NITSCHE-XFEM WITH STREAMLINE DIFFUSION STABILIZATION FOR A TWO-PHASE MASS TRANSPORT PROBLEM

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Abstract. We consider an unsteady convection diffusion equation which models the transport of a dissolved species in two-phase incompressible flow problems. The so-called Henry interface condition leads to a jump condition for the concentration at the interface between the two phases. In [A. Hansbo, P. Hansbo, Comput. Methods Appl. Mech. Engrg. 191 (2002)], for the purely elliptic stationary case, extended finite elements (XFEM) are combined with a Nitsche-type of method, and optimal error bounds are derived. These results were extended to the unsteady case in [A. Reusken, T. Nguyen, J. Fourier Anal. Appl. 15 (2009)]. In the latter paper convection terms are also considered, but assumed to be small. In many two-phase flow applications, however, convection is the dominant transport mechanism. Hence there is a need for a stable numerical method for the case of a convection dominated transport equation. In this paper we address this topic and study the streamline diffusion stabilization for the Nitsche-XFEM method. The method is presented and results of numerical experiments are given that indicate that this kind of stabilization is satisfactory for this problem class. Furthermore, a theoretical error analysis of the stabilized Nitsche-XFEM method is presented that results in optimal a-priori discretization error bounds.

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1. Introduction. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a convex polygonal domain that contains two different immiscible incompressible phases. The (in general time dependent) subdomains containing the two phases are denoted by Ω_1, Ω_2, with $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$. The interface $\Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2$ is assumed to be sufficiently smooth. A model example is a (rising) droplet in a flow field. The fluid dynamics in such a flow problem is usually modeled by the incompressible Navier-Stokes equations combined with suitable conditions at the interface which describe the effect of surface tension. For this model we refer to the literature, e.g. [4, 11, 18, 24, 12]. By $\mathbf{w}$ we denote the velocity field resulting from these Navier-Stokes equations. We assume that $\operatorname{div} \mathbf{w} = 0$ holds. Furthermore, we assume that the transport of the interface is determined by this velocity field, in the sense that $V_\Gamma = \mathbf{w} \cdot \mathbf{n}$ holds, where V_Γ is the normal velocity of the interface and $\mathbf{n}$ denotes the unit normal at Γ pointing from Ω_1 into Ω_2. In this paper we restrict ourselves to the case of a *stationary* interface, i.e., we assume $\mathbf{w} \cdot \mathbf{n} = 0$. This case is (much) easier to handle than the case of an non-stationary interface $\Gamma = \Gamma(t)$. We restrict to this simpler case because even for that the issue of stabilization of the Nitsche-XFEM method for convection-dominated transport problems has not been investigated, yet. The case of a non-stationary interface will be studied in a forthcoming paper. We comment on this further in Remark 7 at the end of the paper. We consider a model which describes the transport of a dissolved species in a two-phase

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flow problem. In strong formulation this model is as follows, cf. Remark 1:

$$\frac{\partial u}{\partial t} + \mathbf{w} \cdot \nabla u - \operatorname{div}(\varepsilon \nabla u) = f \quad \text{in} \quad \Omega_i, \; i = 1, 2, \; t \in [0, T], \tag{1.1}$$

$$[\varepsilon \nabla u \cdot \mathbf{n}]_\Gamma = 0, \tag{1.2}$$

$$[\beta u]_\Gamma = 0, \tag{1.3}$$

$$u(\cdot, 0) = u_0 \quad \text{in} \quad \Omega_i, \; i = 1, 2, \tag{1.4}$$

$$u(\cdot, t) = 0 \quad \text{on} \quad \partial\Omega, \quad t \in [0, T]. \tag{1.5}$$

For a sufficiently smooth function v, $[v] = [v]_\Gamma$ denotes the jump of v across Γ, i.e. $[v] = (v_1)_{|\Gamma} - (v_2)_{|\Gamma}$, where $v_i = v_{|\Omega_i}$ is the restriction of v to Ω_i. In (1.1) we have standard parabolic convection-diffusion equations in the two subdomains Ω_1 and Ω_2. In most applications one has a homogeneous problem, i.e. $f \equiv 0$. The diffusion coefficient $\varepsilon = \varepsilon(x)$ is assumed to be piecewise constant $\varepsilon = \varepsilon_i > 0$ in Ω_i:

$$\varepsilon = \varepsilon_i > 0 \quad \text{in} \quad \Omega_i.$$

In general we have $\varepsilon_1 \neq \varepsilon_2$. The interface condition in (1.2) results from the conservation of mass principle. The condition in (1.3) is the so-called *Henry condition*, cf. [17, 23, 22, 5, 4]. In this condition the coefficient $\beta = \beta(x)$ is strictly positive and piecewise constant $\beta = \beta_i > 0$ in Ω_i:

$$\beta = \beta_i > 0 \quad \text{in} \quad \Omega_i.$$

In general we have $\beta_1 \neq \beta_2$, since species concentration usually has a jump discontinuity at the interface due to different solubilities within the respective fluid phases. Hence, the solution u is *discontinuous across the interface*.

Remark 1. We briefly comment on the physical background of the interface condition (1.3). It originates from Henry's law, which is a constitutive law stating that for a gas in equilibrium with its solution in some (liquid) solvent the solubility of the gas in the solvent is proportional to the pressure of the gas. Since at constant temperature the pressure is proportional to the concentration, one obtains a dimensionless Henry constant as the ratio between the solvent-phase concentration of the solute and the gas-phase concentration. Similar constitutive laws hold for many liquid-liquid-solute systems at equilibrium, where the liquids are immiscible. Such constant concentration ratio constitutive laws are valid only under isothermal conditions and in general the solute concentrations have to be small, i.e. one considers sufficiently dilute solutions. In systems that are *not* in equilibrium one typically assumes a "local" equilibrium at the interface, i.e. a constant concentration ratio at the interface, resulting in the (Henry) condition (1.3). Clearly there is a dependence of the Henry coefficient β on the diffusion coefficient ε in the sense that if one changes the material system, resulting in other diffusion coefficients ε_i, then also the Henry coefficient will be different. We do not consider this dependence here and assume a fixed β. The Dirichlet boundary condition in (1.5) is used to simplify the presentation and the theoretical analysis. Other, from a physical point of view more realistic, boundary conditions are used in section 3 and remark 6.

In recent years it has been shown that for such a transport problem with an (evolving) interface the Nitsche-XFEM method is very well suited [13, 20]. In [14, 15, 16, 1, 8] the application of the Nitsche-XFEM to other classes of problems is studied. In [13] this method is analyzed for a *stationary* heat diffusion problem (no convection) with a conductivity that is discontinuous across the interface ($\varepsilon_1 \neq \varepsilon_2$) but with a solution that is continuous across the interface ($\beta_1 = \beta_2$). In [20] the method is studied for the parabolic problem described above, with $\beta_1 \neq \beta_2$ (discontinuous solution), and with a convection term in (1.1). It is assumed, however, that the transport problem is diffusion dominated. In the papers [25, 7] domain decomposition methods with Nitsche type conditions at the subdomain interfaces and streamline-diffusion or interior penalty stabilization (for strong convection) are studied. In these papers, however, the interfaces (subdomain boundaries) are aligned to the grids and therefore an XFEM technique is not needed. In none of the above-mentioned papers, or in other literature that we know of, the Nitsche-*XFEM* method is considered for a two-phase transport problem as in (1.1)-(1.5) that is *convection-dominated*. In this paper we treat this topic. We combine the Nitsche-XFEM method with one of the most popular FE stabilization techniques for convection-dominated problems, namely the streamline diffusion finite element method (SDFEM), cf. [21]. The resulting method is presented in section 2. In section 3 the method is applied to convection-dominated test problems and its performance is investigated. An error analysis of the Nitsche-XFEM with SD stabilization is given in section 4.

2. The Nitsche-XFEM method with SD stabilization.

Since we restrict to the case of a stationary interface, the discontinuity in the solution is located at a fixed position, independent of t, which then allows a rather standard weak formulation and a corresponding discretization based on the method of lines approach. In this section we present this weak formulation and the stabilized Nitsche-XFEM discretization. In case of an evolving interface a space-time weak formulation and corresponding space-time XFEM discretization is more natural, cf. Remark 7.

We describe the Nitsche-XFEM method as treated in detail in [20]. We first introduce a suitable weak formulation of the transport problem. For this we need the space

$$H_0^1(\Omega_1 \cup \Omega_2) := \{\, v \in L^2(\Omega) \mid v_{|\Omega_i} \in H^1(\Omega_i),\ i = 1,2,\ v_{|\partial\Omega} = 0 \,\}.$$

For $v \in H_0^1(\Omega_1 \cup \Omega_2)$ we write $v_i := v_{|\Omega_i},\ i = 1,2$. Furthermore

$$H := L^2(\Omega), \quad V := \{\, v \in H_0^1(\Omega_1 \cup \Omega_2) \mid [\beta v]_\Gamma = 0 \,\}. \tag{2.1}$$

Note that $v \in V$ iff $\beta v \in H_0^1(\Omega)$. On H we use the scalar product

$$(u,v)_0 := (\beta u, v)_{L^2} = \int_\Omega \beta u v \, dx,$$

which clearly is equivalent to the standard scalar product on $L^2(\Omega)$. The corresponding norm is denoted by $\|\cdot\|_0$. For $u, v \in H^1(\Omega_i)$ we define $(u,v)_{1,\Omega_i} := \beta_i \int_{\Omega_i} \nabla u_i \cdot \nabla v_i \, dx$ and furthermore

$$(u,v)_{1,\Omega_1\cup\Omega_2} := (u,v)_{1,\Omega_1} + (u,v)_{1,\Omega_2}, \quad u,v \in V.$$

The corresponding norm is denoted by $|\cdot|_{1,\Omega_1\cup\Omega_2}$. This norm is equivalent to

$$\left(\|\cdot\|_0^2 + |\cdot|_{1,\Omega_1\cup\Omega_2}^2\right)^{\frac{1}{2}} =: \|\cdot\|_{1,\Omega_1\cup\Omega_2}.$$

We emphasize that the norms $\|\cdot\|_0$ and $\|\cdot\|_{1,\Omega_1\cup\Omega_2}$ depend on β. We define the bilinear form

$$a(u,v) := (\varepsilon u, v)_{1,\Omega_1\cup\Omega_2} + (\mathbf{w}\cdot\nabla u, v)_0, \quad u,v \in V. \tag{2.2}$$

Consider the following weak formulation of the mass transport problem (1.1)-(1.5): Determine $u \in W^1(0,T;V) := \{ v \in L^2(0,T;V) \mid v' \in L^2(0,T;V') \}$ such that $u(0) = u_0$ and for almost all $t \in (0,T)$:

$$(\frac{du}{dt}, v)_0 + a(u,v) = (f,v)_0 \quad \text{for all} \quad v \in V. \tag{2.3}$$

In [20] it is proved that if the velocity field $\mathbf{w}$ satisfies $\operatorname{div} \mathbf{w} = 0$ in $\Omega_i,\ i = 1,2$, $\mathbf{w} \cdot \mathbf{n} = 0$ at Γ, and $\|\mathbf{w}\|_{L^\infty(\Omega)} \leq c < \infty$, then for $f \in H$, and u_0 sufficiently smooth the weak formulation (2.3) has a unique solution. For precise definitions of the generalized time derivatives used in the definition of $W^1(0,T;V)$ and in (2.3) we refer to [20].

