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ANALYSIS OF A NITSCHE XFEM-DG DISCRETIZATION FOR A CLASS OF TWO-PHASE MASS TRANSPORT PROBLEMS*

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Abstract. We consider a standard model for mass transport across an evolving interface. The solution has to satisfy a jump condition across an evolving interface. We present and analyze a finite element discretization method for this mass transport problem. This method is based on a space-time approach in which a discontinuous Galerkin (DG) technique is combined with an extended finite element method (XFEM). The jump condition is satisfied in a weak sense by using the Nitsche method. This Nitsche XFEM-DG method is new. An error analysis is presented. Results of numerical experiments are given which illustrate the accuracy of the method.

Key words. Nitsche method, space-time XFEM, transport problem

AMS subject classifications. 65N12, 65N30

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1. Introduction.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a convex polygonal domain that contains two different immiscible incompressible phases. The (in general, time dependent) subdomains containing the two phases are denoted by Ω_1, Ω_2, with $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$. For simplicity we assume that $\partial\Omega_1 \cap \partial\Omega = \emptyset$, i.e., the phase in Ω_1 is completely surrounded by the one in Ω_2. The interface $\Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2$ is assumed to be sufficiently smooth. A model example is a (rising) droplet in a flow field. The fluid dynamics in such a flow problem is usually modeled by the incompressible Navier–Stokes equations combined with suitable conditions at the interface which describe the effect of surface tension. For this model we refer to the literature, e.g., [2, 9, 17, 23, 10]. By $\mathbf{w}$ we denote the velocity field resulting from these Navier–Stokes equations. We assume that $\operatorname{div} \mathbf{w} = 0$ holds. Furthermore, we assume that the evolution of the interface is determined by this velocity field, in the sense that $V_\Gamma = \mathbf{w} \cdot \mathbf{n}$ holds, where V_Γ is the normal velocity of the interface and $\mathbf{n}$ denotes the unit normal at Γ pointing from Ω_1 into Ω_2. We consider a standard model which describes the transport of a dissolved species in a two-phase flow problem. In strong formulation this model is as follows:

$$\frac{\partial u}{\partial t} + \mathbf{w} \cdot \nabla u - \operatorname{div}(\alpha \nabla u) = f \quad \text{in} \;\; \Omega_i(t), \; i = 1, 2, \; t \in [0, T], \tag{1.1}$$

$$[\alpha \nabla u \cdot \mathbf{n}]_\Gamma = 0, \tag{1.2}$$

$$[\beta u]_\Gamma = 0, \tag{1.3}$$

$$u(\cdot, 0) = u_0 \quad \text{in} \;\; \Omega_i(0), \; i = 1, 2, \tag{1.4}$$

$$u(\cdot, t) = 0 \quad \text{on} \;\; \partial\Omega, \quad t \in [0, T]. \tag{1.5}$$

For a sufficiently smooth function v, $[v] = [v]_\Gamma$ denotes the jump of v across Γ, i.e., $[v] = (v_1)_{|\Gamma} - (v_2)_{|\Gamma}$, where $v_i = v_{|\Omega_i}$ is the restriction of v to Ω_i. In (1.1) we have standard parabolic convection-diffusion equations in the two subdomains Ω_1 and Ω_2.

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In many applications one has a homogeneous problem, i.e., $f \equiv 0$. The diffusion coefficient $\alpha = \alpha(x,t)$ is assumed to be piecewise constant:

$$\alpha = \alpha_i > 0 \quad \text{in} \quad \Omega_i(t).$$

In general, we have $\alpha_1 \neq \alpha_2$. The interface condition in (1.2) results from the conservation of mass principle. The condition in (1.3) is the so-called *Henry condition*; cf. [15, 21, 20, 3, 2]. In this condition the coefficient $\beta = \beta(x,t)$ is strictly positive and piecewise constant:

$$\beta = \beta_i > 0 \quad \text{in} \quad \Omega_i(t).$$

In general we have $\beta_1 \neq \beta_2$, since species concentration usually has a jump discontinuity at the interface due to different solubilities within the respective fluid phases. Hence, the solution u is *discontinuous across the interface*. In this paper we analyze a finite element discretization method for this problem.

Both for the mathematical analysis and numerical treatment of this transport problem there is a big difference between the case with a *stationary interface* and the one with a *nonstationary interface*.

In recent years it has been shown that for such a transport problem with an (evolving) interface the Nitsche-XFEM method is very well suited [11, 19]. In [12, 13, 14, 1, 5] the application of the Nitsche-XFEM to other classes of problems is studied. In [11] this method is analyzed for a *stationary* heat diffusion problem (no convection) with a conductivity that is discontinuous across the interface ($\alpha_1 \neq \alpha_2$) but with a solution that is continuous across the interface ($\beta_1 = \beta_2$). In [19] the method is studied for the parabolic problem described above, with $\beta_1 \neq \beta_2$ (discontinuous solution), and with a convection term in (1.1). It is assumed, however, that the transport problem is diffusion dominated. In the recent paper [18] the convection-dominated case, where the Nitsche XFEM method is combined with a streamline diffusion stabilization, is treated. In all of these papers, and in other literature that we know of, the Nitsche XFEM method is analyzed for a two-phase transport as in (1.1)–(1.5) with a *stationary interface*. In this paper a first analysis for the case of a *nonstationary* interface is presented. We restrict to the diffusion dominated case, i.e., no stabilization with relation to (w.r.t.) convection is needed.

For the weak formulation of the parabolic mass transport problem (1.1)–(1.5) with a nonstationary interface a space-time variational formulation is most natural; cf. chapter 10 in [10]. This suggests an XFEM approach combined with a suitable *space-time discontinuous Galerkin method*. As far as we know, this combination has been considered, without an error analysis, in the literature only in [6]. We explain this method, for the case of *linear* finite elements (in space and time), in section 2. In section 3 an error analysis for this discretization method is presented. To the best of our knowledge this analysis is new. A main result is an error bound as in Theorem 3.15. This result shows that for a transport problem as (1.1)–(1.5), with a solution that is *discontinuous* across an evolving interface, we have a full (i.e., in space and time) discretization that is second order accurate. The space-time XFEM-DG method is an Eulerian method in the sense that the spatial triangulation is not fitted to the interface. We are not aware of other Eulerian type discretization methods for this class of parabolic interface problems that have a guaranteed (i.e., based on an error analysis) second order convergence.

The discretization method, presented in section 2, that is considered in the analysis is often not feasible in practice, due to the fact that it is assumed that integrals over the space-time interface are evaluated exactly. In practice a quadrature rule will be necessary. This leads to a "variational crime." A study of the effect of these quadrature errors is the topic of ongoing research.

2. The Nitsche XFEM-DG discretization. In this section we derive the discretization method. This method will have the form of a variational problem in a certain space-time finite element space. The same space is used for both trial and test functions. We introduce the method for the case of piecewise *bilinear* space-time functions (linear in space and linear in time). In Remark 2.2 we comment on generalizations. We introduce notation. The space-time domain is denoted by $Q = \Omega \times (0,T] \subset \mathbb{R}^{d+1}$. A partitioning of the time interval is given by $0 = t_0 < t_1 < \ldots < t_N = T$, with a uniform time step $\Delta t = T/N$. This assumption of a uniform time step is made to simplify the presentation, but is not essential for the method. Corresponding to each time interval $I_n := (t_{n-1}, t_n]$ we assume a given shape regular simplicial triangulation $\mathcal{T}_n$ of the spatial domain Ω. In general, this triangulation is *not fitted* to the interface $\Gamma(t)$. The triangulation may vary with n. Let V_n be the finite element space of continuous piecewise linear functions on $\mathcal{T}_n$ with zero boundary values on $\partial\Omega$. The spatial mesh size parameter corresponding to V_n is denoted by h_n. Corresponding space-time finite element spaces on the time slab $Q^n := \Omega \times I_n$ are given by

$$\begin{aligned} W_n &:= \{\, v : Q^n \to \mathbb{R} \mid v(x,t) = \phi_0(x) + t\phi_1(x), \quad \phi_0, \phi_1 \in V_n \,\}, \\ W &:= \{\, v : Q \to \mathbb{R} \mid v_{|Q^n} \in W_n \,\}. \end{aligned} \tag{2.1}$$

In the time slab Q^n we define the subdomains $Q_i^n := \cup_{t \in I_n} \Omega_i(t)$, $i = 1,2$, and also $Q_i := \cup_{1 \le n \le N} Q_i^n = \cup_{0 < t \le T} \Omega_i(t)$, $i = 1,2$. We introduce corresponding canonical restrictions R_i^n on $L^2(Q^n)$, R_i on $L^2(Q)$ given by

$$R_i^n v = \begin{cases} v_{|Q_i^n} & \text{on } \; Q_i^n, \\ 0 & \text{on } \; Q^n \setminus Q_i^n, \end{cases} \qquad R_i v = \begin{cases} v_{|Q_i} & \text{on } \; Q_i, \\ 0 & \text{on } \; Q \setminus Q_i. \end{cases}$$

We will also use the notation $v_i := R_i v$. The space-time XFEM spaces are given by

$$\begin{aligned} W_n^\Gamma &:= R_1^n W_n \oplus R_2^n W_n, \\ W^{\Gamma_*} &:= \{\, v :\, Q \to \mathbb{R} \mid v_{|Q^n} \in W_n^\Gamma \,\} = R_1 W \oplus R_2 W. \end{aligned} \tag{2.2}$$

The symbol Γ_*^n denotes the space-time interface in Q^n, i.e., $\Gamma_*^n := \cup_{t \in I_n} \Gamma(t)$, and $\Gamma_* := \cup_{1 \le n \le N} \Gamma_*^n$.

Remark 2.1. In literature on extended finite element spaces the XFEM space W_n^Γ introduced above is usually characterized in a different way [8]. We briefly explain this different characterization. Let $\{q_j\}_{j \in \mathcal{J}}$ be the nodal basis in the finite element space V_n. The vertex corresponding to q_j is denoted by x_j. To each q_j there correspond two space-time basis functions, namely $q_{j,0}(x,t) := \frac{1}{\Delta t}(t_n - t) q_j(x)$ and $q_{j,1}(x,t) := \frac{1}{\Delta t}(t - t_{n-1}) q_j(x)$. The index set of basis functions in the space-time finite element space W_n "close to the interface" is given by

$$\mathcal{J}_{\Gamma_*^n} := \{\, (j,0), (j,1) \mid \text{meas}_3\big(\Gamma_*^n \cap \text{supp}(q_j)\big) > 0 \}.$$

Let $H_{\Gamma_*^n}$ be the characteristic function corresponding to Q_2^n, i.e., $H_{\Gamma_*^n}(x,t) = 1$ if $(x,t) \in Q_2^n$ and zero otherwise. For each space-time node index $(j,\ell) \in \mathcal{J}_{\Gamma_*^n}$ a so-called enrichment function corresponding to the node $(x_j, t_{n-\ell})$ is given by

$$\Phi_{j,\ell}(x,t) := H_{\Gamma_*^n}(x,t) - H_{\Gamma_*^n}(x_j, t_{n-\ell}). \tag{2.3}$$

New basis functions are defined as follows:

$$q_{j,\ell}^{\Gamma_*^n} := q_{j,\ell}\Phi_{j,\ell}, \quad (j,\ell) \in \mathcal{J}_{\Gamma_*^n}. \tag{2.4}$$

The term $H_{\Gamma_*^n}(x_j, t_{n-\ell})$ in the definition of $\Phi_{j,\ell}$ is constant and may be omitted (as it does not introduce new functions in the function space), but ensures that $q_{j,\ell}^{\Gamma_*^n}(x_j, t_{n-\ell}) = 0$ holds in all space-time grid points $(x_j, t_{n-\ell})$. The space-time XFEM space on the time slab $Q^n = \Omega \times I_n$ is given by

$$W_n^\Gamma = W_n \oplus \operatorname{span}\{\, q_{j,\ell}^{\Gamma_*^n} \mid (j,\ell) \in \mathcal{J}_{\Gamma_*^n} \,\}.$$

This characterization shows that the XFEM finite element space W_n^Γ is obtained by adding to the standard space W_n new basis functions that are discontinuous across the space-time interface Γ_*^n; cf. (2.4).

