ANALYSIS OF TIME DISCRETIZATION METHODS FOR A TWO-PHASE STOKES EQUATION WITH SURFACE TENSION

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Abstract. In two-phase incompressible flow problems surface tension effects often play a key role. Due to surface tension the pressure is discontinuous across the interface. In interface capturing methods the grids are typically not aligned to the interface and thus in problems with an evolving interface time dependent pressure spaces should be used. Hence, a method of lines approach is not very suitable for this problem class. We consider a Rothe method with an implicit Euler or a Crank-Nicolson time discretization method. The order of convergence of these methods is not clear, since the surface tension force results in a right-hand side functional in the momentum equation with poor regularity properties. These regularity properties are such that for the Crank-Nicolson method one can not apply error analyses known in the literature. In this paper, for a simplified non-stationary Stokes problem a convergence analysis is presented. The analysis leads to optimal order error bounds. For the Crank-Nicolson method the error analysis uses a norm that is weaker that the L^2-norm. Results of numerical experiments are shown that confirm the analysis.

Key words. two-phase flow, surface tension, time discretization, Crank-Nicolson method

AMS subject classifications. 65M12, 65M15, 76D45

1. Introduction. Two-phase incompressible flow problems with surface tension forces are usually modeled by the Navier-Stokes equations with a surface tension force term on the right-hand side in the momentum equation, cf. section 2. There are several reasons why such a model has a very high numerical complexity. For example, the interface is unknown and due to this the flow problem is strongly nonlinear. Secondly, the surface tension force is localized at the (unknown) interface and often has a major effect on the fluid dynamics. Thirdly, the pressure has a discontinuity across the interface, and also the viscosity and density coefficients are discontinuous across the interface. There are several important issues relevant for the simulation of two-phase flows that are non-existent in *one*-phase incompressible flow problems. To handle these issues, special numerical techniques are required. Concerning the development and analysis of such special numerical methods on ly relatively few (compared to methods for one-phase flows) studies are available in the literature, cf. [6] for a recent overview. In particular only very few papers have appeared in which rigorous analyses (e.g., on discretization errors, rate of convergence of solvers) are presented.

In this paper we consider only one particular aspect that arises in the simulation of two-phase flows and is non-existent in one-phase flow problems. We outline this aspect, which is explained in more detail in the sections 2 and 3. We simplify the Navier-Stokes problem to the Stokes problem. Since the pressure is discontinuous across the interface, in a finite element discretization method for the Stokes problem one then should use a pressure finite element space that is time dependent. Hence, the Rothe approach is more natural than the method of lines. Alternatively one could consider a space-time finite element technique but we do not treat this here.

For a *given* evolution of the interface the non-stationary Stokes problem (with constant viscosity and density coefficients) can be represented as a parabolic equation of the following form: determine $\mathbf{u}(t) \in \mathbf{V}_{\mathrm{div}}$ such that $\mathbf{u}(0) = \mathbf{u}_0$ and for $t \in (0,T)$:

$$\mathbf{u}'(t)(\mathbf{v}) + (\nabla \mathbf{u}(t), \nabla \mathbf{v})_{L^2} = f(t)(\mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in \mathbf{V}_{\mathrm{div}}, \tag{1.1}$$

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with $\mathbf{V}_{\mathrm{div}}$ the space of divergence free velocity fields and $\mathbf{u}'$ a suitable generalized time derivative of $\mathbf{u}$. In the setting of the Rothe method we are interested in time discretization methods for the parabolic problem (1.1). In particular we want to derive error bounds for the very basic (but still very popular) implicit Euler and Crank-Nicolson method. Such an error analysis is presented in this paper. There is extensive literature on error analyses of time discretization methods for parabolic problems (cf. [16] for an overview) but as far as we know there is no analysis that applies to the parabolic problem (1.1). The reason for this is the fact that the surface tension functional $f(t)$ is nonsmooth. As noted above, this surface tension force is an essential feature of most two-phase flows. In the literature there are many studies on time discretization methods for parabolic problems with smooth data, i.e. smooth initial and bo undary conditions and a smooth source term, cf. [16]. Some of these results have been generalized to the (Navier-)Stokes equations [8, 17], with smooth data, in which a certain non-local compatibility condition at $t = 0$ plays an important role. Furthermore, there are analyses of time discretization methods for parabolic problems with nonsmooth initial data, e.g. [10, 7]. In these analyses a so-called parabolic smoothing property plays an important role, cf. also [19]. An analysis of implicit Runge-Kutta time stepping schemes for parabolic problems with incompatible initial-boundary data or nonsmooth boundaries is presented in [9]. In all these studies and the other papers we are aware of, the source term is either zero (homogeneous problem) or assumed to be sufficiently smooth.

In our case the source term $f(t)$ has only low regularity, as specified in section 5. This of course has consequences for the regularity of the solution and also for the accuracy of a higher order discretization method. It turns out that the smoothness is still sufficient to apply a standard analysis to the implicit Euler method, resulting in an optimal $\mathcal{O}(\Delta t)$ error bound in the L^2-norm. The smoothness is too low to apply a similar analysis to the Crank-Nicolson method. Instead, weaker norms are introduced and we obtain an $\mathcal{O}(\Delta t^2)$ error bound for the Crank-Nicolson method in a suitable weaker norm. Results of numerical experiments are presented, which illustrate the error behavior of the implicit Euler and Crank-Nicolson methods applied to a two-phase Stokes problem.

2. Stokes two-phase flow with surface tension. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a domain containing two different immiscible incompressible phases. The time dependent subdomains containing the two phases are denoted by $\Omega_1(t)$ and $\Omega_2(t)$ with $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. We assume that Ω_1 and Ω_2 are connected and $\partial\Omega_1 \cap \partial\Omega = \emptyset$ (i. e., Ω_1 is completely contained in Ω). The interface is denoted by $\Gamma(t) = \overline{\Omega}_1(t) \cap \overline{\Omega}_2(t)$. The standard model for describing incompressible two-phase flows consists of the Navier-Stokes equations in the subdomains with the coupling condition

$$[\boldsymbol{\sigma}\mathbf{n}]_\Gamma = -\tau\kappa\mathbf{n}$$

at the interface, i. e., the surface tension balances the jump of the normal stress on the interface. We use the notation $[v]_\Gamma$ for the jump across Γ, $\mathbf{n} = \mathbf{n}_\Gamma$ is the unit normal at the interface Γ (pointing from Ω_1 into Ω_2), κ the curvature of Γ, τ the surface tension coefficient (assumed to be constant) and $\boldsymbol{\sigma}$ the stress tensor defined by

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mu\mathbf{D}(\mathbf{u}), \qquad \mathbf{D}(\mathbf{u}) = \nabla\mathbf{u} + (\nabla\mathbf{u})^T,$$

with $p = p(x,t)$ the pressure, $\mathbf{u} = \mathbf{u}(x,t)$ the velocity and μ the viscosity. We assume continuity of $\mathbf{u}$ across the interface. Combined with the conservation laws for mass and momentum we obtain the following standard model, cf. for example [11, 14, 13, 6],

$$\begin{cases} \rho_i \frac{\partial \mathbf{u}}{\partial t} - \operatorname{div}(\mu_i \mathbf{D}(\mathbf{u})) + \rho_i (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p = \rho_i \mathbf{g} & \text{in } \Omega_i \times [0,T] \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega_i \times [0,T] \end{cases} \text{for } i = 1,2, \quad (2.1)$$

$$[\boldsymbol{\sigma}\mathbf{n}]_\Gamma = -\tau\kappa\mathbf{n}, \quad [\mathbf{u}]_\Gamma = 0. \quad (2.2)$$

The constants μ_i, ρ_i denote viscosity and density in the subdomain Ω_i, $i = 1,2$, and $\mathbf{g}$ is an external volume force (gravity). To make this problem well-posed we need suitable boundary conditions for $\mathbf{u}$ and an initial condition for $\mathbf{u}$. For simplicity we restrict to homogeneous Dirichlet boundary conditions for $\mathbf{u}$.

The location of the interface $\Gamma(t)$ is in general unknown and is determined by the local flow field which transports the interface. For immiscible fluids this transport of the interface is modeled by $V_\Gamma = \mathbf{u} \cdot \mathbf{n}$, where V_Γ denotes the normal velocity of the interface.

We make the following simplifications. The densities and viscosities are assumed to be constant, i.e., $\rho_1 = \rho_2 = 1$ and $\mu_1 = \mu_2 = 1$. The cases $\rho_1 \neq \rho_2$ and $\mu_1 \neq \mu_2$ are briefly addressed in section 6. Furthermore we restrict to the case of a Stokes flow. For a corresponding weak formulation we introduce the standard spaces $\mathbf{V} = H_0^1(\Omega)^d$, $Q = L_0^2(\Omega)$ and $\hat{W}(I) := \left\{ \mathbf{u} \in L^2(I;\mathbf{V}) : \mathbf{u}' \in L^2(I,\mathbf{V}') \text{ exists} \right\}$, with $I = (0,T)$ a time interval and $\mathbf{u}'$ a suitable generalized time derivative as explained in section 4. An appropriate weak formulation of the Stokes two-phase problem is as follows (cf. [6]): determine $\mathbf{u} \in \hat{W}(I)$, $p \in L^2(I;Q)$ such that for $t \in I$:

$$\mathbf{u}'(t)(\mathbf{v}) + a(\mathbf{u}(t),\mathbf{v}) + b(\mathbf{v},p(t)) = f(t)(\mathbf{v}) + g(\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{V} \quad (2.3)$$

$$b(\mathbf{u}(t),q) = 0 \quad \text{for all } q \in Q \quad (2.4)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad (2.5)$$

with

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} \cdot \nabla\mathbf{v}\, dx = \sum_{i=1}^d \int_\Omega \nabla u_i \cdot \nabla v_i\, dx \quad (2.6)$$

$$b(\mathbf{v},q) = -\int_\Omega q \operatorname{div}\mathbf{v}\, dx \quad (2.7)$$

$$g(\mathbf{v}) = \int_\Omega \mathbf{g} \cdot \mathbf{v}\, dx, \quad f(t)(\mathbf{v}) = -\tau \int_{\Gamma(t)} \kappa\mathbf{n} \cdot \mathbf{v}\, ds. \quad (2.8)$$

The functional $f(t)$ arises due to the interface conditions in (2.2) and models the surface tension force, which acts only at the interface. Due to this, *the pressure has a jump across the evolving interface* $\Gamma(t)$. Related to this, $f(t)$ has only low regularity: $f(t) \in \mathbf{V}'$, but $f(t) \notin L^2(\Omega)^d$. For the derivation and analysis of time discretization methods it is convenient to eliminate the pressure variable and formulate the non-stationary Stokes problem in the spave $\mathbf{V}_{\operatorname{div}} := \{ \mathbf{v} \in \mathbf{V} : \operatorname{div}\mathbf{v} = 0 \}$: determine $\mathbf{u} \in W(I) := \left\{ \mathbf{u} \in L^2(I;\mathbf{V}_{\operatorname{div}}) : \mathbf{u}' \in L^2(I;\mathbf{V}'_{\operatorname{div}}) \right\}$ such that $\mathbf{u}(0) = \mathbf{u}_0$ and

$$\mathbf{u}'(t)(\mathbf{v}) + a(\mathbf{u}(t),\mathbf{v}) = f(t)(\mathbf{v}) + g(\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{V}_{\operatorname{div}}, \quad t \in I. \quad (2.9)$$

As noted above, the interface $\Gamma(t)$ is unknown and implicitly determined by the condition $V_\Gamma = \mathbf{u} \cdot \mathbf{n}$. Due to this, even for a Stokes flow the problem is strongly nonlinear

and only for very special cases results on well-posedness are known, cf. [6]. We make one further (strong) simplification, namely assume that the dynamics of the interface is not driven by $\mathbf{u}$ but by a *given smooth velocity field*, which we denote by $\mathbf{b}$ in the remainder, i.e. $V_\Gamma = \mathbf{b}\cdot\mathbf{n}$. Under this assumption the Stokes two-phase flow is linear and existence and uniqueness results for variational formulations as in (2.3)-(2.5) or in (2.9) are known in the literature, e.g. [2]. In this simplified linear case we still have the same localized, time-dependent, surface tension functional at the interface which causes a jump in the pressure and there is, as far as we know, no analysis of time integration methods for this type of Stokes problems. In section 3 we introduce the implicit Euler and Crank-Nicolson time discretization methods for the problems in (2.3)-(2.5) and in (2.9), and in section 5 we derive discretization error bounds for these.