We describe the Nitsche-XFEM method for spatial discretization of the weak formulation in (2.3). Let $\{\mathcal{T}_h\}_{h>0}$ be a family of shape regular triangulations of Ω. A triangulation $\mathcal{T}_h$ consists of simplices T, with $h_T := \operatorname{diam}(T)$ and $h := \max\{ h_T \mid T \in \mathcal{T}_h\}$. For any simplex $T \in \mathcal{T}_h$ let $T_i := T \cap \Omega_i$ be the part of T in Ω_i. We now introduce the finite element space

$$V_h^\Gamma := \{ v \in H_0^1(\Omega_1 \cup \Omega_2) \mid v_{|T_i} \ \text{ is linear for all } \ T \in \mathcal{T}_h,\ i = 1,2. \}. \tag{2.4}$$

Note that $V_h^\Gamma \subset H_0^1(\Omega_1 \cup \Omega_2)$, but $V_h^\Gamma \not\subset V$, since the Henry interface condition $[\beta v_h] = 0$ does not necessarily hold for $v_h \in V_h^\Gamma$.

Remark 2. In the literature a finite element discretization based on the space V_h^Γ is often called an *extended finite element method* (XFEM), cf. [3, 9]. Furthermore, in the (engineering) literature this space is usually characterized in a different way, which we briefly explain. Let $V_h \subset H_0^1(\Omega)$ be the standard finite element space of continuous piecewise linears, corresponding to the triangulation $\mathcal{T}_h$. Define the index set $\mathcal{J} = \{1, \ldots, n\}$, where $n = \dim V_h$, and let $(\phi_i)_{i \in \mathcal{J}}$ be the nodal basis in V_h. Let $\mathcal{J}_\Gamma := \{ j \in \mathcal{J} \mid \ |\Gamma \cap \operatorname{supp}(\phi_j)| > 0 \}$ be the index set of those basis functions the support of which is intersected by Γ. The Heaviside function H_Γ has the values $H_\Gamma(x) = 0$ for $x \in \Omega_1$, $H_\Gamma(x) = 1$ for $x \in \Omega_2$. Using this, for $j \in \mathcal{J}_\Gamma$ we introduce a so-called *enrichment function* $\Phi_j(x) := H_\Gamma(x) - H_\Gamma(x_j)$, where x_j is the vertex with index j. We introduce new basis functions $\phi_j^\Gamma := \phi_j \Phi_j$, $j \in \mathcal{J}_\Gamma$, and define the space

$$V_h \oplus \operatorname{span}\{ \phi_j^\Gamma \mid j \in \mathcal{J}_\Gamma \}. \tag{2.5}$$

This space is the same as V_h^Γ in (2.4) and the characterization in (2.5) accounts for the name "extended finite element method". The new basis functions ϕ_j^Γ have the property $\phi_j^\Gamma(x_i) = 0$ for all $i \in \mathcal{J}$. An L^2-stability property of the basis $(\phi_j)_{j \in \mathcal{J}} \cup (\phi_j^\Gamma)_{j \in J_\Gamma}$ of V_h^Γ is given in [19].

Define $(\kappa_i)_{|T} = |T_i|/|T|, \quad T \in \mathcal{T}_h,\ i = 1,2$, hence, $\kappa_1 + \kappa_2 = 1$. For v sufficiently smooth such that $(v_i)_{|\Gamma}$, $i = 1,2$, are well-defined, we define the weighted average

$$\{v\} := \kappa_1 (v_1)_{|\Gamma} + \kappa_2 (v_2)_{|\Gamma}.$$

For the average and jump operators the following identity holds for all f, g such that these operators are well-defined:

$$[fg] = \{f\}[g] + [f]\{g\} - (\kappa_1 - \kappa_2)[f][g]. \tag{2.6}$$

Define the scalar products

$$(f,g)_\Gamma := \int_\Gamma fg \, ds, \quad (f,g)_{\frac{1}{2},h,\Gamma} := \sum_{T \in \mathcal{T}_h^\Gamma} h_T^{-1} \int_{\Gamma_T} fg \, ds,$$

where $\mathcal{T}_h^\Gamma$ is the collection of $T \in \mathcal{T}_h$ with $\Gamma_T = T \cap \Gamma \neq \emptyset$. With $\bar{\varepsilon} := \frac{1}{2}(\varepsilon_1 + \varepsilon_2)$ we introduce the bilinear form

$$a_h(u,v) := a(u,v) - ([\beta u], \{\varepsilon \nabla v \cdot \mathbf{n}\})_\Gamma - (\{\varepsilon \nabla u \cdot \mathbf{n}\}, [\beta v])_\Gamma + \lambda \bar{\varepsilon}([\beta u], [\beta v])_{\frac{1}{2},h,\Gamma}, \quad (2.7)$$

with $\lambda > 0$ a parameter that will be specified below. In the literature for a diffusion dominated problem λ is chosen as a "sufficiently large" constant. We will denote this choice as the *diffusive scaling* λ^d. Using the mesh Péclet number $P_h := \frac{1}{2}\|\mathbf{w}\|_\infty h/\bar{\varepsilon}$ with $\|\mathbf{w}\|_\infty := \|\mathbf{w}\|_{L^\infty(\Omega)}$ the analysis in section 4 motivates the condition

$$\lambda^d \leq \lambda \leq \lambda^c := \lambda^d \max(P_h, 1) \quad (2.8)$$

on λ. The choice $\lambda = \lambda^c$ will be denoted as the *convective scaling* as in that case the stabilization term $([\beta u], [\beta v])_{\frac{1}{2},h,\Gamma}$ in (2.7) scales with $\|\mathbf{w}\|_\infty$ in the convection-dominated case $P_h \geq 1$.

REMARK 3. In practice the following localized variant of the parameter choice rule for λ is used. For $T \in \mathcal{T}_h$ we define the element Péclet number $P_h^T := \frac{1}{2}\|\mathbf{w}\|_{\infty,T} h_T/\bar{\varepsilon}$. A generalization of the analysis in section 4 leads to the following condition on $\lambda = \lambda_T$:

$$\lambda_T^d \leq \lambda_T \leq \lambda_T^c \text{ with } \lambda_T^d = c, \;\; \lambda_T^c = c \max(P_h^T, 1) \quad (2.9)$$

The stabilization term $\lambda \bar{\varepsilon}([\beta u], [\beta v])_{\frac{1}{2},h,\Gamma}$ in (2.7) is generalized to

$$\bar{\varepsilon} \sum_{T \in \mathcal{T}_h^\Gamma} \lambda_T h_T^{-1} \int_{\Gamma_T} [\beta u][\beta v] \, ds.$$

In practice this variant typically performs better than the one with a global stabilization parameter λ.

Using the bilinear form $a_h(\cdot,\cdot)$ we define a method of lines discretization of (2.3). Let $\hat{u}_0 \in V_h^\Gamma$ be an approximation of u_0. For $t \in [0,T]$ let $u_h(t) \in V_h^\Gamma$ be such that $u_h(0) = \hat{u}_0$ and

$$(\frac{du_h}{dt}, v_h)_0 + a_h(u_h, v_h) = (f, v_h)_0 \quad \text{for all} \quad v_h \in V_h^\Gamma. \quad (2.10)$$

Opposite to the weak formulation in (2.3), in this discretization method the Henry interface condition $[\beta u_h] = 0$ is *not* treated as an "essential" interface condition in the finite element space V_h^Γ. This interface condition is satisfied only approximately by using a modified bilinear form $a_h(\cdot,\cdot)$, which is a technique due to Nitsche. For this semi-discretization optimal order error bounds are derived in [20]. In the analysis in that paper it is assumed that the transport problem is diffusion-dominated. In the evaluation of the bilinear form $a_h(\cdot,\cdot)$ one has to determine integrals over Γ. In practice the weak formulation will be used with Γ replaced by an approximation Γ_h.

We now add the streamline diffusion stabilization to this semi-discretization. Recall that in a one-phase problem (set $\beta = 1$) in the SD approach one adds a residual term of the form

$$\sum_{T \in \mathcal{T}_h} \gamma_T \int_T \left(\frac{\partial u_h}{\partial t} + \mathbf{w} \cdot \nabla u_h - \operatorname{div}(\varepsilon \nabla u_h) - f \right) \, (\mathbf{w} \cdot \nabla v_h) \;\, dx \quad (2.11)$$

to the variational formulation. The choice of the stabilization parameter value γ_T is discussed below. If, as in our case, one considers linear finite elements then the term $\operatorname{div}(\varepsilon \nabla u_h)$ vanishes.

For the stabilization of the Nitsche-XFEM method we make obvious modifications related to the fact that in the XFEM space, close to the interface we have contributions on elements $T \cap \Omega_i \neq T$. For the stabilization we introduce a locally weighted discrete variant of $(\cdot,\cdot)_0$:

$$(u,v)_{0,h} := \sum_{i=1}^{2} \sum_{T\in\mathcal{T}_h} \beta_i \gamma_T \int_{T\cap\Omega_i} uv\,dx = \sum_{T\in\mathcal{T}_h} \gamma_T (u,v)_{0,T} \tag{2.12}$$

For the choice of γ_T we use a strategy as in the standard finite element method, cf. [21, 10]. We take γ_T as follows:

$$\gamma_T = \begin{cases} \frac{2h_T}{\|\mathbf{w}\|_{\infty,T}} & \text{if} \quad P_h^T > 1 \\ h_T^2/\bar{\varepsilon} & \text{if} \quad P_h^T \leq 1. \end{cases} \tag{2.13}$$

Very similar results (both in the theoretical analysis and in the experiments) are obtained if for the case $P_h^T \leq 1$ one sets $\gamma_T = 0$. Note that the stabilization parameter γ_T does not depend on the position of the interface within the element. We introduce the following Nitsche-XFEM semi-discretization method with SD stabilization: For $t \in [0,T]$ let $u_h(t) \in V_h^\Gamma$ be such that $u_h(0) = \hat{u}_0$ and

$$\begin{aligned} &(\frac{du_h}{dt}, v_h)_0 + (\frac{du_h}{dt}, \mathbf{w}\cdot\nabla v_h)_{0,h} + a_h(u_h,v_h) + (\mathbf{w}\cdot\nabla u_h, \mathbf{w}\cdot\nabla v_h)_{0,h} \\ &= (f,v_h)_0 + (f,\mathbf{w}\cdot\nabla v_h)_{0,h} \quad \text{for all} \quad v_h \in V_h^\Gamma. \end{aligned} \tag{2.14}$$

Clearly, this semi-discretization can be combined with standard methods for time discretization to obtain a fully discrete problem. For example, the θ-scheme takes the following form, where for notational simplicity we assume that f does not depend on t. For $n = 0,1,\ldots,N-1$, with $N\Delta t = T$, set $u_h^0 := \hat{u}_0$ and determine $u_h^{n+1} \in V_h^\Gamma$ such that for all $v_h \in V_h^\Gamma$:

$$\begin{aligned} &\left(\frac{u_h^{n+1}-u_h^n}{\Delta t}, v_h\right)_0 + \left(\frac{u_h^{n+1}-u_h^n}{\Delta t}, \mathbf{w}\cdot\nabla v_h\right)_{0,h} \\ &+ a_h(\theta u_h^{n+1} + (1-\theta)u_h^n, v_h) + (\mathbf{w}\cdot(\theta\nabla u_h^{n+1} + (1-\theta)\nabla u_h^n), \mathbf{w}\cdot\nabla v_h)_{0,h} \\ &= (f,v_h)_0 + (f,\mathbf{w}\cdot\nabla v_h)_{0,h}. \end{aligned} \tag{2.15}$$

In the numerical experiments in section 3 we used this method with $\theta = 1$.

