NITSCHE-XFEM FOR A TRANSPORT PROBLEM IN TWO-PHASE INCOMPRESSIBLE FLOWS

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Summary. We present a finite element discretization method for a mass transport problem with a solution that is discontinuous across an interface. The grids are regular and unfitted. The method is based on a combination of Nitsche’s method and an XFEM approach.

1 INTRODUCTION

Let Ω ⊂ Rd, d = 2, 3, be a convex polygonal domain that contains two different immiscible incompressible phases Ω1 and Ω2. We assume that the interface Γ = Γ(t) = Ω1 ∩ Ω2 is sufficiently smooth. We consider a model which describes the transport of a dissolved species in a divergence-free velocity field w, i.e. div w = 0, as follows:

\[
\frac{\partial u}{\partial t} + w \cdot \nabla u - \text{div}(\alpha \nabla u) = f \quad \text{in} \quad \Omega_i, \quad i = 1, 2, \quad t \in [0, T],
\]

\[
[\alpha \nabla u \cdot n]_\Gamma = 0, \quad [\beta u]_\Gamma = 0,
\]

where n is the unit normal at Γ pointing from Ω1 into Ω2. For a sufficiently smooth function v, [v] denotes the jump of v across Γ. The first interface condition in (2) results from the conservation of mass principle while the second one is the so-called Henry condition. The diffusion coefficient α and the Henry coefficient β are positive and piecewise constant in the two subdomains, so the solution u is in general discontinuous across the interface. For the special case β1 = β2 and with a triangulation which is fitted to the interface, standard finite element spaces have (close to) optimal approximation properties. Here we allow β1 ≠ β2 and use triangulations that are unfitted (as in level set of VOF approaches), i.e. the interface crosses the elements. We will use a variant of Nitsche’s method combined with a special finite element method for the spatial discretization of this problem. From this semi-discrete problem a full discretization is obtained by using a standard θ-scheme for time discretization. We use the same Nitsche method as presented and analyzed for a stationary diffusion problem by Hansbo. We apply this method to the nonstationary problem described above, with discontinuous solution, and furthermore allow a convection term in (1).
2 Weak formulation

In this section we give a weak formulation. For simplicity we only consider homogeneous Dirichlet boundary conditions. Due to the fact that the underlying two-phase fluid dynamics concerns two incompressible immiscible phases it is reasonable to make the following assumption about the velocity field \( w \): \( \text{div } w = 0 \) in \( \Omega_i \), \( i = 1, 2 \), \( w \cdot n = 0 \) at \( \Gamma \), and \( \| w \|_{L^\infty(\Omega)} \leq c < \infty \). We define 
\[ H^1_0(\Omega_1 \cup \Omega_2) := \{ v \in L^2(\Omega) | v_i \in H^1(\Omega_i), \ i = 1, 2, \ v_i|_{\partial \Omega} = 0 \}, \]
where \( v_i := v|_{\Omega_i} \), and 
\[ H := L^2(\Omega), \quad V := \{ v \in H^1_0(\Omega_1 \cup \Omega_2) | [\beta v]_\Gamma = 0 \}, \]
\[ (u, v)_0 := \int_\Omega \beta uv \, dx, \quad u, v \in H, \]
\[ (u, v)_{1,\Omega_1 \cup \Omega_2} := (u, v)_{1,\Omega_1} + (u, v)_{1,\Omega_2} = \sum_{j=1}^2 \left( \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \right)_0, \quad u, v \in V. \]

We now introduce the bilinear form 
\[ a(u, v) := (\alpha u, v)_{1,\Omega_1 \cup \Omega_2} + (w \cdot \nabla u, v)_0, \quad u, v \in V. \]
We have\(^4\) well-posedness of a weak formulation for the case with a stationary interface:

**Lemma 1** Assume that \( \Gamma \) does not depend on \( t \). Take \( f \in H \), \( u_0 \in V_{\text{reg}} := \{ v \in V | v_i \in H^2(\Omega_i), \ i = 1, 2 \} \). There exists a unique \( u \in C([0, T]; V_{\text{reg}}) \) such that \( u(0) = u_0 \) and 
\[ \left( \frac{\partial u}{\partial t}, v \right)_0 + a(u, v) = (f, v)_0 \quad \text{for all } v \in V. \]

(3)
The distributional time derivative satisfies \( \frac{\partial u}{\partial t} \in L^2(0, T; V) \cap C([0, T]; H) \).