3. Discretization of the Stokes two-phase flow problem. We consider a finite element Eulerian setting in which a given (possibly locally refined) triangulation is used for the spatial discretization of the Stokes equations. In two-phase flow problems (with $\Gamma(t)$ unknown) this is typically the case if popular interface capturing methods like VOF or the level set method are applied. In such a setting the triangulation is *not aligned* to the interface and as time evolves, the interface moves through the given triangulation. Since the pressure has a jump across this non-stationary interface, for an accurate discretization of the pressure one should use a finite element space that *depends on time*. Therefore a method of lines approach (first space, then time) is *not* appropriate for (2.3)-(2.5). Hence, we consider the Rothe approach. An alternative would be to use a space-time finite element method, but we do not treat this here. For the derivation and analysis of time discretization methods i t is natural to consider (2.9) instead of (2.3)-(2.5), and introduce the Lagrange multiplier p after time discretization.

For $N \in \mathbb{N}$ define the time step $\Delta t = T/N$. For the implicit Euler method the sequence of approximations $\mathbf{u}^n \in \mathbf{V}_{\mathrm{div}}$, $0 \le n \le N$, is defined as follows: $\mathbf{u}^0 = \mathbf{u}_0$, and for $n \ge 0$:

$$(\mathbf{u}^{n+1} - \mathbf{u}^n, \mathbf{v})_{L^2} + \Delta t\, a(\mathbf{u}^{n+1}, \mathbf{v}) = \Delta t\big(f(t_{n+1})(\mathbf{v}) + g(\mathbf{v})\big) \quad \text{for all } \ \mathbf{v} \in \mathbf{V}_{\mathrm{div}}. \tag{3.1}$$

The Crank-Nicolson method is defined by: $\mathbf{u}^0 = \mathbf{u}_0$ and for $n \ge 0$ the approximation $\mathbf{u}^{n+1} \in \mathbf{V}_{\mathrm{div}}$ is determined by

$$(\mathbf{u}^{n+1} - \mathbf{u}^n, \mathbf{v})_{L^2} + \frac{\Delta t}{2} a(\mathbf{u}^{n+1} + \mathbf{u}^n, \mathbf{v}) = \frac{\Delta t}{2}\left(f(t_{n+1})(\mathbf{v}) + f(t_n)(\mathbf{v}) + 2g(\mathbf{v})\right) \tag{3.2}$$

for all $\mathbf{v} \in \mathbf{V}_{\mathrm{div}}$. In section 5 we prove discretization error bounds for these methods, cf. Theorems 5.3, 5.6, 5.7.

After the time discretization a suitable space discretization has to be applied. Since finite element subspaces of $\mathbf{V}_{\mathrm{div}}$ are inconvenient, in order to satisfy the constraint $\operatorname{div}\mathbf{u}^{n+1} = 0$ one usually introduces a pressure variable p and applies a Galerkin discretization for the pair $(\mathbf{u}^{n+1}, p)$ in the space $\mathbf{V} \times Q$. We outline this for the Crank-Nicolson method. Consider the following problem: $\mathbf{u}^0 = \mathbf{u}_0$ and for $n \ge 0$ the velocity $\mathbf{u}^{n+1} \in \mathbf{V}$ and pressure $p \in Q$ are determined by

$$\begin{aligned}
&(\mathbf{u}^{n+1} - \mathbf{u}^n, \mathbf{v})_{L^2} + \frac{\Delta t}{2} a(\mathbf{u}^{n+1} + \mathbf{u}^n, \mathbf{v}) + \Delta t\, b(\mathbf{v}, p) \\
&= \frac{\Delta t}{2}\big(f(t_{n+1})(\mathbf{v}) + f(t_n)(\mathbf{v}) + 2g(\mathbf{v})\big) \quad \text{for all } \ \mathbf{v} \in \mathbf{V} && (3.3)\\
&b(\mathbf{u}^{n+1}, q) = 0 \quad \text{for all } \ q \in Q. && (3.4)
\end{aligned}$$

From standard analysis, cf. [4], it follows that the problem (3.3)-(3.4) has a unique solution. From (3.4) it follows that $\mathbf{u}^{n+1} \in \mathbf{V}_{\mathrm{div}}$ and by restricting the test functions in (3.3) to $\mathbf{v} \in \mathbf{V}_{\mathrm{div}}$ we see that $\mathbf{u}^{n+1}$ also is the unique solution of (3.2). A spatial discretization can be realized by applying a finite element discretization to the saddle point problem (3.3)-(3.4). Related to this spatial discretization there is the following subtle point concerning the choice of the pressure variable p. Clearly the solution $\mathbf{u}^{n+1}$ doesn't change if p in (3.3) is replaced by $p + \hat{p}$, with $\hat{p}$ any given pressure function from Q. It is important to choose $p + \hat{p}$ in such a way that an accurate discretization of the pressure variable can be obtained. Comparing the Crank-Nicolson method in (3.3)-(3.4) with the continuous saddle point problem in (**??**)-(2.4) we see that p in (3.3) is an approximation of $\frac{1}{2}(p(\cdot, t_{n+1}) + p(\cdot, t_n))$. The exact pressure solution $p(\cdot, t)$ has a jump across $\Gamma(t)$. These observations lead to the following choice for p in (3.3):

$$p = \frac{1}{2}(p^{n+1} + p^n), \tag{3.5}$$

with $p^{n+1} \in Q$ the unknown pressure and p^n the pressure from the previous time step. In the spatial discretization of (3.3)-(3.4) one then has to determine a discrete pressure approximation $p^{n+1} \approx p(\cdot, t_{n+1})$. For this one should use a finite element space $Q_h^{n+1} \subset Q$ that is such that functions with jumps across $\Gamma(t_{n+1})$ can be approximated accurately. As an example we mention the XFEM technique, cf. [5, 12, 1, 3]. As expected, in the spatial discretization of the pressure we then have finite element spaces Q_h^{n+1} that are adapted to the location of the interface $\Gamma(t_{n+1})$ and thus depend on n. Using the choice as in (3.5) one has to determine an initial pressure p^0. This can be achieved by first applying one implicit Euler time step.

4. Discretization of a parabolic problem in a Hilbert space. In this section we first summarize some basic results on parabolic problems in Hilbert spaces, known from the literature, e.g. [18, 15, 20]. Then we analyze the Euler and Crank-Nicolson time discretization methods in this Hilbert space setting. The abstract analysis is applied in section 5 to derive discretization error bounds for these methods applied to the Stokes problem described in section 2.

Let $(Z, (\cdot,\cdot)_Z)$ be a Hilbert and $I := (0, T)$ a time interval. We recall two definitions on generalized derivatives in the Bochner-Lebesque space $L^2(I; Z)$. Take $u \in L^2(I; Z)$. If there exists $w \in L^2(I; Z)$ such that

$$\int_0^T (u(t), z)_Z \phi'(t)\, dt = -\int_0^T (w(t), z)_Z \phi(t)\, dt \quad \text{for all} \;\; \phi \in C_0^\infty(I), \;\; z \in Z,$$

then $w =: u'$ is called *the generalized derivative of u in $L^2(I; Z)$*.

We also need the following weaker notion of a generalized derivative of u. Let $(V, (\cdot,\cdot)_V)$, $(H, (\cdot,\cdot)_H)$ be Hilbert spaces that form a Gelfand triple $V \hookrightarrow H \hookrightarrow V'$. Take $u \in L^2(I; V)$. If there exists $w \in L^2(I; V')$ such that

$$\int_0^T (u(t), v)_H \phi'(t)\, dt = -\int_0^T w(t)(v)\phi(t)\, dt \quad \text{for all} \;\; \phi \in C_0^\infty(I), \;\; v \in V,$$

then $w =: u'$ is called *the generalized derivative of u in $L^2(I; V')$*.

These notions of generalized derivatives can be introduced in the more general setting of distribution theory, cf. [18, 20]. The definitions above mean that if for $u \in L^2(I; V)$ there exists a generalized derivative in $L^2(I; V)$ or in $L^2(I; V')$ then

the distributional derivative of u can be represented as a function, namely u', in $L^2(I;V)$ or in $L^2(I;V')$, respectively. If $u \in L^2(I;V)$ has a generalized derivative $u' \in L^2(I;V)$ then also the generalized derivative of u in $L^2(I;V')$ exists and is given by $t \to (u'(t), \cdot)_H$.

In the remainder we need the following spaces:

$$W(I) = \left\{ u \in L^2(I;V) : \; u' \in L^2(I;V') \;\; \text{exists} \right\},$$
$$W^k(I;Z) = \left\{ u \in L^2(I;Z) : \; u^{(n)} \in L^2(I;Z) \;, 1 \le n \le k, \; \text{exists} \right\}, \quad k \ge 1.$$

Note that $W(I)$ depends on V and H and that $W^1(I;V) \subset W(I)$ holds. Furthermore (cf. [18, 20]) there are continuous embeddings $W(I) \hookrightarrow C(I;H)$ and $W^1(I;Z) \hookrightarrow C(I;Z)$, which implies that for $u \in W(I)$, $v \in W^1(I;Z)$ the values $u(t) \in H$ and $v(t) \in Z$, $t \in I$, are well-defined (in the usual Lebesgue sense).

We consider a standard formulation of a parabolic problem in V. Let $a(\cdot,\cdot)$ be a continuous elliptic bilinear form on V, i.e., there are constants $\gamma > 0$ and $\hat{\gamma}$ such that $a(v,v) \ge \gamma \|v\|_V^2$ for all $v \in V$ and $a(u,v) \le \hat{\gamma} \|u\|_V \|v\|_V$ for all $u, v \in V$.

Let $f \in L^2(I;V')$ and $u_0 \in H$ be given. The abstract parabolic problem is as follows: determine $u \in W(I)$ such that $u(0) = u_0$ and

$$u'(t)(v) + a(u(t),v) = f(t)(v) \quad \text{for all} \;\; v \in V, \;\; t \in I. \tag{4.1}$$

It is well-known (e.g. [18, 20]) that this problem has a unique solution.

Remark 1. We discuss an abstract regularity result, cf. [18] section 27. Assume $u_0 \in V$ and define $f_0(v) := f(0)(v), a_0(v) := a(u_0, v), \; v \in V$. If the data satisfy the smoothness conditions

$$f \in W^1(I;V'), \tag{4.2}$$
$$f_0, \, a_0 \in H' \equiv H, \tag{4.3}$$

then the solution u of (4.1) has the smoothness properties

$$u \in W^1(I;V), \quad u' \in W(I). \tag{4.4}$$

It can be shown that in our application it is reasonable to assume that the conditions (4.2) and (4.3) are satisfied, cf. Remark 2. In the remainder of this section *we assume that the conditions in* (4.2), (4.3) *are satisfied*, hence for the solution u we have $u \in W^1(I;V)$ and $u' \in W(I)$. Then the time derivative in (4.1) can be reformulated as $u'(t)(v) = (u'(t), v)_H$, where u' on the right-hand side is the generalized derivative of u in $L^2(I;V)$. Due to the embeddings $W^1(I;V') \hookrightarrow C(I;V')$, $W^1(I;V) \hookrightarrow C(I;V)$, for $t \in I$ the values $f(t) \in V'$ and $u(t) \in V$ are well-defined.