REMARK 4. Above we considered the case of a stationary interface and an XFEM space based on piecewise *linears*. Both the Nitsche-XFEM method and the SD stabilization method presented above have a straightforward extension to higher order piecewise polynomials. Note that for higher order finite elements in the SD stabilization the term $(\mathrm{div}(\varepsilon\nabla u_h), \mathbf{w}\cdot\nabla v_h)_{0,h}$ has to be taken into account, cf. (2.11).

3. Numerical experiments. In this section we present results of numerical experiments to illustrate properties of the stabilized Nitsche-XFEM method introduced above. We investigate the effect of the choice of the stabilization parameter λ in the Nitsche term. In section 3.1 we consider two *stationary* convection-diffusion problems with a Henry interface condition in two space dimensions. The first problem in section 3.1.1, which has a known smooth solution, is used to illustrate the optimal convergence order of the stabilized method. The second problem in section 3.1.2, which has a known solution with an interior layer, illustrates the effects of SD stabilization. In a

	$\|e_h\|_0$				$\|\mathbf{w}\cdot\nabla e_h\|_0$				$\|[e_h]\|_{L^2(\Gamma)}$			
L	`conv`	eoc	`diff`	eoc	`conv`	eoc	`diff`	eoc	`conv`	eoc	`diff`	eoc
1	1.67e-1	-	1.75e-1	-	1.58e-0	-	1.57e-0	-	1.25e-2	-	5.76e-1	-
2	4.41e-2	1.9	4.41e-2	2.0	7.83e-1	1.0	7.83e-1	1.0	2.06e-3	2.6	1.14e-1	2.3
3	9.58e-3	2.2	9.62e-3	2.2	3.88e-1	1.0	3.88e-1	1.0	5.60e-4	1.9	2.76e-2	2.1
4	2.04e-3	2.2	2.06e-3	2.2	1.93e-1	1.0	1.93e-1	1.0	1.13e-4	2.3	6.03e-3	2.2
5	4.57e-4	2.2	4.60e-4	2.2	9.62e-2	1.0	9.62e-2	1.0	3.48e-5	1.7	1.76e-3	1.8
6	1.07e-4	2.1	1.07e-4	2.1	4.80e-2	1.0	4.80e-2	1.0	1.08e-5	1.7	5.40e-4	1.7

TABLE 3.1
Example of section 3.1.1: Errors on six refinement levels.

third example, which is given in section 3.2, we consider a *transient* spatially 3D transport problem as described in section 1. The example demonstrates the performance of the (un)stabilized Nitsche-XFEM method for a more realistic case. We consider two extreme cases for the penalty term of the Nitsche formulation. The one choice $\lambda_T = \lambda_T^d$ is denoted as the *diffusive scaling*(`diff`) whereas the other choice $\lambda_T = \lambda_T^c$ is denoted as the *convective scaling*(`conv`). Furthermore in all the experiments we use a slightly different stabilization parameter as in (2.13):

$$\gamma_T = \begin{cases} (1 - \frac{1}{P_h^T})\frac{h_T}{2\|\mathbf{w}\|_{\infty,T}} & \text{if} \quad P_h^T > 1 \\ 0 & \text{if} \quad P_h^T \leq 1 \end{cases} \tag{3.1}$$

This choice can also be found in [10].

3.1. Stationary mass transport problems. We start with two examples in two space dimensions with a known solution.

3.1.1. Problem with a smooth solution. We consider a two-dimensional stationary problem with a smooth solution. The interface is $\Gamma = \{y = 0\}$ and the domains are $\Omega_1 = [-1,1]\times[-1,0]$ and $\Omega_2 = [-1,1]\times[0,1]$. The piecewise constant coefficients ε, β are chosen as $\varepsilon = (\varepsilon_1, \varepsilon_2) = (2\cdot 10^{-7}, 1\cdot 10^{-7})$, $\beta = (\beta_1, \beta_2) = (3,2)$ and a stationary velocity field is given by $\mathbf{w} = (1,0)$. We adapt the right hand side f and the Dirichlet boundary conditions such that u^* defines the solution to our problem, with

$$u^*(x,y) = \begin{cases} \frac{2}{3}\sin(\pi(x+y)) & \text{for} \quad (x,y)\in\Omega_1, \\ \sin(\pi(x+\frac{4}{3}y)) & \text{for} \quad (x,y)\in\Omega_2. \end{cases}$$

The problem is solved on an unstructured mesh with 240 elements (on the coarsest level) by the proposed stabilized method. The coarsest mesh ($L = 1$) is uniformly refined five times. The norms used in the error analysis of section 4 for the error $e_h = u - u_h$ are listed in Table 3.1. We observe the expected linear convergence in the norm $\|\mathbf{w}\cdot\nabla e_h\|_0$. The ($\beta$-weighted) L^2-norm converges with $\mathcal{O}(h^2)$ which is half an order better than in the estimates. For the interface jump error the order of convergence appears to be smaller than two, but larger than 3/2 for both scalings of the Nitsche stabilization while the convective scaling leads to an interface error which is roughly 100 times smaller than for the diffusive scaled Nitsche method.

3.1.2. Problem with a sharp layer. In this example a two-dimensional stationary problem with a parabolic layer at the interface is considered. The interface is $\Gamma = \{y = 0\}$ and the domains are $\Omega_1 = [0.25,2]\times[-1,0]$ and $\Omega_2 = [0.25,2]\times[0,1]$.

The piecewise constant coefficients ε, β are chosen as $\varepsilon = (\varepsilon_1, \varepsilon_2) = (9\cdot 10^{-7}, 4\cdot 10^{-7})$, $\beta = (\beta_1, \beta_2) = (27, 11)$ and a stationary velocity field is given by $\mathbf{w} = (1, 0)$. We adapt the right hand side f and the boundary conditions such that the solution to our problem is given by

$$u^*(x,y) = \begin{cases} 1 - \frac{16}{27}\exp(\frac{C_n}{\sqrt{x}}y) & \text{for} \quad (x,y) \in \Omega_1, \\ \exp(\frac{-C_p}{\sqrt{x}}y) & \text{for} \quad (x,y) \in \Omega_2 \end{cases}$$

where the constants C_p and C_n are chosen s.t. the width of the layers at the outflow ($x = 2$) is approximately 10% of the domain size. The solution close to the interface is displayed in Figure 3.1. According to the solution u^* we prescribe Dirichlet boundary conditions on $\partial\Omega_D := \{x = 0.25\}$ and Neumann boundary conditions $\varepsilon\nabla u \cdot \mathbf{n} = g$ on $\partial\Omega \setminus \partial\Omega_D$.

The problem is discretized on an unstructured triangular mesh with 400 elements on the coarsest mesh which is uniformly refined five times. Apart from the error in the weighted interface jump the errors are measured in $\tilde{\Omega} = \{|y| > 0.1\}$ away from the interface. In Figure 3.1 and 3.2 the convergence of the errors in the (β-weighted) L^2-norm of the solution and the streamline derivative as well as the interface jump error are displayed. We observe that the error of the streamline derivative is drastically improved by the stabilized methods. In contrast to the stabilized methods the error of the unstabilized methods are not even monotonously decreasing. In the (β-weighted) L^2-norm one also observes a significant improvement of the stabilization. Concerning the different scalings of the Nitsche stabilization it is expected that the convective scaling leads to a better resolution of the interface jump condition. This is confirmed by the results in Fig. 3.1 (right).

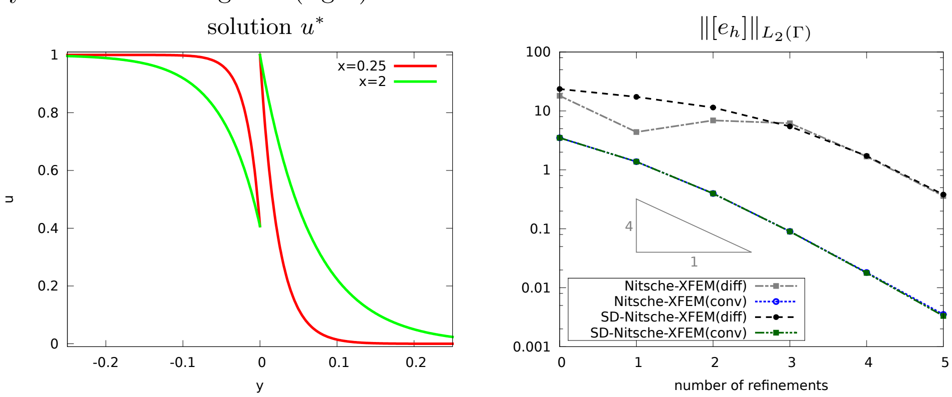

FIG. 3.1. *Example of section 3.1.2: Solution at in- and outflow (left) close to the interface and interface jump error(right).*

3.2. Transient mass transport problem.

3.2.1. Problem description. We consider a time dependent problem with a *stationary* interface. The domain $\Omega := [0, 2]\times[0, 2]\times[0, 1]$ is separated into a cylindrical domain $\Omega_1 := \{(x, y, z) \in \mathbb{R}^3 : (x - 1)^2 + (y - 1)^2 < R^2\}$, with $R = 0.25$, and $\Omega_2 := \Omega \setminus \Omega_1$ by the *stationary* interface $\Gamma := \partial\Omega_1 \setminus \partial\Omega$. The piecewise constant coefficients ε, β are chosen as $\varepsilon = (\varepsilon_1, \varepsilon_2) = (10^{-4}, 2\cdot 10^{-4})$, $\beta = (\beta_1, \beta_2) = (3, 1)$ and a stationary velocity field is given by

$$\mathbf{w}|_{\Omega_1} = (0, 0, 0), \quad \mathbf{w}|_{\Omega_2} = (1 + R^2(d_y^2 - d_x^2)r^{-4},\ -2R^2(d_x d_y)r^{-4},\ 0) \tag{3.2}$$