We will treat the Henry condition $[\beta u]_\Gamma = 0$ using the Nitsche technique (in space-time). For this we need suitable averages and jumps across $\Gamma(t)$, that we now introduce. Take $t \in I_n$, $T \in \mathcal{T}_n$ and let $T_i := T \cap \Omega_i(t)$. We define the weights

$$(\kappa_i(t))_{|T} := \frac{|T_i|}{|T|}.$$

Note that those weights depend only on the *spatial* configuration at a given time t and there holds $\kappa_1(t) + \kappa_2(t) = 1$. We define the weighted average

$$\{v\}_t := \kappa_1(t)(R_1 v)_{|\Gamma(t)} + \kappa_2(t)(R_2 v)_{|\Gamma(t)}.$$

This defines an averaging operator on the space-time interface denoted by

$$\{v\}_{\Gamma_*}(x,t) = \{v\}_t(x), \quad x \in \Gamma(t).$$

We use a similar notation for the jump operators:

$$[v]_t = (R_2 v)_{|\Gamma(t)} - (R_1 v)_{|\Gamma(t)}, \quad [v]_{\Gamma_*}(x,t) = [v]_t(x), \quad x \in \Gamma(t).$$

In the discontinuous Galerkin method we need jump terms across the end points of the time intervals $I_n = (t_{n-1}, t_n]$. We define $u_+^{n-1}(\cdot) := \lim_{\epsilon \downarrow 0} u(\cdot, t_{n-1} + \epsilon)$ and introduce the notation

$$v^n(x) := v(x, t_n), \quad [v]^n(x) := v_+^n(x) - v^n(x), \quad 0 \le n \le N-1, \quad \text{with } \; v^0(x) := 0.$$

On the cross sections $\Omega \times \{t_n\}$, $0 \le n \le N$, of Q we use a weighted L^2 scalar product

$$(u,v)_{0,t_n} := \int_\Omega \beta(\cdot, t_n) uv \, dx = \sum_{i=1}^2 \beta_i \int_{\Omega_i(t_n)} uv \, dx.$$

This scalar product is uniformly (w.r.t. n and N) equivalent to the standard scalar product in $L^2(\Omega)$. Note that we use a weighting with β in this scalar product, which is not reflected in the notation.

In the Nitsche bilinear form introduced below we also use another weighted L^2 scalar product. Related to this we note that the surface measure both on $\Gamma(t) \subset \mathbb{R}^d$ and on $\Gamma_* \subset \mathbb{R}^{d+1}$ play a role. Both measures are denoted by ds. The following transformation formula holds:

$$\int_0^T \int_{\Gamma(t)} f(s,t) \, ds \, dt = \int_{\Gamma_*} f(s) \big(1 + (\mathbf{w} \cdot \mathbf{n}_\Gamma)^2\big)^{-\frac{1}{2}} \, ds =: \int_{\Gamma_*} f(s) \nu(s) \, ds,$$

with $\nu(s) = \left(1 + (\mathbf{w}\cdot\mathbf{n}_\Gamma)^2\right)^{-\frac{1}{2}}$. We assume that the space-time interface is sufficiently smooth such that there is a constant $c_0 > 0$ with

$$c_0 \le \nu(s) \le 1 \quad \text{for all} \quad s \in \Gamma_*.$$

Below in the Nitsche bilinear form we use a weighting with ν.

The notation introduced above is used to define a bilinear form $B(\cdot,\cdot)$, which consists of three parts, namely a term $a(\cdot,\cdot)$ that directly corresponds to the partial differential equation, a term $d(\cdot,\cdot)$ which weakly enforces continuity with respect to t at the time interval end points t_k, and a term $N_{\Gamma_*}(\cdot,\cdot)$ which enforces in a weak sense the Henry condition $[\beta u]_{\Gamma_*} = 0$. These terms are defined per time slab Q^n, i.e., $a(\cdot,\cdot)$ is of the form $a(u,v) = \sum_{n=1}^N a^n(u,v)$ and similarly for the other two terms. We now define the bilinear forms corresponding to each time slab Q^n.

The bilinear form $a^n(\cdot,\cdot)$, $1 \le n \le N$, is given by

$$a^n(u,v) = \sum_{i=1}^{2} \int_{Q_i^n} \left(\frac{\partial u_i}{\partial t} + \mathbf{w}\cdot\nabla u_i\right)\beta_i v_i + \alpha_i\beta_i \nabla u_i \cdot \nabla v_i \, dx\, dt.$$

The bilinear form $d^n(\cdot,\cdot)$, $1 \le n \le N$, is given by

$$d^n(u,v) = ([u]^{n-1}, v_+^{n-1})_{0,t_{n-1}}.$$

The bilinear form $N_{\Gamma_*}^n(\cdot,\cdot)$, $1 \le n \le N$, is as follows:

$$\begin{aligned} N_{\Gamma_*}^n(u,v) = &- \int_{\Gamma_*^n} \nu\{\alpha\nabla u\cdot\mathbf{n}\}_{\Gamma_*}[\beta v]_{\Gamma_*}\, ds - \int_{\Gamma_*^n} \nu\{\alpha\nabla v\cdot\mathbf{n}\}_{\Gamma_*}[\beta u]_{\Gamma_*}\, ds \\ &+ \lambda h_n^{-1} \int_{\Gamma_*^n} \nu[\beta u]_{\Gamma_*}[\beta v]_{\Gamma_*}\, ds, \end{aligned}$$

with a parameter $\lambda \ge 0$. Finally we introduce a right-hand side functional given by

$$\begin{aligned} f^1(v) &= (u_0, v_+^0)_{0,t_0} + \int_{Q^1} f\beta\, v\, dx\, dt, \\ f^n(v) &= \int_{Q^n} f\beta\, v\, dx\, dt, \quad 2 \le n \le N, \end{aligned}$$

with u_0 the initial condition from (1.4) and f the source term in (1.1). Corresponding global (bi)linear forms are obtained by summing over the time slabs:

$$\begin{aligned} a(u,v) &= \sum_{n=1}^N a^n(u,v), \quad d(u,v) = \sum_{n=1}^N d^n(u,v), \\ N_{\Gamma_*}(u,v) &= \sum_{n=1}^N N_{\Gamma_*}^n(u,v), \quad f(v) = \sum_{n=1}^N f^n(v). \end{aligned}$$

These bilinear forms and the functional f are well-defined on the space-time XFEM space W^{Γ_*}. The Nitsche XFEM-DG discretization is defined as follows. Determine $U \in W^{\Gamma_*}$ such that

$$\begin{aligned} &B(U,V) = f(V) \quad \text{for all} \quad V \in W^{\Gamma_*}, \\ &B(U,V) := a(U,V) + d(U,V) + N_{\Gamma_*}(U,V). \end{aligned} \tag{2.5}$$

Note that this formulation still allows us to solve the space-time problem time slab by time slab.

Remark 2.2. We comment on a generalization to a higher order method. On Q, instead of the P_1-P_1 space W as in (2.1), a higher order space-time finite element space can be defined in an obvious manner; cf. [22]. A corresponding higher order XFEM space is then defined as in (2.2) and the higher order discretization is obtained by the variational problem (2.5) with W^{Γ_*} replaced by this higher order XFEM space. We conclude that the method (2.5) has a straightforward generalization to a higher order method. From an implementation point of view there is an important difference between the P_1-P_1 method introduced above and a higher order method. In order to benefit from the higher order accuracy, one needs sufficiently accurate quadrature rules. For the case with an evolving interface such accurate approximations of the space-time integrals are difficult to realize.

3. Error analysis. In this section we present an error analysis of the Nitsche XFEM-DG discretization (2.5). In the subsections below we treat the following four topics: 1. consistency, i.e., a solution of the problem (1.1)–(1.5) satisfies the variational equation in (2.5); 2. stability of the bilinear form $B(\cdot,\cdot)$; 3. interpolation error bounds in the space W^{Γ_*}; 4. discretization error bounds. In the next section we first collect some results that are used in the analysis.

3.1. Preliminaries. Clearly, for the consistency analysis we have to substitute a suitable (weak) solution of the problem (1.1)–(1.5) into the variational formulation. In this paper we do not study a weak formulation of the problem (1.1)–(1.5); cf. [10] for an analysis of this problem. We only assume that a (weak) solution of this problem exists that has certain smoothness properties such that substitution into the bilinear form $B(\cdot,\cdot)$ is well-defined. For describing these smoothness properties it is natural to use anisotropic Sobolev spaces (also called t-anisotropic Sobolev spaces), which are known from the literature on the weak formulation of parabolic problems; cf. [16, 24]. Note that we do not use the (more standard) Bochner space $L^2(0,T;H_0^1(\Omega))$ since we have time-dependent subdomains $\Omega_i(t)$. We need several Sobolev spaces that we now introduce. By $H^2(Q)$ and $H^2(Q_1\cup Q_2)$ we denote the standard Sobolev spaces, i.e., for functions from these spaces all second order derivatives (w.r.t. space and time) exist in the usual weak sense. The corresponding subspaces of all functions that have zero values on the lateral boundary $\partial\Omega\times(0,T)$ of Q are denoted by $H_0^2(Q)$ and $H_0^2(Q_1\cup Q_2)$. We also need anisotropic Sobolev spaces. Let $D^\alpha u$, $\alpha=(\alpha_1,\ldots,\alpha_d)$ denote the usual Sobolev weak derivative w.r.t. to the *spatial* variables in the domain $Q_1^n\cup Q_2^n$ and $D_t u$ the weak derivative w.r.t. $t\in(0,T)$. Let m,k be 0 or 1. The space

$$\left\{\, u \mid D^\alpha u, D_t^r u \in L^2(Q_1^n\cup Q_2^n) \quad \text{for all} \quad |\alpha|\le m,\ 0\le r\le k \,\right\}$$

endowed with the norm

$$\left(\sum_{i=1}^2\int_{Q_i^n}\sum_{|\alpha|\le m}(D^\alpha u)^2+\sum_{r\le k}(D_t^r u)^2\,dx\,dt\right)^{\frac12}$$

is denoted by $H^{m,k}(Q_1^n\cup Q_2^n)$. We write $H^1(Q_1^n\cup Q_2^n):=H^{1,1}(Q_1^n\cup Q_2^n)$. The subspace of $H^{1,0}(Q_1^n\cup Q_2^n)$ consisting of all functions that are zero on the lateral boundary $\partial_l Q^n:=\partial\Omega\times I_n$ (in the trace sense) is denoted by $H_0^{1,0}(Q_1^n\cup Q_2^n)$. The spaces $H^{m,k}(Q_1\cup Q_2)$, $H^1(Q_1\cup Q_2)$, and $H_0^{1,0}(Q_1\cup Q_2)$ are defined in a similar way. Furthermore, the subspace of $H_0^1(Q_1^n\cup Q_2^n)=H_0^{1,1}(Q_1^n\cup Q_2^n)$ consisting of all functions

u with $D^\alpha u \in L^2(Q_1^n \cup Q_2^n)$ for all $|\alpha| \leq 2$ is denoted by $H_0^{2,1}(Q_1^n \cup Q_2^n)$. The subspace $H_0^{2,1}(Q_1 \cup Q_2)$ of $H_0^1(Q_1 \cup Q_2)$ is defined in a similar way. Under (mild) regularity assumptions on the interface Γ_* functions $u \in H_0^{2,1}(Q_1^n \cup Q_2^n)$ have well-defined traces $u_{|\Gamma_*^n}$ and $(\mathbf{n}\cdot\nabla u)_{|\Gamma_*^n}$ in $L^2(\Gamma_*^n)$; cf. [16, 24]. Since the space $H_0^{2,1}(Q_1^n \cup Q_2^n)$ is frequently used below we introduce the compact notation:

$$W_{\mathrm{reg}}^n := H_0^{2,1}(Q_1^n \cup Q_2^n), \quad W_{\mathrm{reg}} := \{\, v : Q \to \mathbb{R} \mid v_{|Q^n} \in W_{\mathrm{reg}}^n \,\}.$$