4.1. Implicit Euler method. We first consider the implicit Euler method and derive an (expected) first order error bound. We assume $u_0 \in V$, hence $a(u_0, v)$, $v \in V$, is well-defined. For $N \in \mathbb{N}$ define the time step $\Delta t := T/N$. The sequence of approximations $u^n \in V$, $0 \le n \le N$, is defined as follows: $u^0 := u_0$ and for $n \ge 0$,

$$(u^{n+1} - u^n, v)_H + \Delta t \, a(u^{n+1}, v) = \Delta t \, f(t_{n+1})(v) \quad \text{for all} \;\; v \in V. \tag{4.5}$$

From the Lax-Milgram lemma it follows that the variational problem (4.5) has a unique solution $u_{n+1} \in V$. The discretization error is denoted by $e^n := u(t_n) - u^n \in V$. From (4.1) and (4.5) we obtain the error formula

$$(e^{n+1} - e^n, v)_H + \Delta t \, a(e^{n+1}, v) = R_a^n(v) + R_f^n(v), \quad v \in V, \;\; 0 \le n \le N-1, \tag{4.6}$$

with

$$R_a^n(v) := \Delta t\, a(u(t_{n+1}), v) - \int_{t_n}^{t_{n+1}} a(u(t), v)\, dt,$$
$$R_f^n(v) := \int_{t_n}^{t_{n+1}} f(t)(v)\, dt - \Delta t\, f(t_{n+1})(v).$$

LEMMA 4.1. *The following holds:*

$$\|R_a^n\|_{V'} \le \hat{\gamma}\Delta t \int_{t_n}^{t_{n+1}} \|u'(t)\|_V\, dt,$$
$$\|R_f^n\|_{V'} \le \Delta t \int_{t_n}^{t_{n+1}} \|f'(t)\|_{V'}\, dt.$$

Proof. For $g \in C^1([a,b])$ we have $\int_a^b g(x)\,dx - (b-a)g(b) = \int_a^b g'(x)(a-x)\,dx$ and thus

$$\left| \int_a^b g(x)\,dx - (b-a)g(b) \right| \le (b-a) \int_a^b |g'(x)|\,dx.$$

A density argument shows that this inequality also holds for g from the Sobolev space $H^1((a,b))$. Take $v \in V$ and define $g_v(t) := a(u(t), v)$. From $u \in W^1(I;V)$ it follows that $g_v \in H^1(I)$ and thus we get

$$\begin{aligned} |R_a^n(v)| &= \left| \int_{t_n}^{t_{n+1}} g_v(t)\, dt - \Delta t\, g_v(t_{n+1}) \right| \le \Delta t \int_{t_n}^{t_{n+1}} |g_v'(t)|\, dt \\ &= \Delta t \int_{t_n}^{t_{n+1}} |a(u'(t), v)|\, dt \le \hat{\gamma}\Delta t \int_{t_n}^{t_{n+1}} \|u'(t)\|_V\, dt\, \|v\|_V, \end{aligned}$$

and thus the bound for $\|R_a^n\|_{V'}$ is proved. The bound for $\|R_f^n\|_{V'}$ follows with the same arguments. ∎

Hence, we easily obtain an optimal error bound for the Euler method.

THEOREM 4.2. *The error bound*

$$\|e^n\|_H \le \frac{1}{\sqrt{\gamma}} \left(\hat{\gamma} \|u'\|_{L^2(I;V)} + \|f'\|_{L^2(I;V')} \right) \Delta t, \quad 0 \le n \le N, \tag{4.7}$$

holds.

Proof. From the error formula (4.6) with $v = e^{n+1}$ we get, using Lemma 4.1,

$$\begin{aligned} &\frac{1}{2}\|e^{n+1}\|_H^2 + \gamma\Delta t \|e^{n+1}\|_V^2 \\ &\le \frac{1}{2}\|e^n\|_H^2 + \Delta t\, \|e^{n+1}\|_V \left(\hat{\gamma} \int_{t_n}^{t_{n+1}} \|u'(t)\|_V\, dt + \int_{t_n}^{t_{n+1}} \|f'(t)\|_{V'}\, dt \right). \end{aligned}$$

Combining this with

$$\begin{aligned} &\Delta t\, \|e^{n+1}\|_V \left(\hat{\gamma} \int_{t_n}^{t_{n+1}} \|u'(t)\|_V\, dt + \int_{t_n}^{t_{n+1}} \|f'(t)\|_{V'}\, dt \right) \\ &\le \gamma\Delta t\, \|e^{n+1}\|_V^2 + \frac{(\Delta t)^2}{2\gamma} \left(\hat{\gamma}^2 \int_{t_n}^{t_{n+1}} \|u'(t)\|_V^2\, dt + \int_{t_n}^{t_{n+1}} \|f'(t)\|_{V'}^2\, dt \right), \end{aligned}$$

and using $e^0 = 0$ we get

$$\begin{aligned}\|e^{n+1}\|_H^2 &\le \frac{(\Delta t)^2}{\gamma}\sum_{k=0}^{n}\left(\hat{\gamma}^2\int_{t_k}^{t_{k+1}}\|u'(t)\|_V^2\,dt + \int_{t_k}^{t_{k+1}}\|f'(t)\|_{V'}^2\,dt\right)\\ &\le \frac{(\Delta t)^2}{\gamma}\big(\hat{\gamma}^2\|u'\|_{L^2(I;V)}^2 + \|f'\|_{L^2(I;V')}^2\big).\end{aligned}$$

Hence the error bound (4.7) holds. □

4.2. Crank-Nicolson method. We apply the Crank-Nicolson method to the problem in (4.1). We assume $u_0 \in V$, hence $a(u_0, v)$, $v \in V$, is well-defined. For $N \in \mathbb{N}$ define the time step $\Delta t := T/N$. The sequence of approximations $u^n \in V$, $0 \le n \le N$, is defined as follows: $u^0 := u_0$ and for $n \ge 0$,

$$(u^{n+1} - u^n, v)_H + \frac{\Delta t}{2}a(u^{n+1} + u^n, v) = \frac{\Delta t}{2}\big(f(t_{n+1})(v) + f(t_n)(v)\big) \quad \forall\, v \in V. \tag{4.8}$$

From the Lax-Milgram lemma it follows that the variational problem (4.8) has a unique solution $u_{n+1} \in V$. The discretization error is denoted by $e^n := u(t_n) - u^n \in V$. From (4.1) and (4.8) we obtain the error formula

$$(e^{n+1} - e^n, v)_H + \frac{\Delta t}{2}a(e^{n+1} + e^n, v) = R_a^n(v) + R_f^n(v), \quad v \in V,\ \ 0 \le n \le N-1, \tag{4.9}$$

with

$$\begin{aligned}R_a^n(v) &:= \frac{\Delta t}{2}a(u(t_n) + u(t_{n+1}), v) - \int_{t_n}^{t_{n+1}} a(u(t), v)\,dt,\\ R_f^n(v) &:= \int_{t_n}^{t_{n+1}} f(t)(v)\,dt - \frac{\Delta t}{2}\big(f(t_n)(v) + f(t_{n+1})(v)\big).\end{aligned}$$

For the analysis of these residual terms we use the following elementary result. Consider $g \in C^2([a,b])$. Using the Peano quadrature error representation formula

$$\int_a^b g(x)\,dx - \frac{b-a}{2}\big(g(a) + g(b)\big) = \frac{1}{2}\int_a^b g^{(2)}(x)(x-a)(x-b)\,dx$$

we obtain

$$\left|\int_a^b g(x)\,dx - \frac{b-a}{2}\big(g(a) + g(b)\big)\right| \le \frac{(b-a)^2}{8}\int_a^b |g^{(2)}(x)|\,dx. \tag{4.10}$$

A density argument shows that this inequality also holds for g from the Sobolev space $H^2((a,b))$.

Using (4.10) one can easily show that *if* the regularity condition $f \in W^2(I;V')$ is satisfied, then

$$|R_f^n(v)| \le \frac{(\Delta t)^2}{8}\int_{t_n}^{t_{n+1}}\|f^{(2)}(t)\|_{V'}\,dt\|v\|_V \quad \text{for all} \ \ v \in V$$

holds. In our applications, however, the assumption $f \in W^2(I;V')$ is *not* realistic, cf. Remark 3. Therefore, in the analysis in this section we use other (weaker) norms which are suitable for our applications.

For this we introduce the Friedrichs operator corresponding to the bilinear form. We briefly recall some elementary properties of this operator, cf. [20]. The bilinear form $a(\cdot,\cdot)$ on V can be represented by a bounded linear self-adjoint operator $A : V \to V'$, given by $(Au)(v) = a(u,v)$ for all $u, v \in V$. The corresponding Friedrichs operator $A_F : D(A_F) \subset H \to H$ is unbounded and has the following properties. Its domain $D(A_F)$ is dense in H and satisfies $D(A_F) \subset V$. Furthermore $(A_F u, v)_H = a(u,v) = (Au)(v)$ holds for all $u \in D(A_F)$, $v \in V$. Hence, there is a constant $c_F > 0$ such that $(A_F u, u)_H \geq c_F \|u\|_H^2$ for all $u \in D(A_F)$. The mapping A_F has a bounded self-adjoint inverse $A_F^{-1} : H \to H$ which satisfies $A_F^{-1} v = A^{-1} v$ for all $v \in H$. We also need the square root $A_F^{\frac{1}{2}} : D(A_F^{\frac{1}{2}}) \to H$ of A_F. This is a positive self-adjoint (unbounded) operator and its domain satisfies $D(A_F^{\frac{1}{2}}) = V$. Furthermore

$$a(u,v) = (A_F^{\frac{1}{2}} u, A_F^{\frac{1}{2}} v)_H \quad \text{for all} \quad u, v \in V$$

holds. Corresponding norms are defined by $\|v\|_{A^{-\alpha}} := (A_F^{-\alpha} v, A_F^{-\alpha} v)_H^{\frac{1}{2}}$, $v \in H$, with $\alpha = \frac{1}{2}$ or $\alpha = 1$. The following holds:

$$\|v\|_{A^{-1}} \leq c_F^{-\frac{1}{2}} \|v\|_{A^{-\frac{1}{2}}} \leq c_F^{-1} \|v\|_H \quad \text{for all} \quad v \in H. \tag{4.11}$$

Using the Friedrichs operator we can avoid the regularity condition $f \in W^2(I; V')$ and derive second order error bounds for the Crank-Nicolson method in a weaker norm than $\|\cdot\|_H$.

THEOREM 4.3. *Assume that the solution u satisfies $u \in W^2(I, H)$ and that there exists a constant c_f independent of n and Δt such that*

$$|R_f^n(A_F^{-1} v)| \leq c_f (\Delta t)^3 \|v\|_H \quad \textit{for all} \quad v \in V. \tag{4.12}$$

Then the error bound

$$\|e^n\|_{A^{-\frac{1}{2}}} \leq \left(\frac{1}{8} \|u^{(2)}\|_{L^2(I;H)} + c_f \sqrt{T} \right) (\Delta t)^2, \quad 0 \leq n \leq N, \tag{4.13}$$

holds.