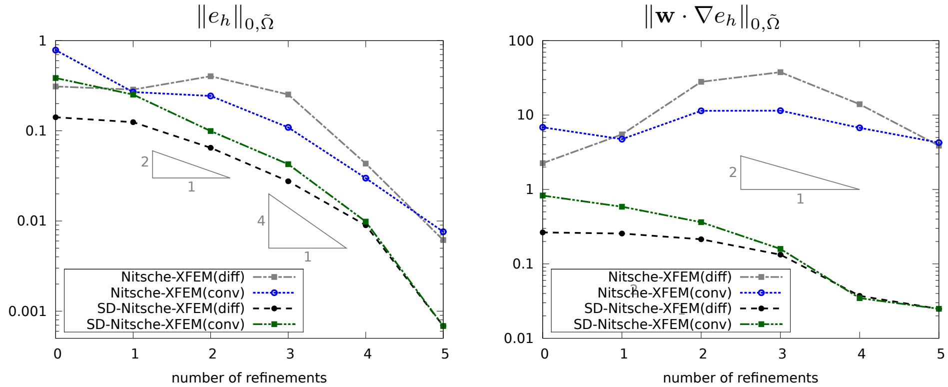

FIG. 3.2. *Example of section 3.1.2: Errors away from the layer on six refinement levels.*

where $d_x := x - 1$, $d_y := y - 1$ and $r := (d_x^2 + d_y^2)^{\frac{1}{2}}$. A sketch of the domains and of $\mathbf{w}$ in term of field-lines is given in Fig. 3.3.

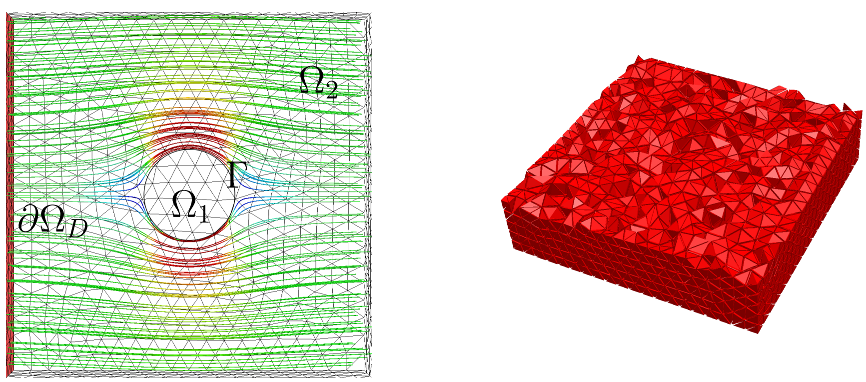

FIG. 3.3. *Sketch of interface position and flow field (left) and mesh (right)*

The assumptions on the velocity field made in section 1 are satisfied: $\operatorname{div}\mathbf{w} = 0$ in both domains and $\mathbf{w}\cdot\mathbf{n} = 0$ on Γ. We impose a Dirichlet boundary condition on $\partial\Omega_D := \{(x,y,z) \in \Omega : x = 0\}$, s.t. $u|_{\partial\Omega_D} = 0.05$ and a homogeneous Neumann boundary condition $\varepsilon\nabla u\cdot\mathbf{n} = 0$ on $\partial\Omega\setminus\partial\Omega_D$. As initial condition we take $u = 0$ on Ω_1, $u = 0.05$ on Ω_2. Note that this initial condition does not satisfy the Henry interface condition (1.3).

This time dependent convection-diffusion problem is strongly convection dominated with a physical Péclet number $P_L := \frac{2\|\mathbf{w}\|}{\bar{\varepsilon}} \approx 2\cdot 10^4$. Furthermore, due to the inconsistent (w.r.t. condition (1.3)) initial condition a *parabolic* boundary layer of thickness $\mathcal{O}(\sqrt{\varepsilon t})$ at the interface will form directly after $t = 0$, independent of the velocity field. For $t \to \infty$ the solution converges to the stationary piecewise constant function $u = 0.05\,\beta^{-1}$. In Fig. 3.4 the solution along a line is displayed, where one observes the predicted boundary layer behavior. In the experiments we consider $t = 1$.

3.2.2. Discretization. We use the mesh with 30000 elements displayed in Fig. 3.3 with an average mesh size $h = 0.05$ and element Péclet numbers up to $P_h^T \approx 250$. Thus, the mesh resolution is too low to resolve the boundary layer (for $t \leq 1$).

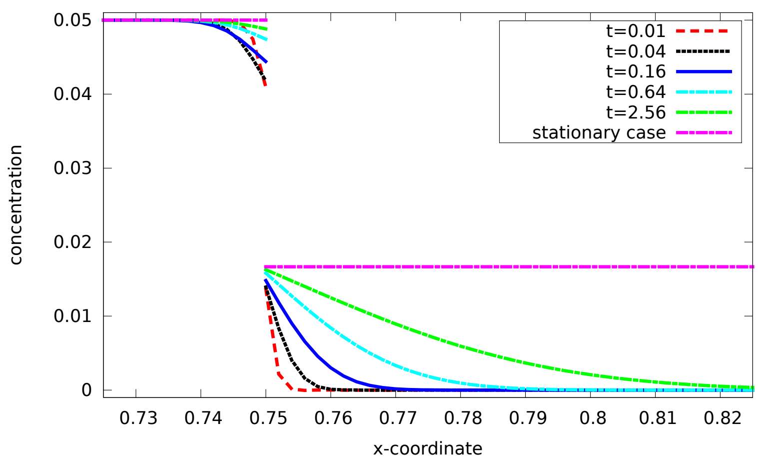

FIG. 3.4. *Concentration profile through line $\gamma_x := \{(x,y,z) \in \Omega : y = 1, z = 0.5\}$ for several values of t. One can observe the predicted $\mathcal{O}(\sqrt{\varepsilon t})$ sized boundary layer.*

In the discretization we use a (sufficiently accurate) polygonal approximation Γ_h of the interface. This introduces an additional error which is not analyzed here but is considered to be sufficiently small and to have negligible effect on the accuracy and stability properties of the (stabilized) Nitsche-XFEM method. We are primarily interested in the accuracy of the spatial discretization. Hence, in the implicit Euler method (2.15) we choose a small time step size $\Delta t = 10^{-4}$, such that the total discretization error is dominated by the spatial discretization error.

Again we consider the same four methods as in section 3.1. We computed a reference solution on a very fine 2D mesh which is aligned to the interface and resolves the boundary layer for $t > 10^{-2}$. This reference solution is used to provide the profiles in Fig. 3.4 and the reference profiles in Fig. 3.6 below.

3.2.3. Numerical results. In Fig. 3.5 the numerical solution in the plane $z = 0.5$ at $t = 1$ (where the boundary layer has a width of approximately 0.01 in Ω_2) is shown for four different methods. Below each picture we also give the L^2 norm of the jump $[\beta u_h]$ on the approximate interface Γ_h.

We observe several effects. The first one also occured in the numerical experiment treated in section 3.1: if one considers the different scalings in the Nitsche method, i.e. the left and the right columns in Fig. 3.5, then the convective scaling results in a better approximation of the interface condition. But it also increases the effect of non-physical oscillations. Comparing the first and the second row in Fig. 3.5, we see that the streamline diffusion stabilization suppresses the oscillations whereas the quality of the approximation of the interface condition is not negatively affected by this stabilization.

In Fig. 3.6 the numerical solutions of the same four methods as in Fig. 3.5 together with the reference solution, on the line $z = 0.5$, $y = 1.0$ in Ω_2 at time $t = 1$ are shown. One can observe that the boundary layer which is represented well by the reference solution is not resolved accurately by any of the four methods. Especially for $x > 1.25$, i.e. downwind of Ω_1 none of the methods yields a discrete solution that is close to the reference solution. The solutions u_h of the SD-Nitsche-XFEM methods are much smoother than the solutions obtained without stabilization and upwind of Ω_1, where the solution is almost constant outside the boundary layer, it is very accurate.

In Fig. 3.7 the results of the SD-Nitsche-XFEM methods on three successively

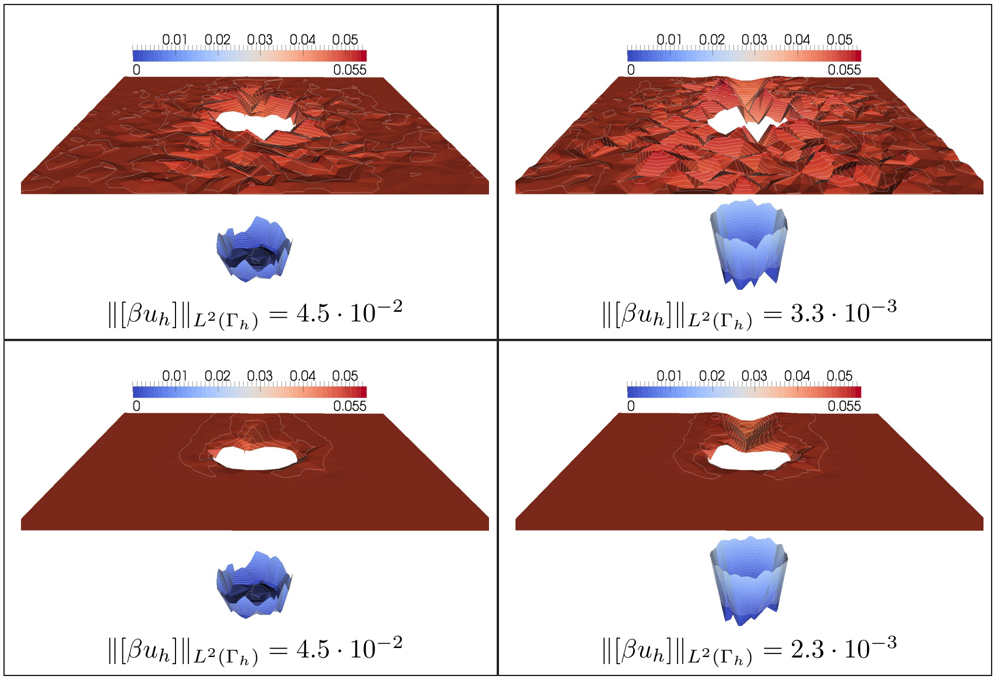

FIG. 3.5. *Numerical solution in the plane $z = 0.5$ at $t = 1$ for Nitsche-XFEM (top) and SD-Nitsche-XFEM (bottom), with diffusive scaling (left) and convective scaling (right) of the Nitsche stabilization.*