The bilinear forms $a^n(\cdot,\cdot)$, $d^n(\cdot,\cdot)$, $N_{\Gamma_*}^n(\cdot,\cdot)$ are well-defined on W_{reg}^n. Hence, $B(\cdot,\cdot)$ is well-defined on W_{reg}. In the analysis below, besides the XFEM space W^{Γ_*} we will also often use the space $W^{\Gamma_*} + W_{\mathrm{reg}}$.

We introduce (semi-)norms and scalar products that are used in the analysis. These are well-defined on the space $W^{\Gamma_*} + W_{\mathrm{reg}}$. Recall the definition $(u,v)_{0,t_n} := \int_\Omega \beta(\cdot,t_n) uv \, dx$. The corresponding norm is denoted by $\|\cdot\|_{0,t_n}$. On Q_i^n, Q^n, and Q we define

$$\begin{aligned} &(u,v)_{1,Q_i^n} = \int_{Q_i^n} \beta_i \nabla u \cdot \nabla v \, dx dt, \quad (u,v)_{1,Q_1^n \cup Q_2^n} = (u,v)_{1,Q_1^n} + (u,v)_{1,Q_2^n}, \\ &(u,v)_{1,Q_1 \cup Q_2} = \sum_{n=1}^N (u,v)_{1,Q_1^n \cup Q_2^n}, \end{aligned} \tag{3.1}$$

with corresponding seminorms denoted by $|\cdot|_{1,Q_i^n}$, $|\cdot|_{1,Q_1^n \cup Q_2^n}$, $|\cdot|_{1,Q_1 \cup Q_2}$. Note that in these scalar products and seminorms there is a scaling with the piecewise constant function β and that opposite to the standard norm $|\cdot|_1$ on $H^1(Q)$ there is no first derivative w.r.t. time in (3.1).

In the analysis in the following sections we need space-time trace operators and a space-time Poincaré–Friedrichs inequality. Under mild smoothness conditions on Γ_* the existence of a bounded trace operator $H^{1,0}(Q_i) \to L^2(\Gamma_*)$ follows from [16]. In the remainder we assume that there exists such a trace operator that is bounded:

$$\|u_i\|_{0,\Gamma_*} \leq c\big(|u|_{1,Q_1 \cup Q_2} + \|u\|_{0,Q}\big) \quad \text{for all} \;\; u \in H_0^{1,0}(Q_1 \cup Q_2), \tag{3.2}$$

with $u_i = u_{|Q_i}$. Furthermore, from standard results it follows that there is a trace operator $H^1(Q_1 \cup Q_2) \to L^2(\Omega)$ that is bounded:

$$\|u(\cdot,t_j)\|_{0,\Omega} \leq c\|u\|_{1,Q_1 \cup Q_2} \quad \text{for all} \;\; u \in H^1(Q_1 \cup Q_2), \; j = 0, N. \tag{3.3}$$

With respect to the Poincaré–Friedrichs inequality we first consider a fixed $t \in [0,T]$. From the Petree–Tartar theorem it follows (cf. Lemma B.63 in [7]) that there exists a constant $c(t)$ such that

$$\|u\|_{0,\Omega} \leq c(t)\big(|u|_{1,\Omega_1 \cup \Omega_2} + \|[\beta u]\|_{0,\Gamma(t)}\big) \quad \text{for all} \;\; u \in H_0^1(\Omega_1(t) \cup \Omega_2(t)) \tag{3.4}$$

holds. In the remainder we assume that there exists a constant c_P such that

$$\|u\|_{0,Q} \leq c_P\big(|u|_{1,Q_1 \cup Q_2} + \|[\beta u]\|_{0,\Gamma_*}\big) \quad \text{for all} \;\; u \in H_0^{1,0}(Q_1 \cup Q_2) \tag{3.5}$$

holds. Note that this follows from the result in (3.4) if $c(t)$ is uniformly bounded with respect to $t \in [0,T]$. We expect that such a uniform boundedness result holds if the space-time interface Γ_* is sufficiently smooth.

LEMMA 3.1. *For $u, v \in W^{\Gamma_*} + W_{\mathrm{reg}}$ the following relations hold:*

$$a^n(u,v) = -a^n(v,u) + (u^n, v^n)_{0,t_n} - (u_+^{n-1}, v_+^{n-1})_{0,t_{n-1}} + 2(\alpha u, v)_{1,Q_1^n \cup Q_2^n}, \tag{3.6}$$

$$a^n(v,v) = \frac{1}{2}\|v^n\|_{0,t_n}^2 - \frac{1}{2}\|v_+^{n-1}\|_{0,t_{n-1}}^2 + |\sqrt{\alpha}\, v|_{1,Q_1^n \cup Q_2^n}^2. \tag{3.7}$$

Proof. From the definition of the bilinear form $a^n(u,v)$ we get

$$\begin{aligned} & a^n(u,v) + a^n(v,u) - 2(\alpha u, v)_{1,Q_1^n \cup Q_2^n} \\ & = \sum_{i=1}^{2} \left\{ \int_{Q_i^n} \left(\frac{\partial u_i}{\partial t} + \mathbf{w} \cdot \nabla u_i \right) \beta_i v_i \, dx \, dt + \int_{Q_i^n} \left(\frac{\partial v_i}{\partial t} + \mathbf{w} \cdot \nabla v_i \right) \beta_i u_i \, dx \, dt \right\}. \end{aligned} \tag{3.8}$$

The normal velocity of the interface $\Gamma(t)$ is given by $V_\Gamma = \mathbf{w} \cdot \mathbf{n}$. From this it follows that the unit normal on Γ_* in $\mathbb{R}^{d+1}$ is given by

$$\hat{\mathbf{n}} := \nu \begin{pmatrix} \mathbf{n} \\ -\mathbf{w} \cdot \mathbf{n} \end{pmatrix} =: \begin{pmatrix} \hat{\mathbf{n}}_x \\ \hat{\mathbf{n}}_t \end{pmatrix}.$$

Note that $\hat{\mathbf{n}}_t + \mathbf{w} \cdot \hat{\mathbf{n}}_x = 0$ holds. We apply partial integration to the first term on the right-hand side in (3.8). Take $i = 1$. We assumed $\partial\Omega_1(t) \cap \partial\Omega = \emptyset$ and thus the boundary of Q_1^n can be decomposed as $\Omega_1(t_{n-1}) \cup \Gamma_*^n \cup \Omega_1(t_n)$. Hence, from partial integration we get

$$\begin{aligned} \int_{Q_1^n} \left(\frac{\partial u_1}{\partial t} + \mathbf{w} \cdot \nabla u_1 \right) \beta_1 v_1 \, dx \, dt &= \int_{\Omega_1(t_n)} \beta_1 u_1^n v_1^n \, dx \\ & - \int_{\Omega_1(t_{n-1})} \beta_1 (u_1)_+^{n-1} (v_1)_+^{n-1} \, dx \\ & + \int_{\Gamma_*^n} \beta_1 u_1 v_1 (\hat{\mathbf{n}}_t + \mathbf{w} \cdot \hat{\mathbf{n}}_x) \, ds - \int_{Q_1^n} \left(\frac{\partial v_1}{\partial t} + \mathbf{w} \cdot \nabla v_1 \right) \beta_1 u_1 \, dx \, dt \\ & = \int_{\Omega_1(t_n)} \beta_1 u_1^n v_1^n \, dx - \int_{\Omega_1(t_{n-1})} \beta_1 (u_1)_+^{n-1} (v_1)_+^{n-1} \, dx \\ & - \int_{Q_1^n} \left(\frac{\partial v_1}{\partial t} + \mathbf{w} \cdot \nabla v_1 \right) \beta_1 u_1 \, dx \, dt. \end{aligned}$$

A similar result is obtained for $i = 2$ (we use that v_2 and u_2 are zero on $\partial\Omega$). This yields

$$\begin{aligned} & \sum_{i=1}^{2} \int_{Q_i^n} \left(\frac{\partial u_i}{\partial t} + \mathbf{w} \cdot \nabla u_i \right) \beta_i v_i \, dx \, dt \\ & = \int_\Omega \beta(\cdot, t_n) u^n v^n \, dx - \int_\Omega \beta(\cdot, t_{n-1}) u_+^{n-1} v_+^{n-1} \, dx \\ & - \sum_{i=1}^{2} \int_{Q_i^n} \left(\frac{\partial v_i}{\partial t} + \mathbf{w} \cdot \nabla v_i \right) \beta_i u_i \, dx \, dt \\ & = (u^n, v^n)_{0,t_n} - (u_+^{n-1}, v_+^{n-1})_{0,t_{n-1}} - \sum_{i=1}^{2} \int_{Q_i^n} \left(\frac{\partial v_i}{\partial t} + \mathbf{w} \cdot \nabla v_i \right) \beta_i u_i \, dx \, dt. \end{aligned}$$

Combining this with the result in (3.8) proves (3.6). The result in (3.7) is a direct consequence of the one in (3.6). ∎

LEMMA 3.2. *For $v \in W^{\Gamma_*} + W_{\mathrm{reg}}$ the following holds:*

$$d^n(v,v) = \frac{1}{2}\|[v]^{n-1}\|^2_{0,t_{n-1}} + \frac{1}{2}\|v_+^{n-1}\|^2_{0,t_{n-1}} - \frac{1}{2}\|v^{n-1}\|^2_{0,t_{n-1}}, \quad 1 \le n \le N,$$

with $[v]^0 = v_+^0$.

