leq c\, h^{\min\{p,2\}} \, \|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} \quad \textit{for all} \ \ v \in W.$$

Proof. Using $\|\nabla d\| = 1$ it follows that $\|\nabla d_h\| = 1 + \mathcal{O}(h^p)$ holds. We can use the same line of reasoning as in the proof of Lemma 4.2. The term in (4.4) remains the same. Instead of the term in (4.3) we now get $\|\mathbf{P}_h(x)\mathbf{P}(x) - \mathbf{P}_h(x)\tilde{\mathbf{P}}_h(x)\|$. We drop x in the notation, and using the assumption we obtain

$$\begin{aligned}
\|\mathbf{P}_h\mathbf{P} - \mathbf{P}_h\tilde{\mathbf{P}}_h\| &= \|\mathbf{P}_h(\mathbf{P} - \tilde{\mathbf{P}}_h)\| \leq \|\mathbf{n}\mathbf{n}^T - \tilde{\mathbf{n}}_h\tilde{\mathbf{n}}_h^T\| \\
&\leq \|(\mathbf{n} - \tilde{\mathbf{n}}_h)\mathbf{n}^T\| + \|\tilde{\mathbf{n}}_h(\mathbf{n} - \tilde{\mathbf{n}}_h)^T\| = 2\|\mathbf{n} - \tilde{\mathbf{n}}_h\| \\
&= 2\left\| \nabla d - \frac{\nabla d_h}{\|\nabla d_h\|} \right\| \\
&\leq 2\left|1 - \|\nabla d_h\|^{-1}\right| \|\nabla d_h\| + 2\|\nabla d - \nabla d_h\| \leq c\, h^p.
\end{aligned}$$

Thus we get an estimate $\|\mathbf{P}_h\mathbf{P} - \mathbf{P}_h\tilde{\mathbf{P}}_h\| \leq c\, h^p$. Combined with the inequality $|d(x)| \|\mathbf{P}_h(x)\mathbf{H}(x)\mathbf{P}(x)\| \leq c\, h^2$ for the term in (4.4) this proves the result. ∎

If we assume that the condition in (5.3) is satisfied for $p = 2$, which is reasonable for a piecewise quadratic approximation d_h of d, we get the following improvement due to the modified functional $\tilde{g}_h$ (cf. Corollary 4.4):

$$|g(v) - \tilde{g}_h(v)| \leq c\, h \, \|v\|_{1,U} + c\, h^2 \, \|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} \quad \text{for all} \ \ v \in W.$$

Combining this with the result in Theorem 4.6 yields (under the assumption as in Theorem 4.6)

$$|g(v) - \tilde{g}_h(v)| \leq c\, h \, \|v\|_{1,U} + c\, h^{\frac{3}{2}} \, \|v\|_1 \quad \text{for all} \ \ v \in V_H.$$

Hence, using this modified functional $\tilde{g}_h$ we have an $\mathcal{O}(h)$ error bound. This significant improvement (compared to the $\mathcal{O}(\sqrt{h})$ error bound for the functional g_h) is confirmed by the numerical experiments in the next section.

6. Numerical experiments. In this section we present results of a numerical experiment which indicates that the $\mathcal{O}(\sqrt{h})$ bound in Corollary 4.8 is sharp. Furthermore, for the improved approximation described in section 5 the $\mathcal{O}(h)$ bound will be confirmed numerically.

We consider the domain $\Omega := [-1, 1]^3$, where the ball $\Omega_1 := \{\mathbf{x} \in \Omega \mid \|\mathbf{x}\| < R\}$ is located in the center of the domain. In our experiments we take $R = \frac{1}{2}$.