3 XFEM space and Nitsche’s method

Let \( \{ T_h \}_{h>0} \) be a family of shape regular triangulations of \( \Omega \). A triangulation \( T_h \) consists of triangles \( T \), with \( h_T := \text{diam}(T) \) and \( h := \max \{ h_T | T \in T_h \} \). Let \( T_i := T \cap \Omega_i \) be the part of \( T \) in \( \Omega_i \). We now introduce the finite element space 
\[ V^\Gamma_h := \{ v \in H^1_0(\Omega_1 \cup \Omega_2) | v|_{T_i} \text{ is linear for all } T \in T_h, \ i = 1, 2 \}. \]

(4)

Note that \( V^\Gamma_h \subset H^1_0(\Omega_1 \cup \Omega_2) \), but \( V^\Gamma_h \not\subset V \), since the Henry interface condition \( [\beta v_h] = 0 \) does not necessarily hold for \( v_h \in V^\Gamma_h \). We define \( (\kappa_i)|_T = \frac{|T_i|}{|T|} \) for all \( T \in T_h, \ i = 1, 2 \), and the weighted average \( \{ v \} := \kappa_1(v_1)|_T + \kappa_2(v_2)|_T \). Let \( (f, g)_\Gamma := \int_\Gamma fg \, ds \) be the \( L^2(\Gamma) \) scalar product. We introduce the bilinear form 
\[ a_h(u, v) := (\alpha u, v)_{1,\Omega_1 \cup \Omega_2} + (w \cdot \nabla u, v)_0 - ([\beta u], \{ \alpha \nabla v \cdot n \})_\Gamma \\
- ([\alpha \nabla u \cdot n], [\beta v])_\Gamma + \lambda h^{-1}([\beta u], [\beta v])_\Gamma, \]

(5)

with a positive parameter \( \lambda \). The following consistency result holds\(^4\) : Let \( u = u(t) \in V_{\text{reg}} \) be the solution defined in lemma 1. Then \( u(t) \) satisfies 
\[ \left( \frac{\partial u}{\partial t}, v_h \right)_0 + a_h(u, v_h) = (f, v_h) \quad \text{for all } v_h \in V^\Gamma_h, \ t \in [0, T]. \]

(6)

For the spatial discretization error we have the following result\(^4\).
**Theorem 1** Assume that \( \Gamma \) does not depend on \( t \). Let \( u = u(t) \in V_{reg} \) be the solution defined in lemma 1 and \( u_h = u_h(t) \in V_h^T \) the solution of (6) with \( u_h(0) = \hat{u}_0 \). The following holds, with \( R_h \) the elliptic projection on \( V_h^T \).

\[
\|u_h(t) - u(t)\|_0 \leq \|\hat{u}_0 - R_h u_0\|_0 + c h^2 \left\{ \|u_0\|_{2,\Omega_{1}\cup\Omega_{2}} + \int_0^t \left\| \frac{\partial u}{\partial t} \right\|_{2,\Omega_{1}\cup\Omega_{2}} \, dt \right\}, \quad 0 \leq t \leq T.
\]

From this result we conclude that for the semi-discretization of our transport problem we have an optimal error bound for the spatial discretization.

4 Numerical experiments

4.1 Experiment with a stationary interface

We consider the problem (1)-(2) in the domain \( \Omega = (0, 1)^3 \), which contains two subdomains \( \Omega_1 := \{(x, y, z) \in \Omega : z < 0.341\} \) and \( \Omega_2 := \Omega \setminus \Omega_1 \), with the coefficients \( \alpha = (\alpha_1, \alpha_2) := (1, 2) \), \( \beta = (\beta_1, \beta_2) := (2, 1) \) and a velocity field \( w := (y(1 - z), x, 0)^T \). The exact solution is chosen as

\[
u(x, y, z, t) := \begin{cases} 
\exp(-t) \cos(\pi x) \cos(2\pi y)az + b & \text{in } \Omega_1, \\
\exp(-t) \cos(\pi x) \cos(2\pi y)z(z - 1) & \text{in } \Omega_2,
\end{cases}
\]