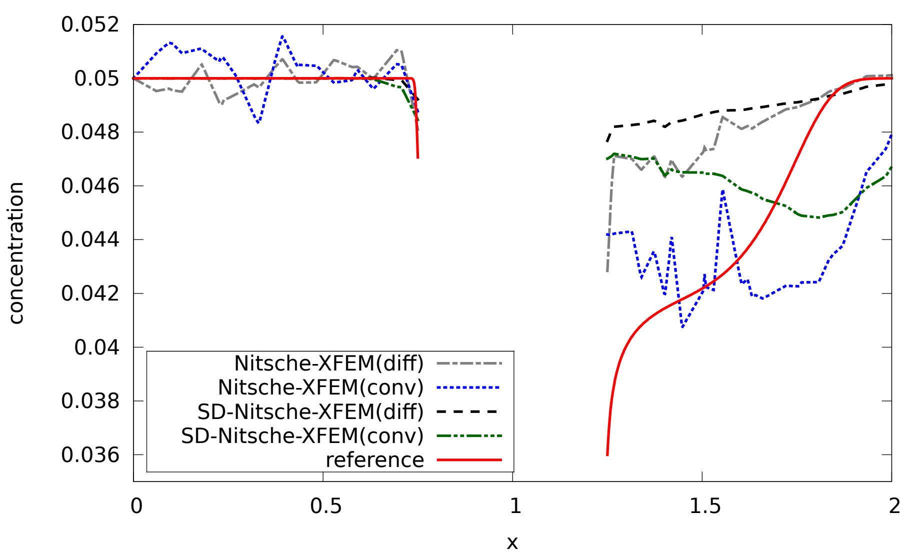

FIG. 3.6. *Numerical solutions on the line $z = 0.5, y = 1.0$ at time $t = 1$ obtained with Nitsche-XFEM, SD-Nitsche-XFEM, and the reference solution.*

(uniformly) refined meshes are shown. The resolution of the boundary layer at $t = 1$ improves if the grid is refined, but on level 3 the discrete solution downwind of Ω_1

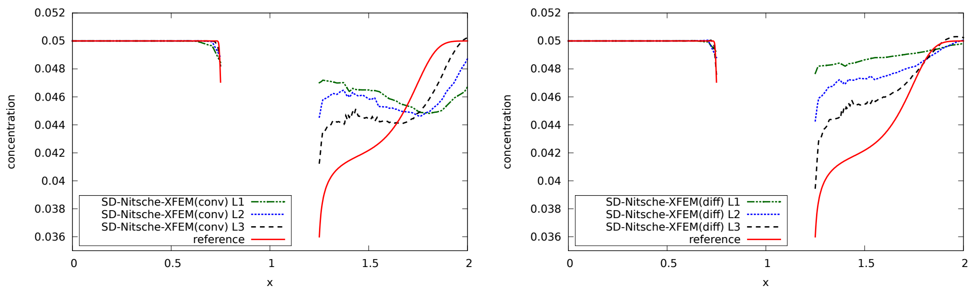

FIG. 3.7. *Numerical solutions on the line $z = 0.5, y = 1.0$ at time $t = 1$ obtained with SD-Nitsche-XFEM applying the convective scaling (left) and the diffusive scaling (right) on three consecutively refined meshes and the reference solution.*

is still not in good agreement with the reference solution. This can be interpreted as follows. For small times t the boundary layers are much smaller, namely $\mathcal{O}(\sqrt{\varepsilon t})$, cf. Fig. 3.4, and cannot be resolved. For small t we thus have (very) large spatial discretization errors. If time evolves until $t = 1$ these large errors are transported in downwind direction and are only mildly damped. This time dependent transport effect causes the large errors downwind of Ω_1 ($x > 1.25$) in Fig. 3.6 and 3.7.

4. Error analysis. In this section we present an error analysis of the Nitsche-XFEM with SD stabilization. We investigate the bilinear form

$$\begin{aligned} a_h(u,v) &:= (\varepsilon u, v)_{1,\Omega_1\cup\Omega_2} + (\mathbf{w}\cdot\nabla u, v)_0 + \xi(u,v)_0 \\ &\quad - ([\beta u], \{\varepsilon\nabla v\cdot\mathbf{n}\})_\Gamma - (\{\varepsilon\nabla u\cdot\mathbf{n}\}, [\beta v])_\Gamma + \lambda\bar{\varepsilon}([\beta u],[\beta v])_{\frac{1}{2},h,\Gamma} \\ &\quad + (\xi u + \mathbf{w}\cdot\nabla u, \mathbf{w}\cdot\nabla u)_{0,h} \end{aligned}$$

on $W_{\mathrm{reg}} + V_h^\Gamma$, with V_h^Γ the XFEM space, cf. (2.4), and $W_{\mathrm{reg}} := \{\, u \in H_0^1(\Omega_1 \cup \Omega_2) \mid u_{|\Omega_i} \in H^2(\Omega_i),\ i = 1,2\,\}$. Compared to the transport problem considered above we introduced an additional zero order term $\xi(u,v)_0$, with a given constant $\xi \geq 0$. This is standard in the analysis of convection-dominated problems (cf. [21]), since only if this zero order term is present ($\xi > 0$) one can derive uniform error bounds in the L^2-norm. We derive an error bound for the Galerkin projection of $u \in W_{\mathrm{reg}}$ on the XFEM space V_h^Γ, cf. Theorem 4.7 below. We start with the main assumptions used and introduce additional notation. To obtain estimates that are uniform with respect to the parameter ξ, we have to generalize the choice of of the stabilization parameter γ_T. If $\xi = 0$ we take γ_T as in (2.13). For the case $\xi > 0$ we take $\gamma_T = \min\{\xi^{-1}, \gamma_T^{(2.13)}\}$. This parameter choice is essentially the same as in [21]. The following estimates can be derived:

$$\gamma_T\xi \leq 1, \quad \gamma_T\|\mathbf{w}\|_{\infty,T} \leq 2h_T, \quad \gamma_T^{-1}h_T^2 \leq \xi h_T^2 + \frac{1}{2}\|\mathbf{w}\|_{\infty,T}h_T + \bar{\varepsilon}. \tag{4.1}$$

The family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is assumed to be shape regular, but not necessarily quasi-uniform. The triangulation $\mathcal{T}_h$ is *not* assumed to be fitted to the interface Γ, but the resolution close to the interface should be sufficiently high such that the interface can be resolved by the triangulation, in the sense that if $\Gamma \cap T =: \Gamma_T \neq \emptyset$ then Γ_T can be represented as the graph of a function on a planar cross-section of T (cf. [13] for precise conditions). In the analysis of the Nitsche-XFEM method an interpolation operator $I_h^\Gamma : W_{\rm reg} \to V_h^\Gamma$ plays an important role. We recall the definition of this operator. For $i = 1, 2$, let R_i be the restriction operator to Ω_i, i.e., $(R_i v)(x) = v(x)$ for $x \in \Omega_i$ and $(R_i v)(x) = 0$ otherwise. Let $\mathcal{E}_i : H^2(\Omega_i) \to H^2(\Omega)$ be a bounded extension operator with $\mathcal{E}_i v = 0$ on $\partial\Omega$, and $I_h : H^2(\Omega) \cap H_0^1(\Omega) \to V_h$ the standard nodal interpolation operator corresponding to the space V_h of continuous linear finite elements. The XFEM interpolation operator is given by

$$I_h^\Gamma = R_1 I_h \mathcal{E}_1 R_1 + R_2 I_h \mathcal{E}_2 R_2.$$

Define $T_i := T \cap \Omega_i$. Note that T_i can be very shape irregular. The constants that occur in the estimates in this section are independent of the shape regularity of T_i. For the interpolation operator I_h^Γ optimal (local) interpolation error bounds can easily be derived (cf. [13, 19]). The following holds:

$$\begin{aligned} \|u - I_h^\Gamma u\|_{H^m(T_i)} &\le \|\mathcal{E}_i R_i u - I_h \mathcal{E}_i R_i u\|_{H^m(T)} \\ &\le c h_T^{2-m} \|\mathcal{E}_i R_i u\|_{H^2(T)}, \quad m = 0, 1, 2, \quad \text{for} \quad u \in W_{\rm reg}. \end{aligned} \tag{4.2}$$

In the analysis below we use the assumptions $\operatorname{div} \mathbf{w} = 0$ on Ω, $\|\mathbf{w}\|_{L^\infty(\Omega)} < \infty$ and $\mathbf{w} \cdot \mathbf{n} = 0$ on Γ.

We are particularly interested in the convection-dominated case, and therefore allow $\bar\varepsilon = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) \downarrow 0$, but we assume the ratio between ε_1 and ε_2 to be bounded, i.e. for $i = 1, 2$, we have $\bar\varepsilon/\varepsilon_i \le c$ with a uniform (for $\bar\varepsilon \downarrow 0$) constant c.

We assume that the constants β_i used in the Henry condition are of order one. We will need several norms related to the Nitsche stabilization and the streamline diffusion stabilization. The inner products $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_{1,\Omega_1\cup\Omega_2}$ (with corresponding norms $\|\cdot\|_0$ and $|\cdot|_{1,\Omega_1\cup\Omega_2}$) have been defined above in section 2. These inner products depend on a weighting with β, but this causes no problem since β is assumed to be of order one. For the streamline diffusion stabilization we introduced the inner product $(u, v)_{0,h} = \sum_{T\in\mathcal{T}_h} \gamma_T (u, v)_{0,T}$ with corresponding norm denoted by $\|\cdot\|_{0,h}$. In the analysis of the Nitsche method the following norms are used:

$$\|v\|^2_{\frac{1}{2},h,\Gamma} = (v, v)_{\frac{1}{2},h,\Gamma} = \sum_{T\in\mathcal{T}_h^\Gamma} h_T^{-1} \|v\|^2_{L^2(\Gamma_T)}, \quad \|v\|^2_{-\frac{1}{2},h,\Gamma} := \sum_{T\in\mathcal{T}_h^\Gamma} h_T \|v\|^2_{L^2(\Gamma_T)}.$$

Recall that $\mathcal{T}_h^\Gamma$ is the collection of $T \in \mathcal{T}_h$ with $\Gamma_T = T \cap \Gamma \ne \emptyset$. We first derive interpolation error bounds in different norms, which turn out to be useful.

The constants used in the results derived below are all independent of λ, ξ, $\bar\varepsilon$, h, $\|\mathbf{w}\|$, and of how the interface Γ intersects the triangulation $\mathcal{T}_h$ (i.e. of the shape regularity of T_i).