Proof. Using $[v]^{n-1} = v_+^{n-1} - v^{n-1}$ we get

$$\begin{aligned} d^n(v,v) &= ([v]^{n-1}, v_+^{n-1})_{0,t_{n-1}} \\ &= \frac{1}{2}([v]^{n-1},[v]^{n-1})_{0,t_{n-1}} + \frac{1}{2}([v]^{n-1}, v_+^{n-1} + v^{n-1})_{0,t_{n-1}} \\ &= \frac{1}{2}\|[v]^{n-1}\|^2_{0,t_{n-1}} + \frac{1}{2}\|v_+^{n-1}\|^2_{0,t_{n-1}} - \frac{1}{2}\|v^{n-1}\|^2_{0,t_{n-1}}, \end{aligned}$$

and thus the result holds. ∎

LEMMA 3.3. *For $u, v \in W^{\Gamma_*} + W_{\mathrm{reg}}$, with $[v]^{n-1} = 0$ for all $2 \le n \le N$, the following holds:*

$$a(u,v) + d(u,v) = -a(v,u) + 2(\alpha u, v)_{1,Q_1\cup Q_2} + (u^N, v^N)_{0,t_N}.$$

Proof. From (3.6) it follows that

$$a(u,v) = -a(v,u) + 2(\alpha u, v)_{1,Q_1\cup Q_2} + \sum_{n=1}^{N}\left[(u^n,v^n)_{0,t_n} - (u_+^{n-1}, v_+^{n-1})_{0,t_{n-1}}\right]. \tag{3.9}$$

We consider the last term in (3.9). Using $[v]^{n-1} = 0$ and with $u^0 := 0$ we get

$$\begin{aligned} &\sum_{n=1}^{N}(u^n,v^n)_{0,t_n} - \sum_{n=1}^{N}(u_+^{n-1},v_+^{n-1})_{0,t_{n-1}} \\ &= (u^N,v^N)_{0,t_N} + \sum_{n=1}^{N}(u^{n-1},v_+^{n-1})_{0,t_{n-1}} - \sum_{n=1}^{N}(u_+^{n-1},v_+^{n-1})_{0,t_{n-1}} \\ &= (u^N,v^N)_{0,t_N} - \sum_{n=1}^{N}([u]^{n-1},v_+^{n-1})_{0,t_{n-1}} = (u^N,v^N)_{0,t_N} - d(u,v). \end{aligned}$$

Combining this with the result in (3.9) proves the claim. ∎

3.2. Consistency analysis. We prove consistency of the variational problem in (2.5).

THEOREM 3.4. *Let u be a solution of* (1.1)–(1.5), *and assume that $u \in H_0^{2,1}(Q_1 \cup Q_2)$. Then*

$$B(u,V) = f(V) \quad \textit{for all} \ \ V \in W^{\Gamma_*}$$

holds.

Proof. Take $V \in W^{\Gamma_*}$. From $u \in H_0^{2,1}(Q_1\cup Q_2)$ it follows that $u \in W_{\mathrm{reg}}$ and $[u]^{n-1} = 0$ for $2 \le n \le N$. Hence,

$$d(u,V) = \sum_{n=1}^{N} d^n(u,V) = d^1(u,V) = (u(\cdot,0), V_+^0)_{0,t_0} = (u_0, V_+^0)_{0,t_0}. \tag{3.10}$$

For the Nitsche bilinear form we obtain, using $[\beta u]_{\Gamma_*} = 0$,

$$N_{\Gamma_*}(u,V) = \sum_{n=1}^{N} N_{\Gamma_*}^n(u,V) = -\int_{\Gamma_*} \nu\{\alpha\nabla u\cdot\mathbf{n}\}_{\Gamma_*}[\beta V]_{\Gamma_*}\,ds. \tag{3.11}$$

For the bilinear form $a^n(\cdot,\cdot)$ we get

$$\begin{aligned} a^n(u,V) &= \sum_{i=1}^{2}\int_{Q_i^n}\left(\frac{\partial u_i}{\partial t} + \mathbf{w}\cdot\nabla u_i\right)\beta_i V_i + \alpha_i\beta_i\nabla u_i\cdot\nabla V_i\,dx\,dt \\ &= \sum_{i=1}^{2}\int_{Q_i^n}\left(\frac{\partial u_i}{\partial t} + \mathbf{w}\cdot\nabla u_i - \alpha_i\Delta u_i\right)\beta_i V_i\,dx\,dt \\ &\quad + \int_{\Gamma_*^n}[\alpha\nabla u\cdot\hat{\mathbf{n}}_x\,\beta V]_{\Gamma_*}\,ds, \end{aligned} \tag{3.12}$$

with $\hat{\mathbf{n}}_x = \nu\mathbf{n}$. Using $[fg]_t = \{f\}_t[g]_t + [f]_t\{g\}_t - [\kappa]_t[f]_t[g]_t$ and the property $[\alpha\nabla u\cdot\mathbf{n}]_t = 0$ for all t (cf. (1.2)), we obtain

$$\int_{\Gamma_*^n}[\alpha\nabla u\cdot\hat{\mathbf{n}}_x\,\beta V]_{\Gamma_*}\,ds = \int_{\Gamma_*^n}\nu\{\alpha\nabla u\cdot\mathbf{n}\}_{\Gamma_*}[\beta V]_{\Gamma_*}\,ds, \tag{3.13}$$

and thus, using the differential equation for u in (1.1) it follows that

$$a(u,V) = \sum_{n=1}^{N} a^n(u,V) = \int_Q f\,\beta V\,dx + \int_{\Gamma_*}\nu\{\alpha\nabla u\cdot\mathbf{n}\}_{\Gamma_*}[\beta V]_{\Gamma_*}\,ds \tag{3.14}$$

holds. From this, the results in (3.10), (3.11), and the definition of the functional $f(\cdot)$, we obtain

$$\begin{aligned} B(u,V) &= a(u,V) + d(u,V) + N_{\Gamma_*}(u,V) \\ &= \int_Q f\,\beta V\,dx + (u_0, V_+^0)_{0,t_0} = f(V), \end{aligned}$$

which proves the consistency result. ∎

3.3. Stability analysis. In this section we study the stability of the bilinear form $B(\cdot,\cdot)$. We summarize *the main assumptions that are used in the analysis* in this and the following sections.

Assumption 3.1. The coefficients α_i, β_i, and $\mathbf{w}$ in the problem should be such that we have a diffusion dominated problem and that there are no singular perturbation effects caused by the Henry coefficient β. More precisely, we assume

$$0 < c_L \leq \alpha_i, \beta_i \quad \text{and} \quad \alpha_i, \beta_i, \|\mathbf{w}\|_{L^\infty(\Omega)} \leq c_U, \quad i = 1, 2, \tag{3.15}$$

with constants c_L and c_U that are of order one. The trace inequality (3.2) and the Poincaré–Friedrichs inequality (3.5) are assumed to hold. Furthermore, the shape regular triangulations $\mathcal{T}_n$ are assumed to be quasi-uniform and all constants related to shape-regularity and quasi-uniformity are assumed to be uniformly bounded both with respect to the spatial mesh size h_n and with respect to Δt (i.e., with respect to N). We assume that the space-time interface Γ_* is a smooth d-manifold and the space-time triangulation is sufficiently fine such that it can resolve Γ_*. More precisely, for each $\omega = T \times I_n$, with $T \in \mathcal{T}_n$ and $\Gamma_\omega := \omega \cap \Gamma_* \neq \emptyset$, we assume that there is a local orthogonal coordinate system $y = (z, \theta)$, $z \in \mathbb{R}^d$, $\theta \in \mathbb{R}$ such that Γ_ω is the graph of a smooth scalar function, say g, i.e., $\Gamma_\omega = \{\, (z, g(z)) \mid z \in U \subset \mathbb{R}^d \,\}$, with $\|\nabla g\|$ uniformly bounded.

First we consider the part $a(\cdot,\cdot) + d(\cdot,\cdot)$.

LEMMA 3.5. *For $v \in W^{\Gamma_*} + W_{\rm reg}$ the following holds (with $[v]^0 = v_+^0$):*

$$a(v,v) + d(v,v) = |\sqrt{\alpha}\, v|^2_{1,Q_1\cup Q_2} + \frac{1}{2}\|v^N\|^2_{0,t_N} + \frac{1}{2}\sum_{n=1}^N \|[v]^{n-1}\|^2_{0,t_{n-1}}.$$

Proof. We recall the results in Lemmas 3.1 and 3.2:

$$\begin{aligned} a^n(v,v) &= \frac{1}{2}\|v^n\|^2_{0,t_n} - \frac{1}{2}\|v_+^{n-1}\|^2_{0,t_{n-1}} + |\sqrt{\alpha}\, v|^2_{1,Q_1^n\cup Q_2^n}, \\ d^n(v,v) &= \frac{1}{2}\|[v]^{n-1}\|^2_{0,t_{n-1}} + \frac{1}{2}\|v_+^{n-1}\|^2_{0,t_{n-1}} - \frac{1}{2}\|v^{n-1}\|^2_{0,t_{n-1}}, \end{aligned}$$

with $v^0 = 0$. Summing these results over $n = 1, \ldots, N$ we obtain the result. ∎

For the analysis of the Nitsche bilinear form we introduce space-time variants of the norms used for the stationary case in [11]:

$$\begin{aligned} \|v\|^2_{\frac{1}{2},h,\Gamma_*^n} &= h_n^{-1}\int_{\Gamma_*^n} \nu v^2\, ds = h_n^{-1}\int_{t_{n-1}}^{t_n} \|v(.,t)\|^2_{L^2(\Gamma(t))}\, dt, \\ \|v\|^2_{-\frac{1}{2},h,\Gamma_*^n} &= h_n \int_{\Gamma_*^n} \nu v^2\, ds = h_n^2 \|v\|^2_{\frac{1}{2},h,\Gamma_*^n}, \\ \|v\|^2_{\frac{1}{2},h,\Gamma_*} &= \sum_{n=1}^N \|v\|^2_{\frac{1}{2},h,\Gamma_*^n}, \quad \|v\|^2_{-\frac{1}{2},h,\Gamma_*} = \sum_{n=1}^N \|v\|^2_{-\frac{1}{2},h,\Gamma_*^n}. \end{aligned}$$

Note that $\int_{\Gamma_*^n} \nu uv\, ds \leq \|v\|_{\frac{1}{2},h,\Gamma_*^n}\|v\|_{-\frac{1}{2},h,\Gamma_*^n}$ holds. The analysis of the Nitsche bilinear form uses essentially the same arguments as in [11] for the stationary case.