Proof. From the error formula (4.9) with $v = A_F^{-1}(e^{n+1} + e^n)$ we get, with $w^n := e^{n+1} + e^n$,

$$\|e^{n+1}\|_{A^{-\frac{1}{2}}}^2 - \|e^n\|_{A^{-\frac{1}{2}}}^2 + \frac{1}{2} \Delta t \|w^n\|_H^2 = R_a^n(A_F^{-1} w^n) + R_f^n(A_F^{-1} w^n).$$

From (4.12) we get

$$|R_f^n(A_F^{-1} w^n)| \leq c_f (\Delta t)^3 \|w^n\|_H \leq c_f^2 (\Delta t)^5 + \frac{1}{4} \Delta t \|w^n\|_H^2.$$

For the other residual term we have

$$\begin{aligned} R_a^n(A_F^{-1} w^n) &= \frac{\Delta t}{2} (A_F^{\frac{1}{2}}(u(t_n) + u(t_{n+1})), A_F^{\frac{1}{2}} A_F^{-1} w^n)_H \\ &\quad - \int_{t_n}^{t_{n+1}} (A_F^{\frac{1}{2}} u(t), A_F^{\frac{1}{2}} A_F^{-1} w^n)_H \, dt \\ &= \frac{\Delta t}{2} (u(t_n) + u(t_{n+1}), w^n)_H - \int_{t_n}^{t_{n+1}} (u(t), w^n)_H \, dt. \end{aligned}$$

Applying (4.10) with $g_v(t) = (u(t), v)_H$ and using $u \in W^2(I;H)$ we get

$$\begin{aligned}|R_a^n(A_F^{-1}w^n)| &\leq \frac{1}{8}(\Delta t)^2 \int_{t_n}^{t_{n+1}} \|u^{(2)}\|_H \, dt \, \|w^n\|_H \\ &\leq \frac{1}{64}(\Delta t)^4 \int_{t_n}^{t_{n+1}} \|u^{(2)}\|_H^2 \, dt + \frac{1}{4}\Delta t \|w^n\|_H^2.\end{aligned}$$

Thus we obtain

$$\|e^{n+1}\|_{A^{-\frac{1}{2}}}^2 \leq \|e^n\|_{A^{-\frac{1}{2}}}^2 + \frac{1}{64}(\Delta t)^4 \int_{t_n}^{t_{n+1}} \|u^{(2)}\|_H^2 \, dt + c_f^2 (\Delta t)^5.$$

Recursive application of this inequality and $e^0 = 0$ yields

$$\|e^{n+1}\|_{A^{-\frac{1}{2}}}^2 \leq \Big(\frac{1}{64}\|u^{(2)}\|_{L^2(I;H)}^2 + T c_f^2\Big)(\Delta t)^4,$$

which completes the proof. □

In Theorem 4.3 we used the smoothness assumption $u \in W^2(I,H)$. From the data smoothness assumptions in (4.2) and (4.3) we can (only) conclude, cf. (4.4), that the generalized derivative $u^{(2)} \in L^2(I;V')$ exists, which is weaker than the property $u^{(2)} \in L^2(I;H)$ that is needed to have $u \in W^2(I,H)$. In concrete applications there may be regularity analyses (using e.g. parabolic smoothing effects) from which $u \in W^2(I,H)$ can be concluded. It is not clear whether in our application in section 5 a regularity property $u \in W^2(I,H)$ can be derived. Therefore, in Theorem 4.5 below we give a result in which this regularity assumption is not needed. The key idea is to replace the norm $\|\cdot\|_{A^{-\frac{1}{2}}}$ used in Theorem 4.3 by the weaker norm $\|\cdot\|_{A^{-1}}$.

LEMMA 4.4. *If $w \in W(I)$ then $A_F^{-\frac{1}{2}} w \in W^1(I;H)$ holds.*

Proof. For $w \in W(I)$ there exists a generalized derivative $w' \in L^2(I;V')$ such that

$$\int_0^T (w(t), v)_H \phi'(t) \, dt = -\int_0^T w'(t)(v)\phi(t) \, dt \quad \text{for all} \;\; \phi \in C_0^\infty(I), \;\; v \in V.$$

Take an arbitrary $z \in H$, hence $A_F^{-\frac{1}{2}} z \in V$ and

$$\int_0^T (A_F^{-\frac{1}{2}} w(t), z)_H \phi'(t) \, dt = \int_0^T (w(t), A_F^{-\frac{1}{2}} z)_H \phi'(t) \, dt = -\int_0^T w'(t)(A_F^{-\frac{1}{2}} z)\phi(t) \, dt$$

for all $\phi \in C_0^\infty(I)$. From

$$\begin{aligned}\int_0^T |w'(t)(A_F^{-\frac{1}{2}} z)|^2 \, dt &\leq \int_0^T \|w'(t)\|_{V'}^2 \, dt \, \|A_F^{-\frac{1}{2}} z\|_V^2 \leq \|w'\|_{L^2(I;V')}^2 \frac{1}{\gamma} a(A_F^{-\frac{1}{2}} z, A_F^{-\frac{1}{2}} z) \\ &= \|w'\|_{L^2(I;V')}^2 \frac{1}{\gamma} \|z\|_H^2\end{aligned}$$

it follows that $w'(t)(A_F^{-\frac{1}{2}} \cdot) \in L^2(I;H') \equiv L^2(I;H)$. Hence, $A_F^{-\frac{1}{2}} w$ has a generalized derivative in $L^2(I;H)$. □

For the solution u we have $u \in W^1(I;V)$, $u' \in W(I)$ and thus, using the result in Lemma 4.4, we get $A_F^{-\frac{1}{2}} u \in W^2(I;H)$.

THEOREM 4.5. *Assume that there exists a constant c_f independent of n and Δt such that*

$$|R_f^n(A_F^{-1}v)| \le c_f(\Delta t)^3\|v\|_H \quad \textit{for all } \ v \in V. \tag{4.14}$$

Then the error bound

$$\|e^n\|_{A^{-1}} \le \left(\frac{1}{8}\|(A_F^{-\frac{1}{2}}u)^{(2)}\|_{L^2(I;H)} + c_f c_F^{-\frac{1}{2}}\sqrt{T}\right)(\Delta t)^2, \quad 0 \le n \le N, \tag{4.15}$$

holds, with c_F as in (4.11).

Proof. We use similar arguments as in the proof of Theorem 4.3. From the error formula (4.9) with $v = A_F^{-2}(e^{n+1}+e^n)$ we get, with $w^n := e^{n+1}+e^n$,

$$\|e^{n+1}\|_{A^{-1}}^2 - \|e^n\|_{A^{-1}}^2 + \frac{1}{2}\Delta t\|w^n\|_{A^{-\frac{1}{2}}}^2 = R_a^n(A_F^{-2}w^n) + R_f^n(A_F^{-2}w^n).$$

From (4.14) and (4.11) we get

$$\begin{aligned}|R_f^n(A_F^{-2}w^n)| &\le c_f(\Delta t)^3\|A_F^{-1}w^n\|_H = c_f(\Delta t)^3\|w^n\|_{A^{-1}}\\ &\le c_f c_F^{-\frac{1}{2}}(\Delta t)^3\|w^n\|_{A^{-\frac{1}{2}}} \le c_f^2 c_F^{-1}(\Delta t)^5 + \frac{1}{4}\Delta t\|w^n\|_{A^{-\frac{1}{2}}}^2.\end{aligned}$$

For the other residual term we have

$$R_a^n(A_F^{-2}w^n) = \frac{\Delta t}{2}\big(A_F^{-\frac{1}{2}}(u(t_n)+u(t_{n+1})), A_F^{-\frac{1}{2}}w^n\big)_H - \int_{t_n}^{t_{n+1}}(A_F^{-\frac{1}{2}}u(t), A_F^{-\frac{1}{2}}w^n)_H\,dt.$$

For the solution u we have $A_F^{-\frac{1}{2}}u \in W^2(I;H)$, and thus applying (4.10) with $g_v(t) = (A_F^{-\frac{1}{2}}u(t), A_F^{-\frac{1}{2}}v)_H$ we get

$$\begin{aligned}|R_a^n(A_F^{-2}w^n)| &\le \frac{1}{8}(\Delta t)^2\int_{t_n}^{t_{n+1}}\|(A_F^{-\frac{1}{2}}u)^{(2)}\|_H\,dt\,\|A_F^{-\frac{1}{2}}w^n\|_H\\ &\le \frac{1}{64}(\Delta t)^4\int_{t_n}^{t_{n+1}}\|(A_F^{-\frac{1}{2}}u)^{(2)}\|_H^2\,dt + \frac{1}{4}\Delta t\|w^n\|_{A^{-\frac{1}{2}}}^2.\end{aligned}$$

Thus we obtain

$$\|e^{n+1}\|_{A^{-1}}^2 \le \|e^n\|_{A^{-1}}^2 + \frac{1}{64}(\Delta t)^4\int_{t_n}^{t_{n+1}}\|(A_F^{-\frac{1}{2}}u)^{(2)}\|_H^2\,dt + c_f^2c_F^{-1}(\Delta t)^5.$$

Recursive application of this inequality and $e^0 = 0$ yields

$$\|e^{n+1}\|_{A^{-1}}^2 \le \Big(\frac{1}{64}\|(A_F^{-\frac{1}{2}}u)^{(2)}\|_{L^2(I;H)}^2 + Tc_f^2c_F^{-1}\Big)(\Delta t)^4,$$

which completes the proof. □

5. Analysis of the Stokes two-phase flow equation. In this section we analyze the Stokes problem introduced in section 2 using the abstract setting presented in section 4. First we introduce suitable function spaces. Let $\mathbf{V} := H_0^1(\Omega)^d$,

$$N(\Omega) := \left\{ \mathbf{v} \in C_0^\infty(\Omega)^d : \operatorname{div} \mathbf{v} = 0 \right\},$$

and

$$\mathbf{H}_{\mathrm{div}} := \overline{N(\Omega)}^{\|\cdot\|_{L^2}} \quad \text{(closure of } N(\Omega) \text{ in } L^2(\Omega)^d \text{)}, \tag{5.1}$$

$$\mathbf{V}_{\mathrm{div}} := \overline{N(\Omega)}^{\|\cdot\|_1} \quad \text{(closure of } N(\Omega) \text{ in } \mathbf{V}). \tag{5.2}$$

The spaces $\left(\mathbf{H}_{\mathrm{div}}, \|\cdot\|_{L^2}\right)$, $\left(\mathbf{V}_{\mathrm{div}}, \|\cdot\|_1\right)$ are Hilbert spaces. From $\|\mathbf{v}\|_{L^2} \leq c\|\mathbf{v}\|_1$ for all $\mathbf{v} \in N(\Omega)$ and a density argument it follows that there is a continuous embedding $\mathbf{V}_{\mathrm{div}} \hookrightarrow \mathbf{H}_{\mathrm{div}}$. Using $N(\Omega) \subset \mathbf{V}_{\mathrm{div}} \subset \mathbf{H}_{\mathrm{div}}$ and the fact that $\mathbf{H}_{\mathrm{div}}$ is the closure of $N(\Omega)$ w.r.t. $\|\cdot\|_{L^2}$ it follows that $\mathbf{V}_{\mathrm{div}}$ is dense in $\mathbf{H}_{\mathrm{div}}$. Thus we have a Gelfand triple

$$\mathbf{V}_{\mathrm{div}} \hookrightarrow \mathbf{H}_{\mathrm{div}} \equiv \mathbf{H}_{\mathrm{div}}' \hookrightarrow \mathbf{V}_{\mathrm{div}}'. \tag{5.3}$$

The space $\mathbf{V}_{\mathrm{div}}$ can also be characterized as, cf. [15],

$$\mathbf{V}_{\mathrm{div}} = \left\{ \mathbf{v} \in \mathbf{V} : \operatorname{div} \mathbf{v} = 0 \right\}.$$

Corresponding to this Gelfand triple we define the space $W(I)$ as in section 4. We consider the variational Stokes problem given in (2.9), which fits in the abstract setting given in section 4. The right-hand side in the Stokes flow consists of two terms. Firstly, a time independent force (gravity), represented by the functional $g(\mathbf{v}) = (\mathbf{g}, \mathbf{v})_{L^2}$ with a given function $\mathbf{g} \in L^2(\Omega)^d$. Secondly, there is a surface tension force located at the (evolving) interface, represented by the time dependent functional

$$f(t)(\mathbf{v}) = -\tau \int_{\Gamma(t)} \kappa \mathbf{n} \cdot \mathbf{v} \, ds.$$

We take $\tau = 1$. The right-hand side term g does not play a role in the error analysis and is neglected in the remainder.