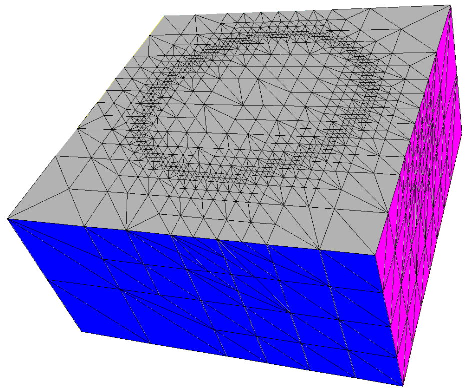

FIG. 6.1. *Lower half of the four times refined mesh $\mathcal{T}_4$.*

For the discretization a uniform tetrahedral mesh $\mathcal{T}_0$ is used, where the vertices form a $6 \times 6 \times 6$ lattice; hence $h_0 = \frac{1}{5}$. This coarse mesh $\mathcal{T}_0$ is locally refined in the vicinity of $\Gamma = \partial\Omega_1$ using an adaptive refinement algorithm presented in [15]. This repeated refinement process yields the gradually refined meshes $\mathcal{T}_1, \mathcal{T}_2, \ldots$ with *local* (i.e., close to the interface) mesh sizes $h_i = \frac{1}{5} \cdot 2^{-i}, i = 1, 2, \ldots$. Part of the tetrahedral triangulation $\mathcal{T}_4$ is shown in Figure 6.1. The corresponding finite element spaces $\mathbf{V}_i := \mathbf{V}_{h_i} = (V_{h_i})^3$ consist of vector functions where each component is a continuous piecewise quadratic function on $\mathcal{T}_i$.

The interface $\Gamma = \partial\Omega_1$ is a sphere, and thus the curvature $\mathcal{K} = \frac{2}{R}$ is constant. If we discretize the flow problem using $\mathbf{V}_i$ as a discrete velocity space, we have to approximate the surface tension force as

$$f_\Gamma(\mathbf{v}) = \frac{2\tau}{R} \int_\Gamma \mathbf{n}_\Gamma \cdot \mathbf{v}\, ds = \tau \int_\Gamma \nabla_\Gamma \operatorname{id}_\Gamma \cdot \nabla_\Gamma \mathbf{v}\, ds, \qquad \mathbf{v} \in \mathbf{V}_i. \tag{6.1}$$

To simplify notation, we take a fixed $i \geq 0$, and the corresponding local mesh size parameter is denoted by $h = h_i$. For the approximation of the interface we use the following approach (cf. Remark 2). The interface Γ is the zero level of the signed distance function d. In this test problem, d is known. For the finite element approximation $d_h \in V_h$ of d we take the continuous piecewise *quadratic* function on $\mathcal{T}_i$ that interpolates d at the vertices and midpoints of edges. Then $I(d_h)$ is the continuous piecewise *linear* function on $\mathcal{T}_i'$ that interpolates d_h at the vertices of all tetrahedra in $\mathcal{T}_i'$; cf. Remark 2 (note that in this test problem, d_h also can be computed by piecewise linear interpolation of d on $\mathcal{T}_i'$). The approximation of Γ is defined by

$$\Gamma_h = \{\, x \in \Omega \mid I(d_h)(x) = 0 \,\}$$

and is illustrated in Figure 2.2. The discrete approximation of the surface tension force is

$$f_{\Gamma_h}(\mathbf{v}) = \tau \int_{\Gamma_h} \nabla_{\Gamma_h} \operatorname{id}_{\Gamma_h} \cdot \nabla_\Gamma \mathbf{v}\, ds, \qquad \mathbf{v} \in \mathbf{V}_i.$$

We are interested in (cf. Corollary 4.8)

$$\|f_\Gamma - f_{\Gamma_h}\|_{\mathbf{V}_i'} := \sup_{\mathbf{v} \in \mathbf{V}_i} \frac{f_\Gamma(\mathbf{v}) - f_{\Gamma_h}(\mathbf{v})}{\|\mathbf{v}\|_1}. \tag{6.2}$$

The evaluation of $f_\Gamma(\mathbf{v})$, for $\mathbf{v} \in \mathbf{V}_i$, requires the computation of integrals on curved triangles or quadrilaterals $\Gamma \cap S$, where S is a tetrahedron from the mesh $\mathcal{T}_i$. We are not able to compute these exactly. Therefore, we introduce an artificial force term which, in this model problem with a known constant curvature, is computable and sufficiently close to f_Γ.