LEMMA 4.1. *For $u \in W_{\rm reg}$ the following interpolation error bounds hold:*

$$\|u - I_h^\Gamma u\|_0 \le c h^2 \|u\|_{2,\Omega_1\cup\Omega_2} \tag{4.3}$$

$$|u - I_h^\Gamma u|_{1,\Omega_1\cup\Omega_2} \le c h \|u\|_{2,\Omega_1\cup\Omega_2} \tag{4.4}$$

$$\sqrt{\xi}\|u - I_h^\Gamma u\|_{0,h} \le c h^2 \|u\|_{2,\Omega_1\cup\Omega_2} \tag{4.5}$$

$$\|\mathbf{w} \cdot \nabla(u - I_h^\Gamma u)\|_{0,h} \le c \|\mathbf{w}\|_\infty^{\frac{1}{2}} h^{1\frac{1}{2}} \|u\|_{2,\Omega_1\cup\Omega_2} \tag{4.6}$$

$$\sum_{i=1}^2 \|R_i(u - I_h^\Gamma u)\|_{\frac{1}{2},h,\Gamma} \le c h \|u\|_{2,\Omega_1\cup\Omega_2} \tag{4.7}$$

$$\sum_{i=1}^2 \|\mathbf{n} \cdot \nabla R_i(u - I_h^\Gamma u)\|_{-\frac{1}{2},h,\Gamma} \le c h \|u\|_{2,\Omega_1\cup\Omega_2}. \tag{4.8}$$

Proof. The results in (4.3), (4.4) are known in the literature, e.g. [13, 19]. Using the choice of the stabilization parameter γ_T we obtain

$$\xi \|u - I_h^\Gamma u\|_{0,h}^2 = \sum_{T\in\mathcal{T}_h} \xi\gamma_T \|u - I_h^\Gamma u\|_{0,T}^2 \le \|u - I_h^\Gamma u\|_0^2 \le ch^4 \|u\|_{2,\Omega_1\cup\Omega_2}^2,$$

and thus the result in (4.5) holds. The result in (4.6) follows from

$$\begin{aligned}\|\mathbf{w}\cdot\nabla(u - I_h^\Gamma u)\|_{0,h}^2 &= \sum_{T\in\mathcal{T}_h} \gamma_T \|\mathbf{w}\cdot\nabla(u - I_h^\Gamma u)\|_{0,T}^2 \le \sum_{i=1}^2 \sum_{T\in\mathcal{T}_h} \gamma_T \|\mathbf{w}\|_{\infty,T}^2 |u - I_h^\Gamma u|_{1,T_i}^2 \\ &\le c\|\mathbf{w}\|_{L^\infty(\Omega)} h |u - I_h^\Gamma u|_{1,\Omega_1\cup\Omega_2}^2 \le c\|\mathbf{w}\|_{L^\infty(\Omega)} h^3 \|u\|_{2,\Omega_1\cup\Omega_2}^2.\end{aligned}$$

The results in (4.7), (4.8), are derived in [13]. The essential ingredient is the following result:

$$\|w\|_{L^2(\Gamma_T)}^2 \le c\big(h_T^{-1}\|w\|_{L^2(T)}^2 + h_T|w|_{1,T}^2\big) \quad \text{for all} \quad w\in H^1(T),$$

which holds for all $T\in\mathcal{T}_h^\Gamma$ and with a constant c that is independent of the shape regularity of T_i, cf. [13, 12]. For completeness we give a proof of (4.7):

$$\begin{aligned}&\sum_{i=1}^2 \|R_i(u - I_h^\Gamma u)\|_{\frac12,h,\Gamma}^2 = \sum_{i=1}^2 \sum_{T\in\mathcal{T}_h^\Gamma} h_T^{-1} \|\mathcal{E}_i R_i u - I_h \mathcal{E}_i R_i u\|_{L^2(\Gamma_T)}^2 \\ &\le c\sum_{i=1}^2 \sum_{T\in\mathcal{T}_h^\Gamma} \Big(h_T^{-2}\|\mathcal{E}_i R_i u - I_h\mathcal{E}_i R_i u\|_{L^2(T)}^2 + |\mathcal{E}_i R_i u - I_h \mathcal{E}_i R_i u|_{1,T}^2\Big) \\ &\le ch^2 \sum_{i=1}^2 \sum_{T\in\mathcal{T}_h^\Gamma} \|\mathcal{E}_i R_i u\|_{2,T}^2 \le ch^2 \sum_{i=1}^2 \|\mathcal{E}_i R_i u\|_{2,\Omega}^2 \le ch^2 \|u\|_{2,\Omega_1\cup\Omega_2}^2.\end{aligned}$$

The result in (4.8) can be proved with similar arguments. □

As we will see below, we can derive an ellipticity and continuity result for the bilinear form $a_h(\cdot,\cdot)$ with respect to a suitable norm. As expected this norm involves terms that come from the Nitsche stabilization and from the streamline diffusion stabilization. To simplify the presentation we split the bilinear form in two parts (corresponding to Nitsche and streamline diffusion stabilization) and first consider these two parts separately. Afterwards the results for these two parts can easily be glued together. We use the splitting

$$\begin{aligned}a_h(u,v) &= a_h^N(u,v) + a_h^{SD}(u,v) \\ a_h^N(u,v) &= \frac12(\varepsilon u, v)_{1,\Omega_1\cup\Omega_2} - ([\beta u], \{\varepsilon\nabla v\cdot\mathbf{n}\})_\Gamma - (\{\varepsilon\nabla u\cdot\mathbf{n}\}, [\beta v])_\Gamma \\ &\quad + \lambda\bar\varepsilon([\beta u],[\beta v])_{\frac12,h,\Gamma} \\ a_h^{SD}(u,v) &= \frac12(\varepsilon u, v)_{1,\Omega_1\cup\Omega_2} + (\mathbf{w}\cdot\nabla u, v)_0 + \xi(u,v)_0 + (\xi u + \mathbf{w}\cdot\nabla u, \mathbf{w}\cdot\nabla u)_{0,h}.\end{aligned}$$

Corresponding norms are defined as

$$\begin{aligned}\|v\|_N^2 &= \frac12\bar\varepsilon|v|_{1,\Omega_1\cup\Omega_2}^2 + \lambda\bar\varepsilon\|[\beta v]\|_{\frac12,h,\Gamma}^2, \\ \|v\|_{SD}^2 &= \frac12\bar\varepsilon|v|_{1,\Omega_1\cup\Omega_2}^2 + \xi\|v\|_0^2 + \|\mathbf{w}\cdot\nabla v\|_{0,h}^2.\end{aligned}$$

LEMMA 4.2. *There exists a constant $c > 0$ such that*

$$a_h^{SD}(v_h, v_h) \geq c\|v_h\|_{SD}^2 \quad \textit{for all} \;\; v_h \in V_h^\Gamma.$$

Proof. We apply partial integration to the term $(\mathbf{w}\cdot\nabla v_h, v_h)_0$. Since v_h may be discontinuous across Γ we have to split the integral. Using $\mathbf{w}\cdot\mathbf{n} = 0$, $\operatorname{div}\mathbf{w} = 0$ and $v_h(x) = 0$ for $x \in \partial\Omega$ we obtain

$$\begin{aligned}(\mathbf{w}\cdot\nabla v_h, v_h)_0 &= \sum_{i=1}^{2}\int_{\Omega_i}\beta_i\mathbf{w}\cdot\nabla v_h\, v_h\,dx = \sum_{i=1}^{2}\int_{\partial\Omega_i\cap\partial\Omega}\beta_i v_h^2\,\mathbf{w}\cdot\mathbf{n}_\Omega\,ds \\ &\quad + \int_\Gamma[\beta v_h^2]\mathbf{w}\cdot\mathbf{n}\,ds - \sum_{i=1}^{2}\int_{\Omega_i}\beta_i\mathbf{w}\cdot\nabla v_h\,v_h + \beta_i(\operatorname{div}\mathbf{w})v_h^2\,dx \\ &= -(\mathbf{w}\cdot\nabla v_h, v_h)_0.\end{aligned} \tag{4.9}$$

Hence, $(\mathbf{w}\cdot\nabla v_h, v_h)_0 = 0$ holds. Furthermore, using $\gamma_T\xi \leq 1$ we get

$$\begin{aligned}\xi(v_h, \mathbf{w}\cdot\nabla v_h)_{0,h} &= \xi\sum_{T\in\mathcal{T}_h}\gamma_T(v_h, \mathbf{w}\cdot\nabla v_h)_{0,T} \\ &\leq \frac{1}{2}\sum_{T\in\mathcal{T}_h}\xi^2\gamma_T\|v_h\|_{0,T}^2 + \gamma_T\|\mathbf{w}\cdot\nabla v_h\|_{0,T}^2 \\ &\leq \frac{1}{2}\xi\|v_h\|_0^2 + \frac{1}{2}\|\mathbf{w}\cdot\nabla v_h\|_{0,h}^2.\end{aligned}$$

Hence,

$$\begin{aligned}a_h^{SD}(v_h, v_h) &\geq \frac{1}{2}\min\{\varepsilon_1, \varepsilon_2\}\|v_h\|_{1,\Omega_1\cup\Omega_2}^2 + \xi\|v_h\|_0^2 + \|\mathbf{w}\cdot\nabla v_h\|_{0,h}^2 + \xi(v_h, \mathbf{w}\cdot\nabla v_h)_{0,h} \\ &\geq c\bar{\varepsilon}\|v_h\|_{1,\Omega_1\cup\Omega_2}^2 + \frac{1}{2}\xi\|v_h\|_0^2 + \frac{1}{2}\|\mathbf{w}\cdot\nabla v_h\|_{0,h}^2,\end{aligned}$$

with a constant $c > 0$ which depends only on the ratio between ε_1 and ε_2, which is assumed to be bounded. □

LEMMA 4.3. *There exists a constant c such that*

$$a_h^{SD}(u - I_h^\Gamma u, v_h) \leq c\big(\sqrt{\bar{\varepsilon}} + \sqrt{\|\mathbf{w}\|_\infty h} + \sqrt{\xi}\,h\big)h\|u\|_{2,\Omega_1\cup\Omega_2}\|v_h\|_{SD} \quad \forall\, u \in W_{\mathrm{reg}},\ v_h \in V_h^\Gamma.$$

Proof. We use the notation $e_h := u - I_h^\Gamma u$ and recall the definition of $a_h^{SD}(\cdot,\cdot)$:

$$a_h^{SD}(e_h, v_h) = \frac{1}{2}(\varepsilon e_h, v_h)_{1,\Omega_1\cup\Omega_2} + (\mathbf{w}\cdot\nabla e_h, v_h)_0 + \xi(e_h, v_h)_0 + (\xi e_h + \mathbf{w}\cdot\nabla e_h, \mathbf{w}\cdot\nabla v_h)_{0,h}.$$