Related to the evolving interface $\Gamma(t)$ we make several smoothness assumptions. In particular, we assume that for all $t \in I = (0, T)$ the interface $\Gamma(t)$ is a C^2-hypersurface in $\mathbb{R}^d$. Furthermore we assume that $\Gamma(t)$ is the boundary of a connected subdomain $\Omega_1(t) \subset \Omega$ with (for simplicity) $\overline{\Omega}_1(t) \cap \partial\Omega = \emptyset$. The interface is assumed to be transported by a smooth velocity field $\mathbf{b} = \mathbf{b}(x, t)$. For convenience we assume $\operatorname{div} \mathbf{b} = 0$. This assumption is not essential for the analysis presented below. Finally, we assume that the curvature function $\kappa = \operatorname{div} \mathbf{n}$, which is defined (only) on $\Gamma(t)$ has a smooth global extension, denoted by $\kappa(x, t)$, $(x, t) \in \Omega \times I$. These smoothness assumptions are reasonable if one considers viscous two-phase flows with a smoothly evolving interface.

A key issue is the regularity with respect to t of the right-hand side functional f. We investigate this regularity issue and start with the following observation:

$$f(t)(\mathbf{v}) = -\int_{\Gamma(t)} \kappa \mathbf{n} \cdot \mathbf{v} \, ds = -\int_{\Omega_1(t)} \nabla \kappa \cdot \mathbf{v} \, dx \quad \text{for all} \quad \mathbf{v} \in \mathbf{V}_{\mathrm{div}}. \tag{5.4}$$

From this result it follows that $f \in L^2(I;\mathbf{V}'_{\rm div})$ holds. From the next lemma it follows, cf. Corollary 5.2, that $f \in W^1(I;\mathbf{V}'_{\rm div})$ holds.

For smooth vector functions $\mathbf{q} = \mathbf{q}(x,t) \in \mathbb{R}^d$ and matrix functions $Q = Q(x,t) \in \mathbb{R}^{d\times d}$ we use the notation $\|\mathbf{q}\|_\infty := \max\{\,|\mathbf{q}(x,t)|_2 : (x,t) \in \Omega \times I\,\}$, where $|\cdot|_2$ denotes the Euclidean vector norm. and $\|Q\|_\infty := \max\{\,|Q(x,t)|_F : (x,t) \in \Omega \times I\,\}$, with $|\cdot|_F$ the Frobenius norm of the matrix Q.

LEMMA 5.1. *Let $\mathbf{q} = \mathbf{q}(x,t) \in \mathbb{R}^d$ be a smooth vector function. Then*

$$\left| \frac{d}{dt} \int_{\Omega_1(t)} \mathbf{q}\cdot\mathbf{v}\,dx \right| \le c_1 \|\mathbf{v}\|_1 \quad \textit{for all} \;\; \mathbf{v} \in C_0^\infty(\Omega)^d$$

holds, with $c_1 = c_1(\|\mathbf{q}\|_\infty, \|\nabla\mathbf{q}\|_\infty, \|\frac{\partial \mathbf{q}}{\partial t}\|_\infty, \|\mathbf{b}\|_\infty)$ independent of t and $\mathbf{v}$.

Proof. Take $\mathbf{v} \in C_0^\infty(\Omega)^d$. From Reynolds' transport theorem and $\operatorname{div}\mathbf{v} = 0$ we get

$$\begin{aligned}\left| \frac{d}{dt} \int_{\Omega_1(t)} \mathbf{q}\cdot\mathbf{v}\,dx \right| &= \left| \int_{\Omega_1(t)} \frac{\partial \mathbf{q}}{\partial t}\cdot\mathbf{v} + \mathbf{b}\cdot\nabla\mathbf{q}\,\mathbf{v} + \mathbf{b}\cdot\nabla\mathbf{v}\,\mathbf{q}\,dx \right| \\ &\le c \left(\int_{\Omega_1(t)} |\mathbf{v}|_2^2 + |\nabla\mathbf{v}|_F^2\,dx \right)^{\frac{1}{2}} \le c\|\mathbf{v}\|_1,\end{aligned}$$

and the result holds. ∎

COROLLARY 5.2. *We use the result in Lemma 5.1 with $\mathbf{q} = \nabla\kappa$ and apply a density argument. This implies that $f \in W^1(I;\mathbf{V}')$ holds.*

REMARK 2. We comment on the data smoothness assumptions discussed in Remark 1, with $V = \mathbf{V}_{\rm div}$ and $H = \mathbf{H}_{\rm div}$. We assume that the initial data $\mathbf{u}_0$ satsfies $\mathbf{u}_0 \in \mathbf{V}_{\rm div}$. From Corollary 5.2 it follows that the condition in (4.2) is satisfied. From $|f(0)(\mathbf{v})| \le c\|\mathbf{v}\|_{L^2}$ for all $\mathbf{v} \in \mathbf{V}_{\rm div}$ and a density argument it follows that $f(0) \in \mathbf{H}'_{\rm div} \equiv \mathbf{H}_{\rm div}$ holds. We assume that the initial condition $\mathbf{u}_0$ is sufficiently smooth, e.g. $\mathbf{u}_0 \in H^2(\Omega)^d$, such that $\mathbf{v} \to a(\mathbf{u}_0, bv)$ is an element of $\mathbf{H}'_{\rm div}$, too. Hence the assumption in (4.3) is also satisfied. From the abstract regularity result in (4.4) we conclude that the solution of the Stokes problem has the regularity properties $\mathbf{u} \in W^1(I;\mathbf{V}_{\rm div})$ and $\mathbf{u}' \in W(I)$, i.e. the generalized second derivative of $\mathbf{u}$ exists in $L^2(I;\mathbf{V}'_{\rm div})$.

From the regularity properties discussed in Remark 2 and the analysis in section 4.1 we obtain the following optimal error bound for the Euler method.

THEOREM 5.3. *The errors $\mathbf{e}^n = \mathbf{u}(t_n) - \mathbf{u}^n$ in the Euler method (3.1) are bounded by*

$$\|\mathbf{e}^n\|_{L^2} \le \frac{1}{\sqrt{\gamma}} \left(\|\mathbf{u}'\|_{L^2(I;\mathbf{V}_{\rm div})} + \|f'\|_{L^2(I;\mathbf{V}'_{\rm div})} \right) \Delta t, \quad 0 \le n \le N, \tag{5.5}$$

where γ is the ellipticity constant of $a(\cdot,\cdot)$ in $\mathbf{V}$.

Proof. Follows immediately from Theorem 4.2. ∎

REMARK 3. We note that $f \in W^2(I;\mathbf{V}'_{\rm div})$ is *not* a realistic assumption. Consider a simplified one-dimensional analogon given by $f(t)(v) := \int_0^{\Gamma(t)} v(x)\,dx$, $v \in V := H_0^1((0,1))$ and $\Gamma(t)$ a smooth function with $0 < \Gamma(t) < 1$. For smooth functions v we have $\frac{d}{dt} f(t)(v) = v(\Gamma(t))$ and $\frac{d^2}{dt^2} f(t)(v) = v'(\Gamma(t))\Gamma'(t)$. From this it follows that $f \in W^1(I;V')$ but $f \notin W^2(I;V')$.

In the abstract error analysis of the Crank-Nicolson method given in Theorem 4.3 we need a bound for the residual term R_f^n, cf. (4.12). Below in Theorem 5.5 we derive such a bound. We first derive a result that will be used in the proof of that theorem.

LEMMA 5.4. *For $f \in W^1(I;\mathbf{V}'_{\rm div})$ as in* (5.4) *the following holds:*

$$\left|\frac{d^2}{dt^2} f(t)(\mathbf{v})\right| \le c\big(\|\mathbf{v}\|_1 + \|\mathbf{v}\|_{H^2(\Omega_1(t))}\big) \quad \textit{for all} \;\; \mathbf{v} \in N(\Omega), \tag{5.6}$$

with a constant c that depends only on (smoothness properties of) κ and $\mathbf{b}$.

Proof. From Reynolds' transport theorem and using $\operatorname{div}\mathbf{b} = 0$ we obtain

$$\begin{aligned}\frac{d}{dt} f(t)(\mathbf{v}) &= \int_{\Omega_1(t)} \nabla\kappa_t \cdot \mathbf{v} + \mathbf{b}\cdot\nabla^2\kappa\,\mathbf{v} + \mathbf{b}\cdot\nabla\mathbf{v}\nabla\kappa\,dx \\ &= \int_{\Omega_1(t)} (\nabla\kappa_t + \nabla^2\kappa\mathbf{b})\cdot\mathbf{v}\,dx + \int_{\Omega_1(t)} \mathbf{b}\cdot\nabla\mathbf{v}\nabla\kappa\,dx.\end{aligned}$$

For the second derivative w.r.t. t we get, using Lemma 5.1,

$$\left|\frac{d}{dt}\int_{\Omega_1(t)} (\nabla\kappa_t + \nabla^2\kappa\mathbf{b})\cdot\mathbf{v}\,dx\right| \le c\|\mathbf{v}\|_1.$$

For the second term on the right-hand side we again apply Reynolds' theorem and obtain

$$\begin{aligned}\left|\frac{d}{dt}\int_{\Omega_1(t)} \mathbf{b}\cdot\nabla\mathbf{v}\nabla\kappa\,dx\right| &\le \int_{\Omega_1(t)} \left|\frac{\partial\mathbf{b}}{\partial t}\cdot\nabla\mathbf{v}\nabla\kappa + \mathbf{b}\cdot\nabla\mathbf{v}\nabla\kappa_t + \mathbf{b}\cdot\nabla(\mathbf{b}\cdot\nabla\mathbf{v}\nabla\kappa)\right| dx \\ &\le c\|\mathbf{v}\|_{H^2(\Omega_1(t))}.\end{aligned}$$

Combination of the two bounds completes the proof. ∎

In order to be able to derive the main result we need a regularity assumption concerning the solution of a *stationary* Stokes problem. For this we introduce the Friedrichs operator A_F as explained in section 4 and corresponding to the Gelfand triple in (5.3) and the Stokes bilinear form $a(\cdot,\cdot)$ as defined in (2.6). The domain of A_F, denoted by $D(A_F) \subset \mathbf{V}_{\rm div}$, equipped with the inner product $(\mathbf{u},\mathbf{v})_A := (A_F\mathbf{u}, A_F\mathbf{v})_{L^2}$ is a Hilbert space. We introduce the subspace of $\mathbf{V}_{\rm div}$ of functions that are in H^2 on the subdomain $\Omega_1(t)$:

$$S = S(t) := \{\, \mathbf{v} \in \mathbf{V}_{\rm div} \mid \mathbf{v}_{|\Omega_1(t)} \in H^2(\Omega_1(t))^d \,\},$$

with norm $\|\mathbf{v}\|_S^2 = \|\mathbf{v}\|_1^2 + \|\mathbf{v}\|_{H^2(\Omega_1(t))}^2$. We introduce the following regularity assumption: there exists a continuous embedding $D(A_F) \hookrightarrow S(t)$ that is uniform in $t \in I$, i.e. there exists a constant c independent of t such that

$$\|\mathbf{v}\|_S \le c\|\mathbf{v}\|_A \quad \text{for all} \;\; \mathbf{v} \in D(A_F). \tag{5.7}$$

REMARK 4. This regularity assumption is a rather weak one, since it only requires, uniformly in t, H^2-regularity of the stationary Stokes equations in the *interior* subdomain $\Omega_1(t) \subset \Omega$.