LEMMA 3.6. *There is a constant c, independent of h_n and Δt, such that for all $V \in W^{\Gamma_*}$ the following holds:*

$$\|\{\alpha\nabla V\cdot\mathbf{n}\}_{\Gamma_*}\|_{-\frac{1}{2},h,\Gamma_*^n} \leq c|V|_{1,Q_1^n\cup Q_2^n}, \tag{3.16}$$

$$\|\{\alpha\nabla V\cdot\mathbf{n}\}_{\Gamma_*}\|_{-\frac{1}{2},h,\Gamma_*} \leq c|V|_{1,Q_1\cup Q_2}. \tag{3.17}$$

Proof. Take $1 \leq n \leq N$. For $V \in W^{\Gamma_*}$ we have

$$\begin{aligned} \|\{\alpha\nabla V\cdot\mathbf{n}\}_{\Gamma_*}\|^2_{-\frac{1}{2},h,\Gamma_*^n} &= \int_{t_{n-1}}^{t_n}\int_{\Gamma(t)} h_n\{\alpha\nabla V\cdot\mathbf{n}\}_t^2\, ds\, dt \\ &\leq \int_{t_{n-1}}^{t_n}\sum_{i=1}^2\int_{\Gamma(t)} 2h_n\kappa_i(t)^2\alpha_i^2\|\nabla V_i\|^2\, ds\, dt, \end{aligned}$$

with $V_i = R_i V$. The finite element function V_i is of the form $V_i(x,t) = \phi_0(x) + t\phi_1(x)$, $x \in \Omega_i(t)$, $t \in I_n$, with $\phi_0, \phi_1 \in V_n$ (space of piecewise linears). Recall that $\kappa_i(t)_{|T} = \frac{|T_i|}{|T|}$ with $T_i = \Omega_i(t) \cap T$. The interface $\Gamma(t)$ can be decomposed as $\Gamma(t) = \cup_{T\in\mathcal{T}_{\Gamma(t)}}\Gamma_T(t)$, with $\Gamma_T(t) = \Gamma(t)\cap T$, and $\mathcal{T}_{\Gamma(t)}$ a subset of simplices of $\mathcal{T}_n$

which have a nonzero intersection with $\Gamma(t)$. Note that due to the dynamics of the interface the set of "interface simplices" $\mathcal{T}_{\Gamma(t)}$ (i.e., those that contain the interface) depends on t. Consider $T \in \mathcal{T}_{\Gamma(t)}$. The functions ϕ_0, ϕ_1 are linear on T and thus $(\nabla V_i)_{|T_i} = \hat{v}_0 + t\hat{v}_1$ for certain vectors $\hat{v}_0$, $\hat{v}_1 \in \mathbb{R}^d$. Thus we have

$$\begin{aligned}\int_{\Gamma_T(t)} 2h_n\kappa_i(t)^2\alpha_i^2\|\nabla V_i\|^2\,ds &= 2h_n\frac{|T_i|^2}{|T|^2}\alpha_i^2\|\hat{v}_0 + t\hat{v}_1\|^2 \int_{\Gamma_T(t)} 1\,ds \\ &= 2h_n\frac{|T_i||\Gamma_T(t)|}{|T|^2}\frac{\alpha_i^2}{\beta_i}\int_{T_i}\beta_i\|\nabla V_i\|^2\,dx \\ &\leq c\int_{T_i}\beta_i\|\nabla V_i\|^2\,dx,\end{aligned}$$

where the constant c is independent of t, Δt, and h_n. In the last inequality we used that the interface is sufficiently resolved by the triangulation (cf. Assumption 3.1) and thus $h_n|\Gamma_T(t)| \leq c|T|$ holds. Summing over all $T \in \mathcal{T}_{\Gamma(t)}$ we obtain

$$\int_{\Gamma(t)} 2h_n\kappa_i(t)^2\alpha_i^2\|\nabla V_i\|^2\,ds \leq c\sum_{T\in\mathcal{T}_{\Gamma(t)}}\int_{T_i}\beta_i\|\nabla V_i\|^2\,dx \leq c\int_{\Omega_i(t)}\beta_i\|\nabla V_i\|^2\,dx,$$

and thus

$$\|\{\alpha\nabla V\cdot\mathbf{n}\}_{\Gamma_*}\|^2_{-\frac{1}{2},h,\Gamma_*^n} \leq c\int_{t_{n-1}}^{t_n}\sum_{i=1}^{2}\int_{\Omega_i(t)}\beta_i\|\nabla V_i\|^2\,dx\,dt = c|V|^2_{1,Q_1^n\cup Q_2^n},$$

which proves the result in (3.16). The result in (3.17) follows after summation over n. ∎

LEMMA 3.7. *There exists a constant $c_0 > 0$, independent of h_n and Δt, such that for all $\delta > 0$*

$$N_{\Gamma_*}(V,V) \geq -\delta c_0|V|^2_{1,Q_1\cup Q_2} + (\lambda - \delta^{-1})\|[\beta V]\|^2_{\frac{1}{2},h,\Gamma_*} \quad \textit{for all} \;\; V \in W^{\Gamma_*}$$

holds.

Proof. From the definition of $N_{\Gamma_*}^n$ and the Cauchy–Schwarz inequality, we get

$$N_{\Gamma_*}^n(V,V) \geq -2\|\{\alpha\nabla V\cdot\mathbf{n}\}_{\Gamma_*}\|_{-\frac{1}{2},h,\Gamma_*^n}\|[\beta V]_{\Gamma_*}\|_{\frac{1}{2},h,\Gamma_*^n} + \lambda\|[\beta V]_{\Gamma_*}\|^2_{\frac{1}{2},h,\Gamma_*^n}. \tag{3.18}$$

For the term $\|\{\alpha\nabla V\cdot\mathbf{n}\}_{\Gamma_*}\|_{-\frac{1}{2},h,\Gamma_*^n}$ we use Lemma 3.6 and obtain, with arbitrary $\delta > 0$,

$$2\|\{\alpha\nabla V\cdot\mathbf{n}\}_{\Gamma_*}\|_{-\frac{1}{2},h,\Gamma_*^n}\|[\beta V]_{\Gamma_*}\|_{\frac{1}{2},h,\Gamma_*^n} \leq c\delta|V|^2_{1,Q_1^n\cup Q_2^n} + \delta^{-1}\|[\beta V]_{\Gamma_*}\|^2_{\frac{1}{2},h,\Gamma_*^n},$$

and thus with (3.18) we have

$$N_{\Gamma_*^n}(V,V) \geq -\delta c|V|^2_{1,Q_1^n\cup Q_2^n} + (\lambda - \delta^{-1})\|[\beta V]\|^2_{\frac{1}{2},h,\Gamma_*^n}.$$

Summation over $n = 1,\ldots,N$ completes the proof. ∎

THEOREM 3.8. *Define $c_B := \frac{1}{2}\min\{\alpha_1,\alpha_2,1\}$. There exists a constant $c^* > 0$ independent of h_n, Δt, and λ such that for all $\lambda > c^*$ the following holds:*

$$B(V,V) \geq c_B\left(|V|^2_{1,Q_1\cup Q_2} + \|V^N\|^2_{0,t_N} + \sum_{n=1}^{N}\|[V]^{n-1}\|^2_{0,t_{n-1}} + \lambda\|[\beta V]\|^2_{\frac{1}{2},h,\Gamma_*}\right)$$

for all $V \in W^{\Gamma_*}$.

Proof. Take $V \in W^{\Gamma_*}$. From Lemma 3.5 we obtain, with $\alpha_{\min} = \min\{\alpha_1, \alpha_2\}$,

$$a(V,V) + d(V,V) \geq \alpha_{\min}|V|^2_{1,Q_1\cup Q_2} + \frac{1}{2}\|V^N\|^2_{0,t_N} + \frac{1}{2}\sum_{n=1}^{N}\|[V]^{n-1}\|^2_{0,t_{n-1}}. \tag{3.19}$$

In Lemma 3.7 we take $\delta = \frac{1}{2}\alpha_{\min}c_0^{-1}$ and $\lambda \geq 4\alpha_{\min}^{-1}c_0 =: c^*$. Hence,

$$N_{\Gamma_*}(V,V) \geq -\frac{1}{2}\alpha_{\min}|V|^2_{1,Q_1\cup Q_2} + \frac{1}{2}\lambda\|[\beta V]\|^2_{\frac{1}{2},h,\Gamma_*}.$$

Combining this with (3.19) completes the proof. ∎

Note that in the stability result in Theorem 3.8 there are both jumps between time slabs (from DG) and jumps across the interface (Nitsche technique) that occur in the lower bound.

COROLLARY 3.9. Take c^* as in Theorem 3.8 and $\lambda > c^*$. Then the Nitsche XFEM-DG discretization (2.5) has a unique solution $U \in W^{\Gamma_*}$.

3.4. Interpolation error bounds. A nice property of the Nitsche XFEM-DG method introduced above is that it allows us to use *non*matching simplicial triangulations $\mathcal{T}_n$ in the time slabs I_n, $n = 1,\dots,N$, i.e., $\mathcal{T}_n \neq \mathcal{T}_m$ for $n \neq m$. For this general setting, consistency and stability results have been derived in the subsections above. For the discretization error analysis we also need suitable interpolation error bounds. This analysis is greatly simplified if we assume *matching* grids. Hence, since in this paper the emphasis is on the Nitsche XFEM space-time discretization, we restrict ourselves to the matching grid case, i.e., in addition to Assumption 3.1 we introduce the following assumption.

Assumption 3.2. We assume that the spatial simplicial triangulation does not depend on n, i.e., we have a family of spatial triangulations $\{\mathcal{T}_h\}_{h>0}$ of Ω and assume $\mathcal{T}_n = \mathcal{T} \in \{\mathcal{T}_h\}$ for all n.

In the analysis presented below, we assume that this assumption holds. Each triangulation $\mathcal{T}$ has a corresponding mesh size parameter $h := \max\{\,\mathrm{diam}(T) \mid T \in \mathcal{T}\,\}$. Recall that we assume the family $\{\mathcal{T}_h\}_{h>0}$ to be *quasi-uniform*.

We now introduce an interpolation operator. For the case $\mathcal{T}_n = \mathcal{T}$ the space-time finite element space as in (2.1) simplifies to

$$W = \{\, v : Q \to \mathbb{R} \mid v(x,t) = \phi_0(x) + t\phi_1(x) \quad \text{on } \ Q^n, \quad \phi_0, \phi_1 \in V_h\},$$

with V_h the space of continuous piecewise linears on $\mathcal{T}$ with zero boundary values on $\partial\Omega$. Below an index 0 used in the notation of spaces (e.g., $C_0(Q)$) always denotes that the functions in this space have zero values on lateral boundary $\partial\Omega \times I$ of Q. For $v \in C_0(Q)$ the piecewise linear nodal interpolation on $\mathcal{T}$ is denoted by $v_h(\cdot,t) \in V_h$. The corresponding space-time linear interpolation is given by $I_W : C_0(Q) \to W$:

$$(I_W v)(x,t) = \frac{t - t_{n-1}}{\Delta t} v_h(x,t_n) + \frac{t_n - t}{\Delta t} v_h(x,t_{n-1}), \quad t \in I_n,\ n = 1,\dots,N.$$

Note that $I_W v \in C_0(Q)$, and thus, in particular, $[I_W v]^n = 0$ holds for $1 \leq n \leq N-1$. Such a globally continuous interpolant would not be available if *non*matching triangulations $\mathcal{T}_n$ are used. From standard interpolation theory we have

$$\|v - I_W v\|_{\ell,Q} \leq c(h^{2-\ell} + \Delta t^{2-\ell})\|v\|_{2,Q} \quad \text{for all } \ v \in H_0^2(Q), \ \ \ell = 0,1. \tag{3.20}$$

We recall that the norm $\|\cdot\|_{1,Q}$ in (3.20) and the (semi-)norm $|\cdot|_{1,Q_1\cup Q_2}$ that corresponds to the scalar product in (3.1) are different. The former is the standard norm on the Sobolev space $H^1(Q)$, and thus contains derivatives w.r.t. x and t, whereas the latter contains only derivatives w.r.t. x. Furthermore, in the latter case these weak x derivatives are w.r.t. $Q_1\cup Q_2$ instead of Q.

In view of the interpolation error bound (3.20) it is natural to introduce the following relation between h and Δt. We assume that there are generic constants $c_2 \geq c_1 > 0$ such that $c_1 \leq \frac{h}{\Delta t} \leq c_2$ holds. This assumption is not essential for the analysis, but is used to simplify the presentation. In the remainder error bounds are formulated in terms of the mesh size parameter h.