Using the interpolation error bounds of lemma 4.1 we obtain

$$\begin{aligned}\frac{1}{2}(\varepsilon e_h, v_h)_{1,\Omega_1\cup\Omega_2} &\leq c\sqrt{\bar{\varepsilon}}\,h\|u\|_{2,\Omega_1\cup\Omega_2}\|v_h\|_{SD} \\ \xi(e_h, v_h)_0 &\leq c\sqrt{\xi}h^2\|u\|_{2,\Omega_1\cup\Omega_2}\|v_h\|_{SD} \\ \xi(e_h, \mathbf{w}\cdot\nabla v_h)_{0,h} &\leq c\sqrt{\xi}h^2\|u\|_{2,\Omega_1\cup\Omega_2}\|v_h\|_{SD} \\ (\mathbf{w}\cdot\nabla e_h, \mathbf{w}\cdot\nabla v_h)_{0,h} &\leq c\|\mathbf{w}\|_\infty^{\frac{1}{2}}h^{1\frac{1}{2}}\|u\|_{2,\Omega_1\cup\Omega_2}\|v_h\|_{SD}.\end{aligned}$$

To the term $(\mathbf{w}\cdot\nabla e_h, v_h)_0$ we apply partial integration as in (4.9), resulting in

$$
\begin{aligned}
(\mathbf{w}\cdot\nabla e_h, v_h)_0 = -(e_h, \mathbf{w}\cdot\nabla v_h)_0 &\leq \big(\sum_{T\in\mathcal{T}_h} \gamma_T^{-1}\|e_h\|_{0,T}^2\big)^{\frac{1}{2}}\|v_h\|_{SD} \\
&\leq c\big(\xi h^2 + \|\mathbf{w}\|_\infty h + \bar{\varepsilon}\big)^{\frac{1}{2}} h\|u\|_{2,\Omega_1\cup\Omega_2}\|v_h\|_{SD},
\end{aligned}
$$

where in the last inequality we used the bound for $\gamma_T^{-1}h_T^2$ given in (4.1). Combining these estimates completes the proof. $\square$

We now turn to the analysis of the Nitsche bilinear form $a_h^N(\cdot,\cdot)$. We need the following inverse inequality given in [13].

LEMMA 4.4. *There exists a constant c_I independent of $\bar{\varepsilon}$ such that*

$$
\|\{\varepsilon\nabla v_h\cdot\mathbf{n}\}\|_{-\frac{1}{2},h,\Gamma} \leq c_I\bar{\varepsilon}|v_h|_{1,\Omega_1\cup\Omega_2} \quad \textit{for all} \;\; v_h \in V_h^\Gamma.
$$

Proof. Lemma 4 in [13]. $\square$

We now derive an ellipticity result for $a_h^N(\cdot,\cdot)$:

LEMMA 4.5. *There exist constants $c_1 > 0$, $c_s > 0$ such that for $\lambda > c_s$*

$$
a_h^N(v_h, v_h) \geq c_1\|v_h\|_N^2 \quad \textit{for all} \;\; v_h \in V_h^\Gamma.
$$

Proof. Define $\hat{c} = \frac{1}{2\bar{\varepsilon}}\min\{\varepsilon_1,\varepsilon_2\} \leq \frac{1}{2}$ and take $\lambda \geq 4c_I^2\hat{c}^{-1}$ with c_I from lemma 4.4. The following holds:

$$
\begin{aligned}
a_h^N(v_h, v_h) &\geq \hat{c}\bar{\varepsilon}|v_h|_{1,\Omega_1\cup\Omega_2}^2 - 2\|[\beta v_h]\|_{\frac{1}{2},h,\Gamma}\|\{\varepsilon\nabla v_h\cdot\mathbf{n}\}\|_{-\frac{1}{2},h,\Gamma} + \lambda\bar{\varepsilon}\|[\beta v_h]\|_{\frac{1}{2},h,\Gamma}^2 \\
&\geq \hat{c}\bar{\varepsilon}|v_h|_{1,\Omega_1\cup\Omega_2}^2 - 2c_I\bar{\varepsilon}\|[\beta v_h]\|_{\frac{1}{2},h,\Gamma}|v_h|_{1,\Omega_1\cup\Omega_2} + \lambda\bar{\varepsilon}\|[\beta v_h]\|_{\frac{1}{2},h,\Gamma}^2 \\
&\geq \frac{1}{2}\hat{c}\bar{\varepsilon}|v_h|_{1,\Omega_1\cup\Omega_2}^2 + (\lambda - 2c_I^2\hat{c}^{-1})\bar{\varepsilon}\|[\beta v_h]\|_{\frac{1}{2},h,\Gamma}^2 \\
&\geq \frac{1}{2}\hat{c}\bar{\varepsilon}|v_h|_{1,\Omega_1\cup\Omega_2}^2 + \frac{1}{2}\lambda\bar{\varepsilon}\|[\beta v_h]\|_{\frac{1}{2},h,\Gamma}^2 \geq \hat{c}\|v_h\|_N^2.
\end{aligned}
$$

$\square$

LEMMA 4.6. *There exists a constant c such that for $\lambda > 0$*

$$
a_h^N(u - I_h^\Gamma u, v_h) \leq c\sqrt{\bar{\varepsilon}}\big(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\big)h\|u\|_{2,\Omega_1\cup\Omega_2}\|v_h\|_N
$$

holds for all $u \in W_{\mathrm{reg}}$, $v_h \in V_h^\Gamma$.

Proof. We use the notation $e_h := u - I_h^\Gamma u$ and recall the definition of $a_h^N(\cdot,\cdot)$:

$$
\begin{aligned}
a_h^N(e_h, v_h) = \frac{1}{2}(\varepsilon e_h, v_h)_{1,\Omega_1\cup\Omega_2} &- ([\beta e_h], \{\varepsilon\nabla v_h\cdot\mathbf{n}\})_\Gamma - (\{\varepsilon\nabla e_h\cdot\mathbf{n}\}, [\beta v_h])_\Gamma \\
&+ \lambda\bar{\varepsilon}([\beta e_h], [\beta v_h])_{\frac{1}{2},h,\Gamma}.
\end{aligned}
$$

Using the interpolation error bounds of lemma 4.1 and the inverse inequality in lemma 4.4 we obtain

$$
\begin{aligned}
\frac{1}{2}(\varepsilon e_h, v_h)_{1,\Omega_1\cup\Omega_2} &\le c\sqrt{\bar\varepsilon}\, h\|u\|_{2,\Omega_1\cup\Omega_2}\|v_h\|_N \\
-([\beta e_h], \{\varepsilon\nabla v_h\cdot\mathbf{n}\})_\Gamma &\le \|[\beta e_h]\|_{\frac12,h,\Gamma}\|\{\varepsilon\nabla v_h\cdot\mathbf{n}\}\|_{-\frac12,h,\Gamma} \\
&\le c\bar\varepsilon\sum_{i=1}^2\|R_i e_h\|_{\frac12,h,\Gamma}|v_h|_{1,\Omega_1\cup\Omega_2} \le c\sqrt{\bar\varepsilon}\, h\|u\|_{2,\Omega_1\cup\Omega_2}\|v_h\|_N \\
-(\{\varepsilon\nabla e_h\cdot\mathbf{n}\}, [\beta v_h])_\Gamma &\le \|\{\varepsilon\nabla e_h\cdot\mathbf{n}\}\|_{-\frac12,h,\Gamma}\|[\beta v_h]\|_{\frac12,h,\Gamma} \\
&\le c\bar\varepsilon\sum_{i=1}^2\|\mathbf{n}\cdot\nabla R_i e_h\|_{-\frac12,h,\Gamma}\|[\beta v_h]\|_{\frac12,h,\Gamma} \\
&\le c\sqrt{\frac{\bar\varepsilon}{\lambda}}\, h\|u\|_{2,\Omega_1\cup\Omega_2}\|v_h\|_N \\
\lambda\bar\varepsilon([\beta e_h],[\beta v_h])_{\frac12,h,\Gamma} &\le \lambda\bar\varepsilon\|[\beta e_h]\|_{\frac12,h,\Gamma}\|[\beta v_h]\|_{\frac12,h,\Gamma} \le c\sqrt{\bar\varepsilon\lambda}\, h\|u\|_{2,\Omega_1\cup\Omega_2}\|v_h\|_N.
\end{aligned}
$$

Combination of these estimates and using $\frac{1}{\sqrt{\lambda}}+\sqrt{\lambda}>1$ proves the result. □

We now combine the estimates derived above for $a_h^N(\cdot,\cdot)$ and $a_h^{SD}(\cdot,\cdot)$. For this we introduce the norm

$$
|\!|\!| v |\!|\!|^2 = \|v\|_N^2 + \|v\|_{SD}^2 = \bar\varepsilon|v|_{1,\Omega_1\cup\Omega_2}^2 + \xi\|v\|_0^2 + \|\mathbf{w}\cdot\nabla v\|_{0,h}^2 + \lambda\bar\varepsilon\|[\beta v]\|_{\frac12,h,\Gamma}^2.
$$

Note that the two terms $\|\mathbf{w}\cdot\nabla v\|_{0,h}^2$ and $\lambda\bar\varepsilon\|[\beta v]\|_{\frac12,h,\Gamma}^2$ originate from the stabilization terms in the streamline diffusion and the Nitsche method, respectively.

We discuss the choice of the stabilization parameter λ in the Nitsche method. For the error analysis in the norm $|\!|\!| \cdot |\!|\!|$ it is natural to balance the upper bounds in lemma 4.3 and in lemma 4.6 to derive an upper bound for λ. For simplicity we make the (weak) assumption that $\xi h^2 \le c(\bar\varepsilon + \|\mathbf{w}\|_\infty h)$ holds, i.e. in the bound in lemma 4.3 the factor $\sqrt{\bar\varepsilon}+\sqrt{\|\mathbf{w}\|_\infty h}+\sqrt{\xi}\,h$ can be replaced by $\sqrt{\bar\varepsilon}+\sqrt{\|\mathbf{w}\|_\infty h}$. The factor $\sqrt{\bar\varepsilon}(\sqrt{\lambda}+1/\sqrt{\lambda})$ in the upper bound in lemma 4.6 should balance the latter factor, i.e., $\sqrt{\bar\varepsilon}+\sqrt{\|\mathbf{w}\|_\infty h}\approx\sqrt{\bar\varepsilon}(\sqrt{\lambda}+1/\sqrt{\lambda})$. This leads to the choice as in (2.8), namely $\lambda=\lambda^c=c\max(\frac12\|\mathbf{w}\|_\infty h/\bar\varepsilon, 1)$. It is, however, not necessary to balance the upper bounds; if we take $\lambda=\lambda^d=c>0$, then the bound in lemma 4.6 is smaller than the one in lemma 4.3. We take $c=c_s$, cf. lemma 4.5, to guarantee ellipticity. A slightly refined error analysis leads to a localized variant of this parameter choice given in remark 3. In the remainder we assume $\lambda^d\le\lambda\le\lambda^c$ as in (2.8), with $c=c_s$, cf. lemma 4.5. To simplify the presentation, we assume $\|\mathbf{w}\|_\infty=\mathcal{O}(1)$ and in the estimates below the terms $\|\mathbf{w}\|_\infty$ are absorbed in the constant c. In convection-dominated problems with $\|\mathbf{w}\|_\infty\gg 1$ one can derive corresponding modified estimates.