THEOREM 5.5. *For f as in* (5.4) *define* $R_f^n(\mathbf{v}) := \int_{t_n}^{t_{n+1}} f(t)(\mathbf{v})\,dt - \frac{\Delta t}{2}\big(f(t_n)(\mathbf{v}) + f(t_{n+1})(\mathbf{v})\big)$, $\mathbf{v} \in \mathbf{V}_{\mathrm{div}}$. *Assume that the regularity assumption* (5.7) *is satisfied. Then the following holds:*

$$|R_f^n(A_F^{-1}\mathbf{v})| \le c_f(\Delta t)^3 \|\mathbf{v}\|_{L^2} \quad \textit{for all} \quad \mathbf{v} \in \mathbf{V}_{\mathrm{div}},$$

with a constant c_f independent of $\mathbf{v}$, n and Δt.

Proof. Take $\mathbf{v} \in N(\Omega)$. Using (4.10) and (5.6) we get

$$\begin{aligned} |R_f^n(A_F^{-1}\mathbf{v})| &\le \frac{1}{8}(\Delta t)^2 \int_{t_n}^{t_{n+1}} \left| \frac{d^2}{dt^2} f(t)(A_F^{-1}\mathbf{v}) \right| dt \\ &\le c(\Delta t)^3 \big(\|A_F^{-1}\mathbf{v}\|_1 + \max_{t\in I} \|A_F^{-1}\mathbf{v}\|_{H^2(\Omega_1(t))} \big). \end{aligned}$$

From the Friedrichs inequality and (4.11) we obtain

$$\|A_F^{-1}\mathbf{v}\|_1 \le c\, a(A_F^{-1}\mathbf{v}, A_F^{-1}\mathbf{v})^{\frac{1}{2}} = c\|A_F^{-\frac{1}{2}}\mathbf{v}\|_{L^2} \le c\|\mathbf{v}\|_{L^2}.$$

Using the regularity assumption (5.7) it follows that

$$\max_{t\in I} \|A_F^{-1}\mathbf{v}\|_{H^2(\Omega_1(t))} \le \max_{t\in I} \|A_F^{-1}\mathbf{v}\|_S \le c\|A_F^{-1}\mathbf{v}\|_A = c\|\mathbf{v}\|_{L^2}.$$

Thus we conclude that $|R_f^n(A_F^{-1}\mathbf{v})| \le c_f(\Delta t)^3\|\mathbf{v}\|_{L^2}$ holds for all $\mathbf{v} \in N(\Omega)$ and, by a density argument, also for all $\mathbf{v} \in \mathbf{V}_{\mathrm{div}}$. ∎

Application of the general results in Theorem 4.3 and Theorem 4.5 result in the following $\mathcal{O}(\Delta t^2)$ error bounds for the Crank-Nicolson method.

THEOREM 5.6. *Assume that the regularity assumption* (5.7) *is satisfied and that the solution* $\mathbf{u}$ *satisfies* $\mathbf{u} \in W^2(I, \mathbf{H}_{\mathrm{div}})$. *Then the errors* $\mathbf{e}^n = \mathbf{u}(t_n) - \mathbf{u}^n$ *in the Crank-Nicolson method* (3.2) *are bounded by*

$$\|\mathbf{e}^n\|_{A^{-\frac{1}{2}}} \le \left(\frac{1}{8}\|\mathbf{u}^{(2)}\|_{L^2(I;\mathbf{H}_{\mathrm{div}})} + c_f\sqrt{T} \right) (\Delta t)^2, \quad 0 \le n \le N. \tag{5.8}$$

Proof. Follows immediately from Theorem 4.3 and Theorem 5.5. ∎

In general we only have, cf. Remark 2, the regularity properties $\mathbf{u} \in W^1(I; \mathbf{V}_{\mathrm{div}})$ and $\mathbf{u}' \in W(I)$, i.e. $\mathbf{u}^{(2)} \in L^2(I; \mathbf{V}'_{\mathrm{div}})$. In Theorem 5.6 we need the (slightly) stronger regularity condition $\mathbf{u} \in W^2(I, \mathbf{H}_{\mathrm{div}})$. It is not clear, whether under realistic assumptions this condition holds. In the next theorem this regularity condition is circumvented.

THEOREM 5.7. *Assume that the regularity assumption* (5.7) *is satisfied. Then the errors* $\mathbf{e}^n = \mathbf{u}(t_n) - \mathbf{u}^n$ *in the Crank-Nicolson method* (3.2) *are bounded by*

$$\|\mathbf{e}^n\|_{A^{-1}} \le \left(\frac{1}{8}\|(A_F^{-1}\mathbf{u})^{(2)}\|_{L^2(I;\mathbf{H}_{\mathrm{div}})} + c_f c_F^{-\frac{1}{2}}\sqrt{T} \right) (\Delta t)^2, \quad 0 \le n \le N. \tag{5.9}$$

Proof. Follows immediately from Theorem 4.5 and Theorem 5.5. ∎

6. Generalizations. We briefly address a few generalizations of the analysis presented above.

Instead of the Stokes problem one can consider a linearized Navier-Stokes problem (Oseen equation) of the following form. Let $\hat{\mathbf{u}} = \hat{\mathbf{u}}(x,t)$ be a given (sufficiently smooth) velocity field with $\operatorname{div} \hat{\mathbf{u}} = 0$ and consider a bilinear form $\tilde{a}(\cdot,\cdot)$ on $\mathbf{V}$ given by

$$\tilde{a}(\mathbf{u},\mathbf{v}) = a(\mathbf{u},\mathbf{v}) + (\hat{\mathbf{u}}\cdot\nabla\mathbf{u},\mathbf{v})_{L^2} = a(\mathbf{u},\mathbf{v}) - (\mathbf{u},\hat{\mathbf{u}}\cdot\nabla\mathbf{v})_{L^2},$$

with $a(\cdot,\cdot)$ as in (2.6). For the convection term we have

$$\begin{aligned}
|(\mathbf{u},\hat{\mathbf{u}}\cdot\nabla\mathbf{v})_{L^2}| &\le \|\hat{\mathbf{u}}\|_\infty \|\mathbf{u}\|_{L^2} a(\mathbf{v},\mathbf{v})^{\frac{1}{2}},\\
|\frac{d}{dt}(\mathbf{u},\hat{\mathbf{u}}\cdot\nabla\mathbf{v})_{L^2}| &\le c(\|\mathbf{u}\|_{L^2} + \|\mathbf{u}'\|_{L^2}) a(\mathbf{v},\mathbf{v})^{\frac{1}{2}},\\
|\frac{d^2}{dt^2}(\mathbf{u},\hat{\mathbf{u}}\cdot\nabla\mathbf{v})_{L^2}| &\le c(\|\mathbf{u}\|_{L^2} + \|\mathbf{u}'\|_{L^2} + \|\mathbf{u}^{(2)}\|_{L^2}) a(\mathbf{v},\mathbf{v})^{\frac{1}{2}},
\end{aligned}$$

with constants $c = c(\hat{\mathbf{u}})$ independent of t. The Oseen generalization of the Stokes problem in (2.9) is as follows: determine $\mathbf{u} \in W(I)$ such that $\mathbf{u}(0) = \mathbf{u}_0$ and

$$\mathbf{u}'(t)(\mathbf{v}) + \tilde{a}(\mathbf{u}(t),\mathbf{v}) = f(t)(\mathbf{v}) + g(\mathbf{v}) \quad \text{for all } \ \mathbf{v} \in \mathbf{V}_{\mathrm{div}}, \ \ t \in I.$$

For this problem, discretization error bounds for the implicit Euler and Crank-Nicolson method can be derived based on a slight generalization of the abstract analysis in section 4. For this we introduce the bilinear form

$$\tilde{a}(u,v) = a(u,v) + a_c(u,v)$$

with $a(\cdot,\cdot)$ as in section 4 and $a_c(u,v) = a_c(t;u,v)$ a (time-dependent) continuous bilinear form on V such that

$$\begin{aligned}
&a_c(u,u) = 0 \quad \text{for all } \ u \in V,\\
&a_c(u,v) \le c_d \|u\|_H \, a(v,v)^{\frac{1}{2}}, \quad \text{for all } \ u,v \in V, \ t \in I,\\
|\frac{d^k}{dt^k} a_c(u,v)| &\le c\big(\sum_{j=0}^{k} (\|u^{(j)}\|_H\big) \, a(v,v)^{\frac{1}{2}}, \quad k = 1,2, \ u \in W^k(I;H), \ v \in V, \ t \in I,
\end{aligned}$$

with constants c_d and c independent of t. We consider the abstract parabolic problem as in (4.1) with $a(\cdot,\cdot)$ replaced by $\tilde{a}(\cdot,\cdot)$. The regularity results formulated in Remark 1 still hold. The abstract analysis of the implicit Euler method in section 4.1 can be applied with only minor modifications. In the error formula (4.6) we now have $\tilde{a}(e^{n+1},v)$ and $R^n_{\tilde{a}}(v) = R^n_a(v) + R^n_{a_c}(v)$. Taking $v = e^{n+1}$ we get $\tilde{a}(e^{n+1},e^{n+1}) = a(e^{n+1},e^{n+1}) \ge \gamma \|e^{n+1}\|^2_V$. The additional (due to convection) residual term $R^n_{a_c}(v)$ can be bounded by

$$|R^n_{a_c}(v)| \le \Delta t \int_{t_n}^{t_{n+1}} \left| \frac{d}{dt} a_c(u,v) \right| dt \le c\Delta t \int_{t_n}^{t_{n+1}} \|u(t)\|_H + \|u'(t)\|_H \, dt \, \|v\|_V.$$

Using these results the same analysis as in the proof of Theorem 4.2 can be applied, leading to a first order error bound of the form $\|e^n\|_H \le c\Delta t$. The analysis of the Crank-Nicolson method in section 4.2 can also be generalized. We briefly address the

analysis in Theorem 4.3. In the error formula we now have, with $v = A_F^{-1}w^n$, the term $\tilde{a}(w^n, A_F^{-1}w^n)$ which can be estimated by

$$\tilde{a}(w^n, A_F^{-1}w^n) = a(w^n, A_F^{-1}w^n) + a_c(w^n, A_F^{-1}w^n) = \|w^n\|_H^2 + a_c(w^n, A_F^{-1}w^n) \geq$$
$$\|w^n\|_H^2 - c_d\|w^n\|_H a(A_F^{-1}w^n, A_F^{-1}w^n)^{\frac{1}{2}} \geq (1 - c_d c_F^{-\frac{1}{2}})\|w^n\|_H^2.$$

We assume that $1 - c_d c_F^{-\frac{1}{2}} > 0$ holds (which in the application is fulfilled if $\|\hat{\mathbf{u}}\|_\infty$ is sufficiently small). The residual term $R_a^n(A_F^{-1}v) = R_a^n(A_F^{-1}v) + R_{a_c}^n(A_F^{-1}v)$ can be treated by using

$$\begin{aligned}
|R_{a_c}^n(A_F^{-1}v)| &\leq \frac{(\Delta t)^2}{8}\int_{t_n}^{t_{n+1}} \left|\frac{d^2}{dt^2}a_c(u, A_F^{-1}v)\right| dt \\
&\leq c(\Delta t)^2 \int_{t_n}^{t_{n+1}} \sum_{j=0}^{2} \|u^{(j)}(t)\|_H \, dt \, a(A_F^{-1}v, A_F^{-1}v)^{\frac{1}{2}} \\
&\leq c(\Delta t)^2 \int_{t_n}^{t_{n+1}} \sum_{j=0}^{2} \|u^{(j)}(t)\|_H \, dt \, \|v\|_H.
\end{aligned}$$

Using these estimates the analysis in Theorem 4.3 applies with minor modifications. A similar generalization of Theorem 4.5 can be obtained.