We now introduce an interpolation operator in the space-time XFEM space $W^{\Gamma_*} = R_1 W + R_2 W$. This operator is analogous to the operator used in the analysis of the spatial XFEM; cf. [11, 10]. We assume we have linear extension operators $\mathcal{E}_i : H_0^2(Q_i) \to H_0^2(Q)$, $i=1,2$, that are bounded: $\|\mathcal{E}_i v\|_{2,Q} \leq c\|v\|_{2,Q_i}$ for all $v \in H_0^2(Q_i)$. The space-time XFEM interpolation operator $I_{\Gamma_*} : H_0^2(Q_1\cup Q_2) \to W^{\Gamma_*}$ is given by

$$I_{\Gamma_*} = I_{W^{\Gamma_*}} := R_1 I_W \mathcal{E}_1 R_1 v + R_2 I_W \mathcal{E}_2 R_2 v.$$

In the remainder of this subsection we derive error bounds for this interpolation operator. These bounds are used in the discretization error analysis in subsection 3.6.

We need some further notations. The space-time triangulation of $Q = \Omega \times [0,T]$ consists of cylindrical space-time elements ω and is given by

$$\mathcal{T}_Q := \{\, T \times I_n \mid T \in \mathcal{T},\ 1 \leq n \leq N \,\} =: \{\omega\}.$$

Let $\mathcal{T}_{\Gamma_*} \subset \mathcal{T}_Q$ be the subset of cylindrical elements that have a nonzero intersection with the space-time interface Γ_*. Hence Γ_* can be partitioned as

$$\Gamma_* = \cup_{\omega\in\mathcal{T}_{\Gamma_*}} \Gamma_\omega, \quad \text{with } \ \Gamma_\omega := \Gamma_* \cap \omega \neq \emptyset.$$

For the cylindrical elements the following trace result holds (see [4, Thm. 1.1.6]), with a constant c that depends only on the shape regularity of $\mathcal{T}$:

$$\|v\|_{0,\partial\omega}^2 \leq c(h^{-1}\|v\|_{0,\omega}^2 + h\|v\|_{1,\omega}^2) \quad \text{for all } \ v \in H^1(\omega). \tag{3.21}$$

The following lemma is a space-time variant of a result derived in [11, 12]. For completeness we include a proof.

LEMMA 3.10. *There is a constant c, depending only on the shape regularity of $\mathcal{T}$ and the smoothness of Γ_*, such that*

$$\|v\|_{0,\Gamma_\omega}^2 \leq c(h^{-1}\|v\|_{0,\omega}^2 + h\|v\|_{1,\omega}^2) \quad \text{for all } \ v \in H^1(\omega),\ \omega \in \mathcal{T}_{\Gamma_*} \tag{3.22}$$

holds.

Proof. Take $\omega \in \mathcal{T}_{\Gamma_*}$ and let $\Gamma_\omega = \{\,(z, g(z)) \mid z \in U \subset \mathbb{R}^d\,\}$ be as in Assumption 3.1. If Γ_ω coincides with one of the faces of ω, then (3.22) follows from (3.21). We consider the situation that the interface Γ_ω divides ω into two nonempty subdomains $\omega_i := \omega \cap Q_i$. Take i such that $\Gamma_\omega \subset \partial\omega_i$. Let $\mathbf{n} = (n_1, \ldots, n_{d+1})^T$ be the unit outward pointing normal on $\partial\omega_i$. For $v \in H^1(\omega)$ the following holds, where div_y denotes the divergence operator in the y-coordinate system,

$$\begin{aligned} 2\int_{\omega_i} v\frac{\partial v}{\partial \theta}\,dy &= \int_{\omega_i} \mathrm{div}_y \begin{pmatrix} 0 \\ v^2 \end{pmatrix} dy = \int_{\partial\omega_i} \mathbf{n}\cdot \begin{pmatrix} 0 \\ v^2 \end{pmatrix} ds = \int_{\partial\omega_i} n_{d+1} v^2\,ds \\ &= \int_{\Gamma_\omega} n_{d+1} v^2\,ds + \int_{\partial\omega_i\setminus\Gamma_\omega} n_{d+1} v^2\,ds. \end{aligned}$$

The normal $\mathbf{n}$ has direction $(-\nabla_z g(z), 1)^T$ and thus $n_{d+1}(y) = (\|\nabla_z g(z)\|^2 + 1)^{-\frac{1}{2}}$ holds. From smoothness assumptions it follows that there is a generic constant c, depending only on the smoothness of Γ_*, such that $1 \leq n_{d+1}(z)^{-1} \leq c$ holds. Using this we obtain

$$\begin{aligned}\int_{\Gamma_\omega} v^2\,ds \leq c\int_{\Gamma_\omega} n_{d+1}v^2\,ds &\leq c\|v\|_{0,\omega_i}\|v\|_{1,\omega_i} + c\int_{\partial\omega_i\setminus\Gamma_\omega} v^2\,ds \\ &\leq c\|v\|_{0,\omega}\|v\|_{1,\omega} + c\int_{\partial\omega} v^2\,ds \\ &\leq c(h^{-1}\|v\|_{0,\omega}^2 + h\|v\|_{1,\omega}^2),\end{aligned}$$

where in the last inequality we used (3.21). ◻

In the next theorem we derive error bounds for the space-time XFEM interpolation operator I_{Γ_*} that will be used in the discretization error analysis in section 3.6.

THEOREM 3.11. *For $u \in H_0^2(Q_1 \cup Q_2)$ define the interpolation error $e_u := u - I_{\Gamma_*}u$. There exists a constant c, independent of h, such that for all $u \in H_0^2(Q_1 \cup Q_2)$ the following holds:*

$$\|e_u\|_{\ell,Q_1\cup Q_2} \leq ch^{2-\ell}\|u\|_{2,Q_1\cup Q_2}, \quad \ell = 0,1, \tag{3.23}$$

$$\|e_u(\cdot,t_j)\|_{0,t_j} \leq ch\|u\|_{2,Q_1\cup Q_2}, \quad j = 0, N, \tag{3.24}$$

$$\sum_{i=1}^{2}\|R_i e_u\|_{\frac{1}{2},h,\Gamma_*} \leq ch\|u\|_{2,Q_1\cup Q_2}, \tag{3.25}$$

$$\sum_{i=1}^{2}\|\mathbf{n}\cdot\nabla R_i e_u\|_{-\frac{1}{2},h,\Gamma_*} \leq ch\|u\|_{2,Q_1\cup Q_2}. \tag{3.26}$$

Proof. The result in (3.23) follows from the definition of the interpolation operator I_{Γ_*} and the result in (3.20) for the standard nodal interpolation operator I_W:

$$\|e_u\|_{\ell,Q_1\cup Q_2}^2 = \sum_{i=1}^{2}\|e_u\|_{\ell,Q_i}^2 = \sum_{i=1}^{2}\|u - I_W\mathcal{E}_iR_iu\|_{\ell,Q_i}^2$$

$$\leq \sum_{i=1}^{2}\|(I - I_W)\mathcal{E}_iR_iu\|_{\ell,Q}^2 \leq ch^{2(2-\ell)}\sum_{i=1}^{2}\|\mathcal{E}_iR_iu\|_{2,Q}^2 \tag{3.27}$$

$$\leq ch^{2(2-\ell)}\sum_{i=1}^{2}\|R_iu\|_{2,Q_i}^2 = ch^{2(2-\ell)}\|u\|_{2,Q_1\cup Q_2}^2. \tag{3.28}$$

The result in (3.24) follows from the trace operator bound in (3.3) and the result in (3.23) for $\ell = 1$. For the proof of (3.25) we use the result in Lemma 3.10:

$$\begin{aligned}\sum_{i=1}^{2}\|R_ie_u\|_{\frac{1}{2},h,\Gamma_*}^2 &= \sum_{i=1}^{2}\sum_{\omega\in\mathcal{T}_{\Gamma_*}} h^{-1}\|\nu^{\frac{1}{2}}(I - I_W)\mathcal{E}_iR_iu\|_{0,\Gamma_\omega}^2 \\ &\leq c\sum_{i=1}^{2}\sum_{\omega\in\mathcal{T}_{\Gamma_*}}\left(h^{-2}\|(I - I_W)\mathcal{E}_iR_iu\|_{0,\omega}^2 + \|(I - I_W)\mathcal{E}_iR_iu\|_{1,\omega}^2\right) \\ &\leq c\sum_{i=1}^{2}\left(h^{-2}\|(I - I_W)\mathcal{E}_iR_iu\|_{0,Q}^2 + \|(I - I_W)\mathcal{E}_iR_iu\|_{1,Q}^2\right) \\ &\leq ch^2\|u\|_{2,Q_1\cup Q_2}^2,\end{aligned}$$

where in the last inequality we used (3.27)–(3.28).

The result in (3.26) can be proved with similar arguments:

$$\sum_{i=1}^{2} \|\mathbf{n}\cdot\nabla R_i e_u\|_{-\frac{1}{2},h,\Gamma_*} = \sum_{i=1}^{2}\sum_{\omega\in\mathcal{T}_{\Gamma_*}} h\|\nu^{\frac{1}{2}}\mathbf{n}\cdot\nabla(I-I_W)\mathcal{E}_i R_i u\|_{0,\Gamma_\omega}^2$$
$$\le c\sum_{i=1}^{2}\sum_{\omega\in\mathcal{T}_{\Gamma_*}} \big(\|\mathbf{n}\cdot\nabla(I-I_W)\mathcal{E}_i R_i u\|_{0,\omega}^2 + h^2\|\mathbf{n}\cdot\nabla(I-I_W)\mathcal{E}_i R_i u\|_{1,\omega}^2\big)$$
$$\le c\sum_{i=1}^{2}\big(\|(I-I_W)\mathcal{E}_i R_i u\|_{1,Q}^2 + h^2\|\mathcal{E}_i R_i u\|_{2,Q}^2\big) \le ch^2\|u\|_{2,Q_1\cup Q_2}^2,$$

which completes the proof. ∎

3.5. Continuity results. We derive continuity results for the bilinear forms $a(\cdot,\cdot)$, $d(\cdot,\cdot)$, and $N_{\Gamma_*}(\cdot,\cdot)$.

THEOREM 3.12. *There exists a constant c, depending only on $\|\mathbf{w}\|_{\infty,Q}$ and α_i,β_i, $i=1,2$, such that for all $u\in H_0^1(Q_1\cup Q_2)$, $v\in H_0^{1,0}(Q_1\cup Q_2)$ the following holds:*

$$|a(u,v)| \le c\|u\|_{1,Q_1\cup Q_2}\big(|v|_{1,Q_1\cup Q_2} + \|[\beta v]\|_{0,\Gamma_*}\big). \tag{3.29}$$

Let $C(\bar{Q}_1\cup\bar{Q}_2)$ denote the space of continuous functions on $Q_1\cup Q_2$ such that $v_i = v_{|Q_i}$ has a continuous extension to $\bar{Q}_i$, $i=1,2$. For all $u\in C(\bar{Q}_1\cup\bar{Q}_2)$, $v\in W^{\Gamma_*}$ the following holds:

$$|d(u,v)| \le \|u(\cdot,0)\|_{0,t_0}\|v(\cdot,0)\|_{0,t_0}. \tag{3.30}$$

For all $u,v\in W_{\mathrm{reg}} + W^{\Gamma_*}$ the following holds:

$$\begin{aligned} N_{\Gamma_*}(u,v) \le (1+\lambda)&\big(\|\{\alpha\nabla u\cdot\mathbf{n}\}\|_{-\frac{1}{2},h,\Gamma_*} + \|[\beta u]\|_{\frac{1}{2},h,\Gamma_*}\big)\\ &\cdot\big(\|\{\alpha\nabla v\cdot\mathbf{n}\}\|_{-\frac{1}{2},h,\Gamma_*} + \|[\beta v]\|_{\frac{1}{2},h,\Gamma_*}\big).\end{aligned} \tag{3.31}$$

Proof. Take $u\in H_0^1(Q_1\cup Q_2)$, $v\in H_0^{1,0}(Q_1\cup Q_2)$. From the definition of $a^n(\cdot,\cdot)$ it follows that there exists a constant, depending only on $\|\mathbf{w}\|_{\infty,Q}$ and α_i, β_i, such that

$$|a^n(u,v)| \le c\|u\|_{1,Q_1^n\cup Q_2^n}\big(|v|_{1,Q_1^n\cup Q_2^n} + \|v\|_{0,Q^n}\big).$$

Summation over n and with $\sum_{n=1}^N \sqrt{\eta_n}(\sqrt{\gamma_n}+\sqrt{\xi_n}) \le \sqrt{2}\big(\sum_{n=1}^N \eta_n\big)^{\frac{1}{2}}\big(\sum_{n=1}^N \gamma_n + \sum_{n=1}^N \xi_n\big)^{\frac{1}{2}}$ yields

$$|a(u,v)| \le c\|u\|_{1,Q_1\cup Q_2}\big(|v|_{1,Q_1\cup Q_2} + \|v\|_{0,Q}\big).$$

From this and using the Poincaré–Friedrichs inequality (3.5) we obtain the result in (3.29).