LEMMA 6.1. *For* $\mathbf{v} \in \mathbf{V} = (H_0^1(\Omega))^3$ *define*

$$\hat{f}_{\Gamma_h}(\mathbf{v}) := \frac{2\tau}{R} \int_{\Gamma_h} \mathbf{n}_h \cdot \mathbf{v}\, ds$$

($\mathbf{n}_h$ denotes the piecewise constant outward unit normal on Γ_h). Then the following inequality holds:

$$\|f_\Gamma - \hat{f}_{\Gamma_h}\|_{\mathbf{V}'} \le ch. \tag{6.3}$$

Proof. Let $\Omega_{1,h} \subset \Omega$ be the domain enclosed by Γ_h, i.e., $\partial\Omega_{1,h} = \Gamma_h$. We define $D_h^+ := \Omega_1 \setminus \Omega_{1,h}$, $D_h^- := \Omega_{1,h} \setminus \Omega_1$, and $D_h := D_h^+ \cup D_h^-$. Due to the Stokes theorem, for $\mathbf{v} \in \mathbf{V}$ we have

$$|f_\Gamma(\mathbf{v}) - \hat{f}_{\Gamma_h}(\mathbf{v})| = \frac{2\tau}{R} \left| \int_{\Omega_1} \operatorname{div} \mathbf{v}\, dx - \int_{\Omega_{1,h}} \operatorname{div} \mathbf{v}\, dx \right| \tag{6.4}$$

$$= \frac{2\tau}{R} \left| \int_{D_h^+} \operatorname{div} \mathbf{v}\, d\mathbf{x} - \int_{D_h^-} \operatorname{div} \mathbf{v}\, dx \right| \tag{6.5}$$

$$\le \frac{2\tau}{R} \int_{D_h} |\operatorname{div} \mathbf{v}|\, dx. \tag{6.6}$$

Using the Cauchy–Schwarz inequality, we get the estimate

$$|f_\Gamma(\mathbf{v}) - \hat{f}_{\Gamma_h}(\mathbf{v})| \le c\sqrt{|D_h|}\|\mathbf{v}\|_1 \quad \text{for all} \ \ \mathbf{v} \in \mathbf{V}.$$

For the piecewise planar approximation Γ_h of the interface Γ we have $|D_h| = \mathcal{O}(h^2)$, and thus (6.3) holds. $\square$

From Lemma 6.1 we obtain $\|f_\Gamma - \hat{f}_{\Gamma_h}\|_{\mathbf{V}_j'} \le c\, h$ with a constant c independent of j. Thus we have

$$\|\hat{f}_{\Gamma_h} - f_{\Gamma_h}\|_{\mathbf{V}_i'} - ch \le \|f_\Gamma - f_{\Gamma_h}\|_{\mathbf{V}_i'} \le \|\hat{f}_{\Gamma_h} - f_{\Gamma_h}\|_{\mathbf{V}_i'} + ch. \tag{6.7}$$

The term $\|\hat{f}_{\Gamma_h} - f_{\Gamma_h}\|_{\mathbf{V}_i'}$ can be evaluated as follows. Since Γ_h is piecewise planar and $\mathbf{v} \in \mathbf{V}_i$ is a piecewise quadratic function, for $\mathbf{v} \in \mathbf{V}_i$, both $\hat{f}_{\Gamma_h}(\mathbf{v})$ and $f_{\Gamma_h}(\mathbf{v})$ can be computed exactly (up to machine accuracy) using suitable quadrature rules. For the evaluation of the dual norm $\|\cdot\|_{\mathbf{V}_i'}$ we proceed as follows. Let $\{\boldsymbol{\phi}_j\}_{j=1,\dots,n}$ be the standard nodal basis in $\mathbf{V}_i$ and $J : \mathbb{R}^n \to \mathbf{V}_i$ the isomorphism $J\vec{x} = \sum_{k=1}^n x_k \boldsymbol{\phi}_k$. Let M_h be the mass matrix and A_h the stiffness matrix of the Laplacian:

$$\left.\begin{aligned} (M_h)_{i,j} &:= \int_\Omega \boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j\, dx, \\ (A_h)_{i,j} &:= \int_\Omega \nabla\boldsymbol{\phi}_i \cdot \nabla\boldsymbol{\phi}_j\, dx. \end{aligned}\right. \qquad 1 \le i,\ j \le n.$$