We return to the Stokes problem and consider the case with piecewise constant viscosities $\mu_i(t)$ in the subdomains $\Omega_i(t)$, with $\mu_1 \neq \mu_2$. The corresponding bilinear form is denoted by

$$a_\mu(\mathbf{u}, \mathbf{v}) = \int_\Omega \mu(t)\nabla\mathbf{u} \cdot \nabla\mathbf{v} \, dx = \sum_{i=1}^{2} \int_{\Omega_i(t)} \mu_i \nabla\mathbf{u} \cdot \nabla\mathbf{v} \, dx.$$

Note that $\mu_{\min} a(\mathbf{u}, \mathbf{u}) \leq a_\mu(\mathbf{u}, \mathbf{u}) \leq \mu_{\max} a(\mathbf{u}, \mathbf{u})$ for all $\mathbf{u} \in \mathbf{V}$, with $\mu_{\min} := \min\{\mu_1, \mu_2\}$, $\mu_{\max} := \max\{\mu_1, \mu_2\}$. The Stokes problem considered is as in (2.9), with $a(\cdot,\cdot)$ replaced by $a_\mu(\cdot,\cdot)$. Existence and uniqueness of a solution of this problem follow from the literature, cf. [18]. In the analysis of the implicit Euler method as in section 4.1 the following complication arises. We have to control the time derivative of $g_{\mathbf{v}}(t) := a_{\mu(t)}(\mathbf{u}(t), \mathbf{v})$. For sufficiently smooth $\mathbf{u}, \mathbf{v}$ we have

$$\frac{d}{dt}a_\mu(\mathbf{u}(t), \mathbf{v}) = \sum_{i=1}^{2} \int_{\Omega_i(t)} \mu_i \nabla\mathbf{u}'(t) \cdot \nabla\mathbf{v} + \mu_i \mathbf{b} \cdot \nabla(\nabla\mathbf{u} \cdot \nabla\mathbf{v}) \, dx.$$

Hence there occur second order spatial derivatives of $\mathbf{v}$ and due to this a bound of the form $|R_{a_\mu}^n(\mathbf{v})| \leq c(\mathbf{u})\|\mathbf{v}\|_V$, similar to the one in Lemma 4.1, does not hold. To deal with this problem one could use a weaker norm (as in the analysis of the Crank-Nicolson method) and, instead of $\mathbf{e}^{n+1}$, insert $A_F^{-1}\mathbf{e}^{n+1}$ in the error formula (4.6), with A_F a Friedrichs operator. Obvious possibilities for this A_F are the Friedrichs operator correpsonding to the Stokes bilinear form $a(\cdot,\cdot)$ (as in section 5), or the one corresponding to the bilinear form $a_{\mu(t_{n+1})}(\cdot,\cdot)$. A further investigation of this is left for future research.

Finally we comment on the case of a two-phase Stokes problem with a constant viscosity but piecewise constant densities $\rho_i(t)$ in the subdomains $\Omega_i(t)$, with $\rho_1 \neq \rho_2$. This case is more involved and can not be analyzed by a simple modification of the analysis presented in this paper. Difficulties are caused by the fact that one has to control the error that arises due to the discretization of the term $\int_{t_n}^{t_{n+1}} (\rho(t)\mathbf{u}'(t), \mathbf{v})_{L^2} \, dt$ by e.g., $(\rho(t_{n+\frac{1}{2}})(\mathbf{u}^{n+1} - \mathbf{u}^n), \mathbf{v})_{L^2}$.

7. Numerical experiments. We present results of a numerical experiment for a non-stationary Stokes problem as in (2.9). We take $\Omega = (0,1)^3$ and consider a time dependent interface $\Gamma(t)$ which is an oscillating ellipsoid with major axes given by

$$\begin{pmatrix} 0.25(1.0 + t \cdot 0.1 \sin(4\pi t)) \\ 0.25(1.0 + t \cdot 0.1 \cos(4\pi t)) \\ 0.25 \end{pmatrix}, \quad t \in [0,1].$$

On Γ we have a surface tension force as in (2.8) with $\tau = 1$ and gravity $\mathbf{g} = 0$. The viscosities and densities are equal to one on Ω, the boundary conditions are of homogeneous Dirichlet type. The initial conditon is given by the velocity solution of the stationary Stokes problem at $t = 0$.
A cross section of the (size of the) computed discrete velocity solution for $t = \frac{1}{8}i$, $i = 0, \ldots, 8$ is given in Fig. 7.1.

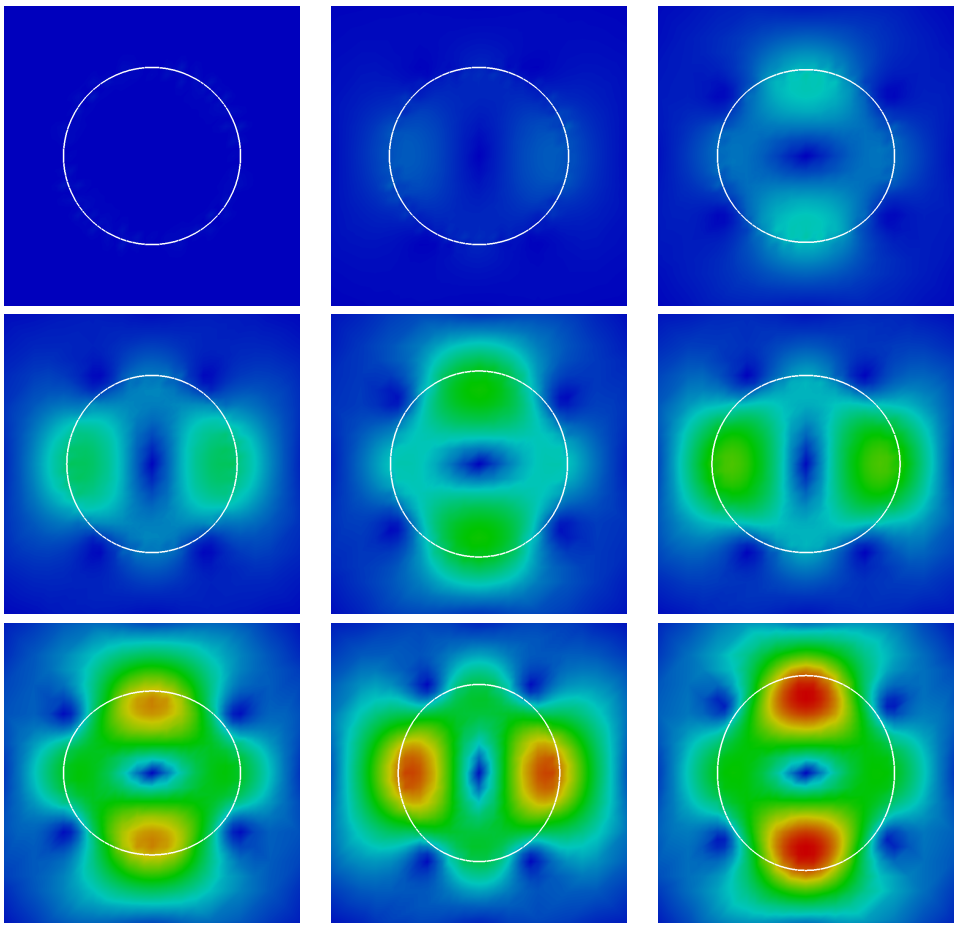

FIG. 7.1. *x-z-plane: velocity magnitude for* $t = \frac{1}{8}i$, $i = 0, \ldots, 8$

We solve the problem in a fully discrete, i.e. time *and* space, form. We use the Rothe approach and consider the implicit Euler and the Crank-Nicolson time discretization. The space discretization is based on a variational formulation in the

space $H^1_0(\Omega)^3 \times L^2_0(\Omega)$, i.e. a formulation that uses the pressure variable to satisfy the divergence free constraint. For the Crank-Nicolson method this variational problem is given in (3.3)-(3.4). For an accurate discretization it is important to use pressure spaces that adapt to the discontinuity of the pressure across the interface. We use the XFEM (extended finite element method) approach, based on the extension of the standard linear finite element space with suitable discontinuous basis functions corresponding to vertices close to the interface. We refer to the literature for an explanation of this method [5]. It can be shown that this space has optimal approximation properties for the discontinuous piecewise smooth pressure function. For the velocity we use continuous piecewise quadratics. These finite elements are used on a fixed (i.e. time independent) locally refined triangulation as illustrated in Fig. 7.2, with 18936 tetrahedra (62256 velocity unknowns).

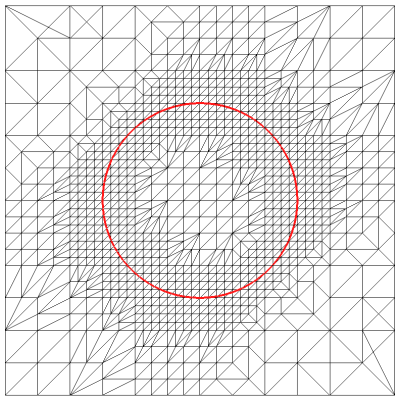

FIG. 7.2. *Cross section of tetrahedral triangulation.*

In the remainder the triangulation and the corresponding velocity finite element space (piecewise quadratics) are fixed. We vary the time discretization method (implicit Euler or Crank-Nicolson) and the time step Δt. Note that due to the non-stationary interface the pressure XFEM space varies during the time integration. For the Crank-Nicolson method the (discrete) pressure is treated in the trapezoidal form $p = \frac{1}{2}(p^{n+1} + p^n)$, cf. (3.5). For the implicit Euler method we use $p = p^{n+1}$.

In the experiments below we computed a (discrete) reference solution $\mathbf{u}_h(t)$, $t \in [0,1]$, by using a sufficiently small time step Δt. Given this reference solution, a time discretization method and a time step we can then determine the error $\mathbf{e}^n_h = \mathbf{u}_h(t_n) - \mathbf{u}^n_h$, where $\mathbf{u}^n_h$ is the computed discrete solution at time $t_n = n\Delta t$.

We measure errors $\|\mathbf{e}^n_h\|$ using the L^2-norm and the following weaker norm, denoted by $\|\cdot\|_{\mathbf{A}_h^{-1}}$, which in a certain sense (cf. Remark 5) resembles the norm $\|\cdot\|_{A^{-1}}$ used in Theorem 5.7. Let $\mathbf{V}_h \subset \mathbf{V}$ be the velocity finite element space of piecewise quadratics. For $\mathbf{f} \in L^2(\Omega)^3$ we consider the variational Poisson equation: determine $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_{L^2} \quad \text{for all} \quad \mathbf{v}_h \in \mathbf{V}_h. \tag{7.1}$$

Let m be the number of degrees of freedom in this discretization and $\mathbf{A}_h$, $\mathbf{M}_h \in \mathbb{R}^{m\times m}$ the stiffness and mass matrix corresponding to the problem (7.1). The norm we use is given by

$$\|\mathbf{x}\|_{\mathbf{A}_h^{-1}} := \|\mathbf{M}_h^{\frac{1}{2}} \mathbf{A}_h^{-1} \mathbf{M}_h \mathbf{x}\|_2, \quad \mathbf{x} \in \mathbb{R}^m, \tag{7.2}$$

where $\|\cdot\|_2$ is the (scaled) Euclidean norm on $\mathbb{R}^m$.