From the interpolation error bounds in lemma 4.1, and using $\lambda\bar\varepsilon\le c(\bar\varepsilon+\|\mathbf{w}\|_\infty h)\le c(\bar\varepsilon+h)$, we obtain

$$
|\!|\!| u - I_h^\Gamma u |\!|\!| \le c\big(\sqrt{\bar\varepsilon}+\sqrt{h}+\sqrt{\xi}\,h\big)h\|u\|_{2,\Omega_1\cup\Omega_2} \quad \text{for all } \ u\in W_{\mathrm{reg}}. \tag{4.10}
$$

THEOREM 4.7. *For $u\in W_{\mathrm{reg}}$ let $R_G u\in V_h^\Gamma$ be the Galerkin projection for the bilinear form $a_h(\cdot,\cdot)$, i.e. $a_h(R_G u, v_h)=a_h(u,v_h)$ for all $v_h\in V_h^\Gamma$. The following holds:*

$$
|\!|\!| u - R_G u |\!|\!| \le c\big(\sqrt{\bar\varepsilon}+\sqrt{h}+\sqrt{\xi}\,h\big)h\|u\|_{2,\Omega_1\cup\Omega_2} \quad \textit{for all } \ u\in W_{\mathrm{reg}}. \tag{4.11}
$$

The constant c is independent of $\bar{\varepsilon}$, h, ξ and of how the interface Γ intersects the triangulation $\mathcal{T}_h$.

Proof. The proof uses standard arguments. Define $\chi_h = R_G u - I_h^\Gamma u \in V_h^\Gamma$. Using the results in the lemmas above we obtain, with a suitable $c > 0$,

$$\begin{aligned}
\|\!|\chi_h|\!\|^2 &= \|\chi_h\|_N^2 + \|\chi_h\|_{SD}^2 \le c\big(a_h^N(\chi_h,\chi_h) + a_h^{SD}(\chi_h,\chi_h)\big) \\
&= c\,a_h(\chi_h,\chi_h) = c\,a_h(u - I_h^\Gamma u, \chi_h) = c\,a_h^N(u - I_h^\Gamma u, \chi_h) + c\,a_h^{SD}(u - I_h^\Gamma u, \chi_h) \\
&\le c\big(\sqrt{\bar{\varepsilon}} + \sqrt{h} + \sqrt{\xi}\,h\big)h\|u\|_{2,\Omega_1\cup\Omega_2}\|\!|\chi_h|\!\|.
\end{aligned}$$

The result follows from a triangle inequality and the interpolation error bound in (4.10). ∎

We comment on the bound derived in (4.11). For the diffusion dominated case, i.e. $\bar{\varepsilon} \sim 1$, this result reduces to results known in the literature. We discuss the convection dominated case $\bar{\varepsilon} \le \|\mathbf{w}\|_\infty h$ with $\xi \in [0,1]$ and write $e_h := u - R_G u$. Furthermore we assume $h \le c h_T$ (quasi-uniformity of the family of triangulations). Using $h \le c\gamma_T$ for all $T \in \mathcal{T}_h$ we obtain from (4.11)

$$\|\mathbf{w}\cdot\nabla e_h\|_{L^2(\Omega)} \le ch\|u\|_{2,\Omega_1\cup\Omega_2}.$$

Hence, as for the streamline diffusion finite element method with the standard linear finite element space, we have an optimal error bound (uniformly in $\bar{\varepsilon}$) for the derivative of the error in streamline direction. The estimate (4.11) also implies

$$\lambda\bar{\varepsilon}\|[\beta e_h]\|_{\frac{1}{2},h,\Gamma}^2 \le ch^3\|u\|_{2,\Omega_1\cup\Omega_2}^2.$$

For the *convective* scaling we have $\lambda\bar{\varepsilon} \sim ch$ and thus obtain $\|[\beta e_h]\|_{L^2(\Gamma)} \le ch^{1\frac{1}{2}}\|u\|_{2,\Omega_1\cup\Omega_2}$ uniformly in $\bar{\varepsilon}$. For the *diffusive* scaling we have $\lambda \sim c$ and thus obtain a worse bound $\|[\beta e_h]\|_{L^2(\Gamma)} \le ch^2\bar{\varepsilon}^{-\frac{1}{2}}\|u\|_{2,\Omega_1\cup\Omega_2}$. Finally, if we take $\xi > 0$ we obtain an L^2-norm error bound that is the same as for the streamline diffusion finite element method with the standard linear finite element space, namely

$$\|e_h\|_{L^2(\Omega)} \le \frac{c}{\sqrt{\xi}}\,h^{1\frac{1}{2}}\|u\|_{2,\Omega_1\cup\Omega_2}.$$

Remark 5. As noted in Remark 4, the SD-Nitsche-XFEM method has a straightforward extension to finite elements of higher order. We comment on the generalization of the error analysis presented above to the higher order case. The interpolation error bounds in Lemma 4.1 can easily be generalized to higher order extended finite elements. The result in Lemma 4.4 also holds for higher order elements, cf. [2]. Using this the results for the Nitsche bilinear form in the Lemmas 4.5 and 4.6 can be generalized. In the analysis of the streamline diffusion bilinear form, however, a difficulty arises related to an inverse inequality needed in the analysis. For higher order finite elements the term $(\mathrm{div}(\varepsilon\nabla u_h), \mathbf{w}\cdot\nabla v_h)_{0,h}$ arises in the streamline diffusion stabilization. In the analysis of the streamline diffusion method for a standard higher order finite element space V_h one uses an inverse inequality of the form $\|\Delta v_h\|_{0,T} \le \mu_{\mathrm{inv}} h_T^{-1}|v_h|_{1,T}$ for all $v_h \in V_h$, cf. [21]. Such a result does not hold in a higher order XFEM space, since the supports $T_i = T\cap\Omega_i$ of the additional (discontinuous) basis functions can be very shape irregular. We only have $\|\Delta v_h\|_{0,T_i} \le \mu(T_i)h_{T_i}^{-1}|v_h|_{1,T_i}$ with a factor $\mu(T_i)$ that depends on the shape regularity of T_i. To control this, instead of $\min\{\xi^{-1}, \gamma_T^{(2.13)}\}$, one can choose a stabilization parameter γ_{T_i} that is sufficiently small. This would yield a stability result as in Lemma 4.2. If, however, this parameter is “too small” it is not likely that a result as in Lemma 4.3, which uses the third inequality in (4.1), still holds. We did not investigate this further.

Remark 6. For the proposed stabilized semi-discretization (2.14) for the case of a stationary interface, i.e. $\mathbf{w}\cdot\mathbf{n}=0$, there holds a mass conservation property. Define the inflow boundary by $\partial\Omega_- := \{\, x\in\partial\Omega \mid \mathbf{w}\cdot\mathbf{n}_\Omega < 0\,\}$, where $\mathbf{n}_\Omega$ is the outward pointing unit normal on $\partial\Omega$, and $\partial\Omega_+ := \partial\Omega\setminus\partial\Omega_-$. Instead of (1.5) we consider

$$\begin{aligned}(u\mathbf{w}-\varepsilon\nabla u)\cdot\mathbf{n}_\Omega = u_I\mathbf{w}\cdot\mathbf{n}_\Omega \quad &\text{on}\ \ \partial\Omega_-,\ t\in[0,T]\\ \varepsilon\nabla u\cdot\mathbf{n}_\Omega = 0 \quad &\text{on}\ \ \partial\Omega_+,\ t\in[0,T],\end{aligned} \tag{4.12}$$

with u_I a given mass inflow function. Integrating the equations in (1.1) over Ω, and using the relations in (1.2), (4.12), we obtain the global mass conservation property

$$\frac{d}{dt}\int_\Omega u\,dx + \int_{\partial\Omega_-} u_I\mathbf{w}\cdot\mathbf{n}_\Omega\,ds + \int_{\partial\Omega_+} u\mathbf{w}\cdot\mathbf{n}_\Omega\,ds = \int_\Omega f\,dx. \tag{4.13}$$

We show that an analogon of the conservation law (4.13) holds for the Nitsche-XFEM stabilized discretization. Due to the modification of the boundary condition the XFEM space we use is given by $\tilde V_h^\Gamma := \{\, v\in H^1(\Omega_1\cup\Omega_2) \mid v_{|T_i}$ is linear for all $T\in\mathcal{T}_h,\ i=1,2.\,\}$ and the discretization in (2.14) is modified by adding the boundary integral $-\int_{\partial\Omega_-}\beta u_h v_h\mathbf{w}\cdot\mathbf{n}_\Omega\,ds$ to the l.h.s. and $-\int_{\partial\Omega_-}\beta u_I v_h\mathbf{w}\cdot\mathbf{n}_\Omega\,ds$ to the r.h.s.. Taking the test function $\beta^{-1}\in\tilde V$ in the modified version of (2.14) all terms with ∇v_h vanish and for the Nitsche bilinear form, cf. (2.7), we have $a_h(u_h,\beta^{-1}) = (\mathbf{w}\cdot\nabla u_h,\beta^{-1})_0$. Partial integration for the term $(\mathbf{w}\cdot\nabla u_h,\beta^{-1})_0 = \int_\Omega \mathbf{w}\cdot\nabla u_h\,dx$ results in

$$\frac{d}{dt}\int_\Omega u_h\,dx + \int_{\partial\Omega_-} u_I\mathbf{w}\cdot\mathbf{n}_\Omega\,ds + \int_{\partial\Omega_+} u_h\mathbf{w}\cdot\mathbf{n}_\Omega\,ds = \int_\Omega f\,dx,$$

which is the discrete global mass conservation analogon of the one in (4.13).

Remark 7. In the error analysis in section 4 we only studied the bilinear form for the quasi-stationary problem. Based on the techniques presented in the recent paper [6] it may be possible to derive, for the case of a *stationary* interface, error bounds for the semi-discrete problem (2.14).

In view of applications the case of a *non*-stationary interface $\Gamma(t)$ is much more interesting than that of a stationary one. We comment on a generalization of the method in (2.15) to the former case. For an evolving interface $\Gamma(t)$, instead of the weak formulation in (2.3), one has to consider a space-time variational formulation to obtain a well-posed problem, cf. [12]. A discretization of the time derivative by means of finite difference approximations (as done here for a stationary interface) does no longer lead to a consistent discretization if the interface $\Gamma(t)$ is moving in time. A discretization based on a space-time formulation using a suitable space-time extended finite element space should be used. This can be combined with a space-time streamline diffusion stabilization. The development and analysis of such a space-time SD-Nitsche-XFEM method is a topic of current research.

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