where the constants \( a \) and \( b \) are determined from the interface conditions (2). For the spatial discretization, we create a uniform grid with mesh size \( h = \frac{1}{N} \) \( (N = 8, 16, 32) \) then refine the elements near the interface two times further. The semi-discretization \( u_h(t) \) is approximated by \( u_h^*(t) \) using the implicit Euler time-stepping scheme with a (sufficiently small) time step size \( \Delta t = 10^{-4} \). In Table 1, the errors \( \|u_h^*(T) - u(T)\|_{L^2} \) for \( T = 0.15 \) are displayed, which are consistent with the theoretical bound \( O(h^2) \) given in theorem 1. For a stationary elliptic problem the bound \( \|\beta u_h\|_{L^2(\Gamma)} \leq c h^{1/2} \|u\|_{2,\Omega_{1}\cup\Omega_{2}} \) holds. For the time dependent case we were not able to derive a theoretical bound for this error quantity. The errors \( \|\beta u_h^*(T)\|_{L^2(\Gamma)} \) are given in Table 2, which seems to behave like \( O(h) \). The numerical solution for \( N = 16 \) at \( T = 0.15 \) in the plane \( x = 0.25 \) is shown in Figure 1. To investigate the time discretization

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
\( N \) & \( \|u_h^*(T) - u(T)\|_{L^2} \) & \text{factor} & \text{order} \\
\hline
8 & 0.00738506 & - & - \\
16 & 0.0020308 & 3.65 & 1.87 \\
32 & 0.0005228 & 3.87 & 1.95 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
\( N \) & \( \|\beta u_h^*(T)\|_{L^2(\Gamma)} \) & \text{factor} & \text{order} \\
\hline
8 & 1.565e - 4 & - & - \\
16 & 7.975e - 05 & 1.96 & 0.972 \\
32 & 3.900e - 05 & 2.05 & 1.03 \\
\hline
\end{tabular}
\end{table}

To investigate the time discretization error, we use a fixed mesh with \( N = 16 \) and compute a reference solution \( u_h^*(t) \) with \( \Delta t = 10^{-4} \) in the time interval \([0, 0.2]\). The Euler discretization with time step \( \Delta t = \frac{T}{n} \) results in approximations \( u_h^*(T) \) of \( u_h^*(T) \). For the cases \( n = 5, 10, 20 \) the temporal errors in the \( L^2 \)-norm are given in Table 3. We observe the expected first order of convergence in \( \Delta t \).

4.2 Experiment with a nonstationary interface

We consider the problem (1)-(2) in the unit cube \( \Omega \) and with \( \Omega_1(0) \) a sphere of radius \( R = 0.2 \) centered at the barycenter of \( \Omega \). This sphere is moved in with constant velocity \( w = (0, 1, 0)^T \), i.e.,
Arnold Reusken and Trung Hieu Nguyen

<table>
<thead>
<tr>
<th>$n$</th>
<th>$|u_h^n - u^*<em>h(0.2)|</em>{L^2}$</th>
<th>factor</th>
<th>order</th>
<th>$N$</th>
<th>$|u_h^*(T) - u(T)|_{L^2}$</th>
<th>factor</th>
<th>order</th>
</tr>
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<td>-</td>
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<td>16</td>
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<td>3.9</td>
<td>1.963</td>
</tr>
</tbody>
</table>

Table 3: Planar interface: Time discretization error in $L^2$ norm and convergence order at $T = 0.2$

Table 4: Moving interface: Spatial discretization error in $L^2$-norm and convergence order at $T = 0.1$

$\Omega_1(t) = \Omega_1(0) + tw$. Let $d(x, t)$ be the distance from the point $x \in \Omega$ to the center of $\Omega_1(t)$. We take the piecewise quadratic solution

$$u(x, t) := \begin{cases} 
\alpha_2 (d(x, t)^2 - R^2) + 0.1 \cdot \beta_2 & \text{in } \Omega_1, \\
\alpha_1 (d(x, t)^2 - R^2) + 0.1 \cdot \beta_1 & \text{in } \Omega_2,
\end{cases}$$

with coefficients $(\alpha_1, \alpha_2) := (1, 5), (\beta_1, \beta_2) := (2, 1)$. As the XFEM space now is time dependent, we discretize the problem first in time using the implicit Euler method with the time step size $\Delta t = 10^{-4}$. The resulting convection-diffusion-reaction problem is discretized with the Nitsche method. We use a uniform grid with the mesh size $h = \frac{1}{N}$, where $N = 16, 32, 64$. The errors $\|u_h^*(T) - u(T)\|_{L^2}$ for $T = 0.1$ are displayed in Table 4 with the expected convergence order 2.

Figure 1: Planar interface: Numerical solution at $T = 0.15$ in the plane $x = 0.25$.

Figure 2: Moving interface: Numerical solution at $T = 0.1$ in the plane $x = 0.5$.

REFERENCES