REMARK 5. The norm in (7.2) is related to the $\|\cdot\|_{A^{-1}}$-norm as follows. The inverse Friedrichs operator $A_F^{-1} : L^2(\Omega)^3 \to D(A_F) \subset \mathbf{V}_{\mathrm{div}}$ used in the analysis in section 5 is the solution operator corresponding to the stationary Stokes problem, i.e., for $\mathbf{f} \in L^2(\Omega)^3$ we have $\mathbf{u} = A_F^{-1}\mathbf{f}$ iff $\mathbf{u} \in \mathbf{V}_{\mathrm{div}}$ satisfies

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{L^2} \quad \text{for all} \quad \mathbf{v} \in \mathbf{V}_{\mathrm{div}}.$$

Then we have $\|\mathbf{f}\|_{A^{-1}} = \|A_F^{-1}\mathbf{f}\|_{L^2} = \|\mathbf{u}\|_{L^2}$. We assume this stationary Stokes problem to be H^2-regular, i.e., there is a constant c such that $\|A_F^{-1}\mathbf{g}\|_{H^2} \leq c\|\mathbf{g}\|_{L^2}$ for all $\mathbf{g} \in L^2(\Omega)^3$. A corresponding Poisson problem, with Friedrichs operator denoted by A_P, is defined as follows: determine $\hat{\mathbf{u}} = A_P^{-1}\mathbf{f} \in \mathbf{V}$ such that

$$a(\hat{\mathbf{u}}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{L^2} \quad \text{for all} \quad \mathbf{v} \in \mathbf{V}.$$

For the solutions $\mathbf{u}$ and $\hat{\mathbf{u}}$ of the Stokes and Poisson problem, respectively, we have $a(\mathbf{u}, \mathbf{v}) = a(\hat{\mathbf{u}}, \mathbf{v})$ for all $\mathbf{v} \in \mathbf{V}_{\mathrm{div}}$. For $\mathbf{w} := A_F^{-1}\mathbf{u}$ we get

$$\|\mathbf{u}\|_{L^2}^2 = a(\mathbf{w}, \mathbf{u}) = a(\mathbf{u}, \mathbf{w}) = a(\hat{\mathbf{u}}, \mathbf{w}) \leq \|\mathbf{w}\|_{H^2}\|\hat{\mathbf{u}}\|_{L^2} \leq c\|\mathbf{u}\|_{L^2}\|\hat{\mathbf{u}}\|_{L^2}.$$

From this and from the boundedness of $A_P^{-1} : L^2(\Omega)^3 \to L^2(\Omega)^3$ we conclude

$$\|\mathbf{f}\|_{A^{-1}} = \|A_F^{-1}\mathbf{f}\|_{L^2} \leq c\|A_P^{-1}\mathbf{f}\|_{L^2} \leq \hat{c}\|\mathbf{f}\|_{L^2} \quad \text{for all} \quad \mathbf{f} \in L^2(\Omega)^3,$$

with constants c and $\hat{c}$ independent of $\mathbf{f}$. Hence $\|A_P^{-1}\cdot\|_{L^2}$ defines a norm on $L^2(\Omega)^3$ that is weaker than $\|\cdot\|_{L^2}$ and stronger than $\|\cdot\|_{A^{-1}}$. The norm in (7.2) is the vector representation of this norm $\|A_P^{-1}\cdot\|_{L^2}$. To be more precise, for $\mathbf{f}_h \in \mathbf{V}_h$ and with $P_h : \mathbb{R}^m \to \mathbf{V}_h$ the finite element isomorphism we have

$$\|A_P^{-1}\mathbf{f}_h\|_{L^2} = \|\mathbf{M}_h^{\frac{1}{2}}\mathbf{A}_h^{-1}\mathbf{M}_h P_h^{-1}\mathbf{f}_h\|_2 = \|P_h^{-1}\mathbf{f}_h\|_{\mathbf{A}_h^{-1}}.$$

We finally note that we use $\|A_P^{-1}\cdot\|_{L^2}$ instead of $\|A_F^{-1}\cdot\|_{L^2}$ since the former is much easier to compute.

Implicit Euler method. We applied the implicit Euler method with $N\Delta t = 1$ and vary N between $N = 32$ and $N = 4096$. Results for the relative L^2-error $\|\mathbf{e}_h^n\|_{L^2}/\|\mathbf{u}_h(t_n)\|_{L^2}$ are shown in Figure 7.3 and Table 7.1. The results in Figure 7.3 are obtained by evaluating, for given $N \in \{512, 1024, 2048, 4096\}$, the error in equidistant time steps $t_i = i/32$, $0 < i \leq 32$. The mean error, denoted by $\|\mathbf{e}_h^{\mathrm{mean}}\|_{L^2}$, is obtained by computing the mean of these relative errors at the time points t_i, $0 < i \leq 32$. The results and corresponding numerical convergence order are given in Table 7.1. These results clearly show the expected first order convergence of the implicit Euler method.

Crank-Nicolson method. We repeated the experiment but now with the Crank-Nicolson method instead of the Euler method. The error behavior in the L^2-norm is shown in Figure 7.4. The mean L^2-errors and corresponding numerical order are given in Table 7.2.

Note the very irregular error behavior in Figure 7.4. Furthermore, from the results in that figure and in Table 7.2 it is clear that the Crank-Nicolson method does

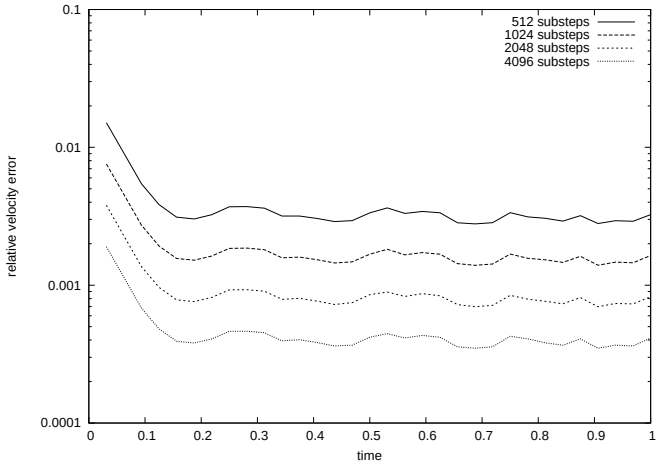

FIG. 7.3. *L^2-errors in implicit Euler method:* $||\mathbf{e}_h^n||_{L^2}/\|\mathbf{u}_h(t_n)\|_{L^2}$

N	$\|\|\mathbf{e}_h^{\text{mean}}\|\|_{L^2}$	order
32	3.20e-2	-
64	1.67e-2	0.94
128	8.57e-3	0.97
256	4.34e-3	0.98
512	2.18e-3	0.99
1024	1.09e-3	1.00
2048	5.48e-4	1.00
4096	2.74e-4	1.00

TABLE 7.1
L^2-errors in implicit Euler method.

N	$\|\|\mathbf{e}_h^{\text{mean}}\|\|_{L^2}$	order
32	7.06e-3	-
64	8.74e-4	3.01
128	2.10e-4	2.06
256	5.54e-5	1.92
512	2.04e-5	1.44
1024	1.26e-5	0.69
2048	7.90e-6	0.67
4096	4.77e-6	0.73

TABLE 7.2
L^2-errors in Crank-Nicolson method.

not show second order convergence with respect to the L^2-norm. We repeated the experiment, but now with the weaker $\|\cdot\|_{\mathbf{A}_h^{-1}}$-norm instead of the L^2-norm. The results are shown in Figure 7.5 and Table 7.3 (first three columns).

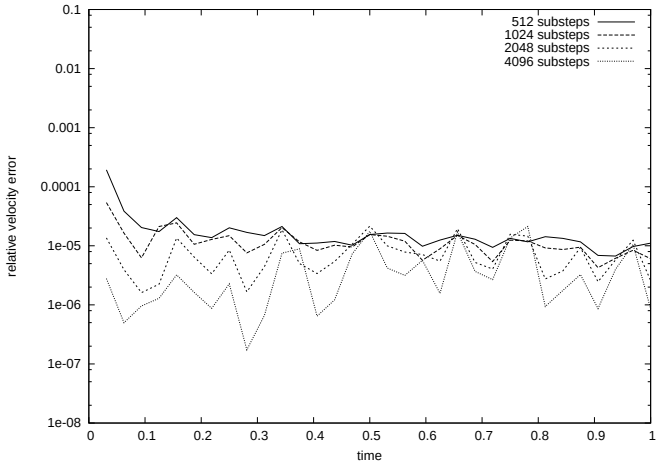

FIG. 7.4. *L^2-errors in Crank-Nicolson method:* $||\mathbf{e}_h^n||_{L^2}/\|\mathbf{u}_h(t_n)\|_{L^2}$

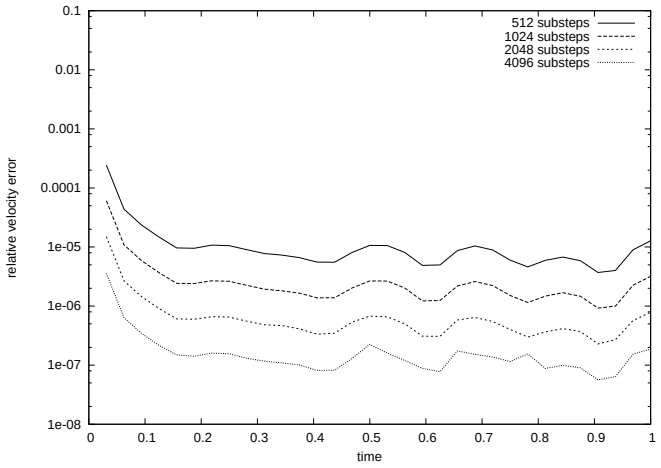

FIG. 7.5. *$\mathbf{A}_h^{-1}$-errors in Crank-Nicolson method:* $||\mathbf{e}_h^n||_{\mathbf{A}_h^{-1}}/\|\mathbf{u}_h(t_n)\|_{\mathbf{A}_h^{-1}}$

We now observe a much smoother error behavior and a second order convergence. This is consistent to the theoretical error analysis presented in this paper.

Finally we address an issue related to the choice of the pressure Lagrange multiplier, which is introduced to satisfy the divergence free constraint for the velocity.. Based on heuristic arguments we used the choice $p = \frac{1}{2}(p^{n+1} + p^n)$. An alternative is to use $p = p^{n+1}$. We repeated the experiment with the latter choice; the results are given in Table 7.3 (columns 4,5). We see that using $p = p^{n+1}$ leads to a strong deterioration of the rate of convergence of the Crank-Nicolson method. A theoretical

N	$p=\frac{1}{2}(p^{n+1}+p^n)$ $\|\|\mathbf{e}_h^{\text{mean}}\|\|_{\mathbf{A}_h^{-1}}$	order	$p=p^{n+1}$ $\|\|\mathbf{e}_h^{\text{mean}}\|\|_{\mathbf{A}_h^{-1}}$	order
32	6.86e-3	-	6.92e-3	-
64	1.11e-3	2.63	1.15e-3	2.58
128	2.71e-4	2.03	3.15e-4	1.87
256	6.77e-5	2.00	1.01e-4	1.63
512	1.69e-5	2.00	3.91e-5	1.37
1024	4.22e-6	2.00	1.75e-5	1.16
2048	1.05e-6	2.01	8.42e-6	1.05
4096	2.58e-7	2.02	4.17e-6	1.01

TABLE 7.3
$\mathbf{A}_h^{-1}$-errors in Crank-Nicolson method.

analysis which explains this effect, for this class of problems, is not known, yet.

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