TABLE 6.1
Error norms and numerical order of convergence for different refinement levels.

i	$\|\hat{f}_{\Gamma_h} - f_{\Gamma_h}\|_{\mathbf{V}_i'}$	order	$\|\hat{f}_{\Gamma_h} - \tilde{f}_{\Gamma_h}\|_{\mathbf{V}_i'}$	order
0	1.79 E-1	–	1.32 E-1	–
1	1.40 E-1	0.35	4.43 E-2	1.57
2	1.03 E-1	0.45	1.46 E-2	1.61
3	7.22 E-2	0.51	5.06 E-3	1.52
4	5.02 E-2	0.53	1.78 E-3	1.51

Define $C_h = A_h + M_h$. Note that for $\mathbf{v} = J\vec{x} \in \mathbf{V}_i$ we have $\|\mathbf{v}\|_1^2 = \langle C_h\vec{x}, \vec{x}\rangle$. Take $e \in \mathbf{V}_i'$ and define $\vec{e} \in \mathbb{R}^n$ by $e_j := e(\boldsymbol{\phi}_j)$, $j = 1, \ldots, n$. Due to

$$\|e\|_{\mathbf{V}_i'} = \sup_{\mathbf{v}\in\mathbf{V}_i} \frac{|e(\mathbf{v})|}{\|\mathbf{v}\|_1} = \sup_{\vec{x}\in\mathbb{R}^n} \frac{|\sum_{j=1}^n x_j e(\boldsymbol{\phi}_j)|}{\sqrt{\langle C_h\vec{x}, \vec{x}\rangle}}$$

we obtain

$$\|e\|_{\mathbf{V}_i'} = \sup_{\vec{x}\in\mathbb{R}^n} \frac{\langle \vec{x}, \vec{e}\rangle}{\sqrt{\langle C_h\vec{x}, \vec{x}\rangle}} = \|C_h^{-1/2}\vec{e}\| = \sqrt{\langle C_h^{-1}\vec{e}, \vec{e}\rangle}. \tag{6.8}$$

Thus for the computation of $\|e\|_{\mathbf{V}_i'}$ we proceed in the following way:

1. Compute $\vec{e} = \big(e(\boldsymbol{\phi}_j)\big)_{j=1}^n$.
2. Solve the linear system $C_h\vec{z} = \vec{e}$ up to machine accuracy.
3. Compute $\|e\|_{\mathbf{V}_i'} = \sqrt{\langle \vec{z}, \vec{e}\rangle}$.

We applied this strategy to $e := \hat{f}_{\Gamma_h} - f_{\Gamma_h}$. The results are given in the second column in Table 6.1. The numerical order of convergence in the third column of this table clearly indicates an $\mathcal{O}(\sqrt{h})$ behavior. Due to (6.7) this implies the same $\mathcal{O}(\sqrt{h})$ convergence behavior for $\|f_\Gamma - f_{\Gamma_h}\|_{\mathbf{V}_i'}$. This indicates that the $\mathcal{O}(\sqrt{h})$ bound in Corollary 4.8 is sharp.

The same procedure can be applied with f_{Γ_h} replaced by the modified (improved) approximate surface tension force

$$\tilde{f}_{\Gamma_h}(\mathbf{v}) = \tau \sum_{i=1}^{3} \tilde{g}_{h,i}(v_i)$$

with $\tilde{g}_{h,i}$ as defined in (5.2). This yields the results in the fourth column in Table 6.1. For this modification the numerical order of convergence is significantly better, namely, at least first order in h. From (6.7) it follows that for $\|f_\Gamma - \tilde{f}_{\Gamma_h}\|_{\mathbf{V}_i'}$ we can expect $\mathcal{O}(h^p)$ with $p \geq 1$.

Summarizing, we conclude that the results of these numerical experiments confirm the theoretical $\mathcal{O}(\sqrt{h})$ error bound derived in the analysis in section 4 and show that the modified approximation indeed leads to (much) better results.

Results of numerical experiments for a Stokes two-phase flow problem using both f_{Γ_h} and $\tilde{f}_{\Gamma_h}$ are presented in [16].

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