Take $u\in C(\bar{Q}_1\cup\bar{Q}_2)$ and $v\in W^{\Gamma_*}$. From the continuity of u w.r.t. t it follows that

$$d(u,v) = \sum_{n=1}^N d^n(u,v) = d^1(u,v) = ([u]^0, v)_{0,t_0} = (u(\cdot,0), v(\cdot,0))_{0,t_0},$$

from which the result in (3.30) follows.

Take $u, v \in W_{\rm reg} + W^{\Gamma_*}$. Using the definition of $N^n_{\Gamma_*}$ and of the trace norms we get

$$\begin{aligned}|N^n_{\Gamma_*}(u,v)| &\le \|\{\alpha\nabla u\cdot\mathbf{n}\}\|_{-\frac12,h,\Gamma^n_*}\|[\beta v]\|_{\frac12,h,\Gamma^n_*} + \|\{\alpha\nabla v\cdot\mathbf{n}\}\|_{-\frac12,h,\Gamma^n_*}\|[\beta u]\|_{\frac12,h,\Gamma^n_*}\\ &\quad + \lambda\|[\beta u]\|_{\frac12,h,\Gamma^n_*}\|[\beta v]\|_{\frac12,h,\Gamma^n_*}.\end{aligned}$$

Summation over n results in

$$\begin{aligned}|N_{\Gamma_*}(u,v)| &\le \|\{\alpha\nabla u\cdot\mathbf{n}\}\|_{-\frac12,h,\Gamma_*}\|[\beta v]\|_{\frac12,h,\Gamma_*} + \|\{\alpha\nabla v\cdot\mathbf{n}\}\|_{-\frac12,h,\Gamma_*}\|[\beta u]\|_{\frac12,h,\Gamma_*}\\ &\quad + \lambda\|[\beta u]\|_{\frac12,h,\Gamma_*}\|[\beta v]\|_{\frac12,h,\Gamma_*}\\ &\le (1+\lambda)\big(\|\{\alpha\nabla u\cdot\mathbf{n}\}\|_{-\frac12,h,\Gamma_*} + \|[\beta u]\|_{\frac12,h,\Gamma_*}\big)\\ &\quad\cdot\big(\|\{\alpha\nabla v\cdot\mathbf{n}\}\|_{-\frac12,h,\Gamma_*} + \|[\beta v]\|_{\frac12,h,\Gamma_*}\big),\end{aligned}$$

and thus the result in (3.31) holds. ∎

3.6. Discretization error bounds. Assume that (1.1)–(1.5) has a solution $u \in H^2_0(Q_1\cup Q_2) \subset W_{\rm reg}$. We assume a fixed $\lambda > c^*$ as in Theorem 3.8. Let $U \in W^{\Gamma_*}$ be the unique solution of the Nitsche XFEM-DG discretization (2.5); cf. Corollary 3.9. In this section we derive bounds for the discretization error $e := u - U$. From the consistency result in Theorem 3.4 we obtain the Galerkin property

$$B(e,V) = 0 \quad \text{for all} \quad V \in W^{\Gamma_*}.$$

Based on the stability analysis we introduce the norm (with $v^N = v(\cdot, t_N)$)

$$|\!|\!| v |\!|\!|^2 = |v|^2_{1,Q_1\cup Q_2} + \|v^N\|^2_{0,t_N} + \sum_{n=1}^N \|[v]^n\|^2_{0,t_{n-1}} + \lambda\|[\beta v]\|^2_{\frac12,h,\Gamma_*}, \quad v \in W_{\rm reg} + W^{\Gamma_*}.$$

THEOREM 3.13. *Let the conditions in the Assumptions* 3.1 *and* 3.2 *be satisfied and assume that* (1.1)–(1.5) *has a solution* $u \in H^2_0(Q_1\cup Q_2)$. *For the discretization error* $u-U$ *the following holds:*

$$|\!|\!| u - U |\!|\!| \le ch\|u\|_{2,Q_1\cup Q_2}.$$

Proof. We decompose the error $e = u - U$ as

$$e = (u - I_{\Gamma_*}u) + (I_{\Gamma_*}u - U) =: e_u + e_W. \tag{3.32}$$

Note that e_u is continuous with respect to t and thus $[e_u]^n = 0$ for $n = 2, \dots, N$, and $[e_u]^1 = e_u(\cdot, 0)$. Hence,

$$|\!|\!| e_u |\!|\!|^2 \le |e_u|^2_{1,Q_1\cup Q_2} + \|e_u(\cdot,t_N)\|^2_{0,t_N} + \|e_u(\cdot,t_0)\|^2_{0,t_0} + \lambda\max_{i=1,2}\beta_i\sum_{i=1}^2\|R_ie_u\|^2_{\frac12,h,\Gamma_*}.$$

Using the interpolation error bounds in Theorem 3.11 yields

$$|\!|\!| e_u |\!|\!| \le ch\|u\|_{2,Q_1\cup Q_2}.$$

For $e_W \in W^{\Gamma_*}$ we get, using the stability result in Theorem 3.8 and the Galerkin property,

$$\begin{aligned}|\!|\!| e_W |\!|\!|^2 &\le cB(e_W,e_W) = cB(I_{\Gamma_*}u - U, e_W) = cB(I_{\Gamma_*}u - u, e_W)\\ &= -c\big(a(e_u,e_W) + d(e_u,e_W) + N_{\Gamma_*}(e_u,e_W)\big).\end{aligned}$$

The errors e_u and e_W have smoothness properties such that Theorem 3.12 can be applied. In combination with the interpolation error bounds in Theorem 3.11 this results in

$$\begin{aligned}
&- a(e_u, e_W) - d(e_u, e_W) \\
&\leq c\|e_u\|_{1,Q_1\cup Q_2}\big(|e_W|_{1,Q_1\cup Q_2} + \|[\beta e_W]\|_{0,\Gamma_*}\big) + \|e_u(\cdot,0)\|_{0,t_0}\|e_W(\cdot,0)\|_{0,t_0} \\
&\leq c\big(\|e_u\|_{1,Q_1\cup Q_2} + \|e_u(\cdot,0)\|_{0,t_0}\big)\|\!|e_W|\!\| \leq ch\|u\|_{2,Q_1\cup Q_2}\|\!|e_W|\!\|.
\end{aligned}$$

For the term $N_{\Gamma_*}(e_u, e_W)$ we use Theorem 3.12, Lemma 3.6 and the interpolation error bounds in Theorem 3.11, and obtain

$$\begin{aligned}
N_{\Gamma_*}(e_u, e_W) &\leq (1+\lambda)\big(\|\{\alpha\nabla e_u\cdot\mathbf{n}\}\|_{-\frac{1}{2},h,\Gamma_*} + \|[\beta e_u]\|_{\frac{1}{2},h,\Gamma_*}\big) \\
&\qquad \cdot \big(\|\{\alpha\nabla e_W\cdot\mathbf{n}\}\|_{-\frac{1}{2},h,\Gamma_*} + \|[\beta e_W]\|_{\frac{1}{2},h,\Gamma_*}\big) \\
&\leq ch\|u\|_{2,Q_1\cup Q_2}\big(|e_W|_{1,Q_1\cup Q_2} + \|[\beta e_W]\|_{\frac{1}{2},h,\Gamma_*}\big) \\
&\leq ch\|u\|_{2,Q_1\cup Q_2}\|\!|e_W|\!\|.
\end{aligned} \tag{3.33}$$

The combination of these results yields

$$\|\!|e_W|\!\| \leq ch\|u\|_{2,Q_1\cup Q_2}. \tag{3.34}$$

The application of the triangle inequality completes the proof. ∎

In the following lemma we derive a result that will be used in a duality argument in Theorem 3.15.

Lemma 3.14. *Let $\hat{u}_T \in H^1\big(\Omega_1(T)\cup\Omega_2(T)\big)$ with $(\hat{u}_T)_{|\partial\Omega} = 0$ be given. Assume that the homogeneous backward problem*

$$-\frac{\partial\hat{u}}{\partial t} - \mathbf{w}\cdot\nabla\hat{u} - \operatorname{div}(\alpha\nabla\hat{u}) = 0 \quad \text{in} \;\; \Omega_i(t),\; i=1,2,\; t\in[0,T], \tag{3.35}$$

$$[\alpha\nabla\hat{u}\cdot\mathbf{n}]_\Gamma = 0, \tag{3.36}$$

$$[\beta\hat{u}]_\Gamma = 0, \tag{3.37}$$

$$\hat{u}(\cdot,T) = \hat{u}_T \quad \text{in} \;\; \Omega_i(T),\; i=1,2, \tag{3.38}$$

$$u(\cdot,t) = 0 \quad \text{on} \;\; \partial\Omega, \quad t\in[0,T], \tag{3.39}$$

has a solution $\hat{u} \in H_0^{2,1}(Q_1\cup Q_2)$. This solution satisfies

$$B(w,\hat{u}) = (w^N, \hat{u}_T)_{0,T} \quad \text{for all} \;\; w \in W^{\Gamma_*} + W_{\text{reg}}. \tag{3.40}$$

Proof. For the Nitsche bilinear form we obtain, using $[\beta\hat{u}]_{\Gamma_*} = 0$,

$$N_{\Gamma_*}(w,\hat{u}) = \sum_{n=1}^{N} N_{\Gamma_*}^n(w,\hat{u}) = -\int_{\Gamma_*} \nu\{\alpha\nabla\hat{u}\cdot\mathbf{n}\}_{\Gamma_*}[\beta w]_{\Gamma_*}\,ds. \tag{3.41}$$

The solution $\hat{u}$ satisfies $[\hat{u}]^{n-1} = 0$ for all $2 \leq n \leq N$, and thus from Lemma 3.3 we get

$$a(w,\hat{u}) + d(w,\hat{u}) = -a(\hat{u},w) + 2(\alpha\hat{u},w)_{1,Q_1\cup Q_2} + (w^N,\hat{u}^N)_{0,t_N}. \tag{3.42}$$

Using $[\alpha\nabla\hat{u}\cdot\mathbf{n}]_t = 0$ we obtain, with a partial integration as in (3.12)–(3.14), and with (3.35),

$$\begin{aligned}
&- a(\hat{u}, w) + 2(\alpha\hat{u}, w)_{1,Q_1\cup Q_2}\\
&= \sum_{n=1}^{N}\sum_{i=1}^{2}\int_{Q_i^n}\left(-\frac{\partial\hat{u}_i}{\partial t} - \mathbf{w}\cdot\nabla\hat{u}_i - \alpha_i\Delta\hat{u}_i\right)\beta_i w\,dx\,dt\\
&+ \sum_{n=1}^{N}\int_{\Gamma_*^n}\nu\{\alpha\nabla\hat{u}\cdot\mathbf{n}\}_{\Gamma_*}[\beta w]_{\Gamma_*}\,ds\\
&= \int_{\Gamma_*}\nu\{\alpha\nabla\hat{u}\cdot\mathbf{n}\}_{\Gamma_*}[\beta w]_{\Gamma_*}\,ds.
\end{aligned}$$

Combining this with the results in (3.41), (3.42) we obtain

$$B(w,\hat{u}) = a(w,\hat{u}) + d(w,\hat{u}) + N_{\Gamma_*}(w,\hat{u}) = (w^N, \hat{u}_T)_{0,T},$$

which proves the result in (3.40). ∎

THEOREM 3.15. *Let the conditions in the Assumptions* 3.1 *and* 3.2 *be satisfied and assume that* (1.1)–(1.5) *has a solution* $u \in H_0^2(Q_1\cup Q_2)$. *Assume that the homogeneous backward problem as in Lemma* 3.14 *has a solution* $\hat{u} \in H_0^2(Q_1\cup Q_2)$ *that has the regularity property* $\|\hat{u}\|_{2,Q_1\cup Q_2} \le c\|\hat{u}_T\|_{0,t_N}$ *with a constant* c *independent of the initial data* $\hat{u}_T \in L^2(\Omega)$. *For the discretization error* $u-U$ *the following holds:*

$$\|(u-U)(\cdot,t_N)\|_{0,t_N} \le ch^2\|u\|_{2,Q_1\cup Q_2}.$$

Proof. The error is denoted by $e = u - U$. In (3.38) we take $\hat{u}_T = e(\cdot,t_N)$. Note that $\hat{u}_T \in H^1\big(\Omega_1(T)\cup\Omega_2(T)\big)$ and $(\hat{u}_T)_{|\partial\Omega} = 0$ holds. The induced solution is denoted by $\hat{u}$. In (3.40) we take $w = e$. This yields, with $e_{\hat{u}} := \hat{u} - I_{\Gamma_*}\hat{u}$,

$$\|e(\cdot,t_N)\|_{0,t_N}^2 = B(e,\hat{u}) = B(e,e_{\hat{u}}) = a(e,e_{\hat{u}}) + d(e,e_{\hat{u}}) + N_{\Gamma_*}(e,e_{\hat{u}}). \tag{3.43}$$

In the term $d(e,e_{\hat{u}})$ the second argument $e_{\hat{u}}$ is continuous with respect to t, whereas this is not necessarily true for the first argument. Therefore the boundedness result in (3.30) is not applicable. Instead, however, we can apply Lemma 3.3 and thus, using the boundedness result in (3.29), the trace bound in (3.3), the interpolation error bound in (3.23) and the discretization error bound in Theorem 3.13, we obtain

$$\begin{aligned}
a(e,e_{\hat{u}}) + d(e,e_{\hat{u}}) &= -a(e_{\hat{u}},e) + 2(\alpha e, e_{\hat{u}})_{1,Q_1\cup Q_2} + (e(\cdot,t_N), e_{\hat{u}}(\cdot,t_N))_{0,t_N}\\
&\le c\|e_{\hat{u}}\|_{1,Q_1\cup Q_2}|\!|\!|e|\!|\!| \le ch\|\hat{u}\|_{2,Q_1\cup Q_2}|\!|\!|e|\!|\!|\\
&\le ch\|\hat{u}_T\|_{0,t_N}|\!|\!|e|\!|\!| = ch\|e(\cdot,t_N)\|_{0,t_N}|\!|\!|e|\!|\!|\\
&\le ch^2\|u\|_{2,Q_1\cup Q_2}\|e(\cdot,t_N)\|_{0,t_N}.
\end{aligned} \tag{3.44}$$

We consider the term $N_{\Gamma_*}(e,e_{\hat{u}})$. Decompose $e = e_u + e_W$ as in (3.32); hence,

$$N_{\Gamma_*}(e,e_{\hat{u}}) = N_{\Gamma_*}(e_{\hat{u}},e) = N_{\Gamma_*}(e_{\hat{u}},e_u) + N_{\Gamma_*}(e_{\hat{u}},e_W).$$

For the last term we have (cf. (3.33) and (3.34))

$$N_{\Gamma_*}(e_{\hat{u}},e_W) \le ch\|\hat{u}\|_{2,Q_1\cup Q_2}|\!|\!|e_W|\!|\!| \le ch^2\|u\|_{2,Q_1\cup Q_2}\|e(\cdot,t_N)\|_{0,t_N}. \tag{3.45}$$

Finally, from the boundedness result in Theorem 3.12 and the interpolation error bounds in Theorem 3.11, we get

$$
\begin{aligned}
N_{\Gamma_*}(e_{\hat{u}}, e_u) &\leq (1+\lambda)\big(\|\{\alpha\nabla e_{\hat{u}}\cdot\mathbf{n}\}\|_{-\frac{1}{2},h,\Gamma_*} + \|[\beta e_{\hat{u}}]\|_{\frac{1}{2},h,\Gamma_*}\big) \\
&\qquad \cdot \big(\|\{\alpha\nabla e_u\cdot\mathbf{n}\}\|_{-\frac{1}{2},h,\Gamma_*} + \|[\beta e_u]\|_{\frac{1}{2},h,\Gamma_*}\big) \\
&\leq ch^2\|\hat{u}\|_{2,Q_1\cup Q_2}\|u\|_{2,Q_1\cup Q_2} \leq ch^2\|u\|_{2,Q_1\cup Q_2}\|e(\cdot,t_N)\|_{0,t_N}.
\end{aligned} \tag{3.46}
$$

Inserting the bounds (3.44), (3.45), (3.46) into (3.43) completes the proof. ∎

Remark 3.1. We comment on a few aspects related to the analysis presented in this section. In the main theorems, Theorems 3.13 and 3.15, we used the regularity assumption that (1.1)–(1.5) has a solution $u \in H_0^2(Q_1 \cup Q_2)$. We consider this to be a reasonable assumption but we do not know of any literature where this regularity issue is studied. A similar statement holds for the regularity assumption $\hat{u} \in H_0^{2,1}(Q_1 \cup Q_2)$ used in Lemma 3.14. Note that the function $\hat{u}_T$ used in that lemma is consistent with the homogeneous boundary condition on $\partial\Omega$.

Furthermore, we note that the error bound derived in Theorem 3.13 for P_1-P_1 space-time finite elements is of optimal order w.r.t. the space variable but suboptimal w.r.t. time, since the norm $|\!|\!|\cdot|\!|\!|$ contains no derivatives w.r.t. t. In Theorem 3.15 we derived a second order error bound. This result might be suboptimal, since for the standard P_1-P_1 FEM-DG (not XFEM!) better bounds are known in the literature. In [22, Theorem 12.7] for the P_1-P_1 DG-FEM method applied to the standard heat equation an error bound of the form

$$
\|(u-U)(\cdot,t_N)\|_{0,t_N} \leq c\big(h^2 + (\Delta t)^3\big) \tag{3.47}
$$

is derived, i.e., an error bound with third order convergence w.r.t. Δt. For other polynomial degrees, say k and m w.r.t. space and time, respectively, based on the analysis in [22] one expects that the bound in (3.47) can be generalized to $c\big(h^{k+1} + (\Delta t)^{2m+1}\big)$. The analysis in [22] is very different from the one presented in this section. A key idea in the analysis in [22] is (as is usual in the derivation of error bounds of finite element methods for parabolic problems) to use an error splitting

$$
u - U = (u - R_h\tilde{u}) + (R_h\tilde{u} - U),
$$

with $R_h : H_0^1(\Omega) \to V_h$ the Ritz projection corresponding to the stationary (i.e., elliptic) problem and $\tilde{u}(x,t)$ is suitable space-time interpolant of u. For this idea to work it is essential that the space V_h (and thus R_h) is *in*dependent of t. Although we did not investigate this, we expect that this analysis is applicable to the Nitsche XFEM-DG method for the case of a *stationary* interface, since then the XFEM space can be taken independent of t. This analysis, however, is not applicable to the (more interesting) case with a nonstationary interface, which explains why we present a different analysis. With our analysis, however, we are not able to derive a bound as in (3.47) with a term $(\Delta t)^3$. The numerical experiments in section 4, which are for a spatially one-dimensional problem, yield only results which show an error behavior similar to the one in (3.47).

In Remark 2.2 we noted that the method has a straightforward extension to higher order finite elements. Concerning the analysis for higher order polynomials we note that all arguments used in the analysis above, except for the proof of Lemma 3.6, can be applied (with minor modifications) to higher order space-time finite elements as well. We claim that estimates as in Lemma 3.6 also hold for higher order finite elements. We sketch how this might be proved. For an arbitrary tetrahedron S and natural number k there exists a constant c, independent of the shape regularity of S, such that $p(\tilde{x})^2 \leq \frac{c}{|S|}\int_S p(x)^2\,dx$ for all $\tilde{x} \in \partial S$ and all polynomials p of degree k. From this one obtains (cf. proof of Lemma 3.6 for notation), for the case that T_i is a tetrahedron,

$$\int_{\Gamma_T(t)} h_n \kappa_i(t)^2 p^2\,ds \leq c|T_i| \max_{x\in\Gamma_T(t)} p(x)^2 \leq c\int_{T_i} p^2\,dx$$

for polynomials of degree k, which is the main estimate in the proof of Lemma 3.6.

Finally, we recall that in order to derive useful interpolation error bounds we assumed matching triangulations $\mathcal{T}_n = \mathcal{T}_m$ for all m, n; cf. the introduction to section 3.4. It is not clear whether the analysis can be extended to the nonmatching case, because we do not know how to construct an appropriate global interpolation of u in the XFEM-DG space W.

4. Numerical experiments. In this section we present results of numerical experiments. We restrict ourselves to the spatially one-dimensional case. We illustrate the convergence behavior, in particular the order of convergence w.r.t. h and Δt for XFEM finite element spaces with different polynomial degrees. The results are consistent with the theoretical analysis presented above. The relatively simple one-dimensional setting allows a very accurate approximation of the integrals that occur in the bilinear form and thus the method can be implemented in such a way that quadrature errors are negligible. For an efficient implementation, however, in particular in higher dimensions, one has to use a suitable quadrature rule for the approximation of the integrals over the space-time interface and of integrals over cylindrical elements cut by the space-time interface. Numerical experiments with the Nitsche XFEM-DG method in higher dimensions and a study of the effect of quadrature errors will be presented in a forthcoming paper.

4.1. Problem description. We consider the spatial domain $\Omega = [-1, 1]$ and instead of the Dirichlet boundary conditions in (1.5) we consider periodic ones. The reason for this is that we want to avoid (nonphysical) effects caused by interfaces that intersect a Dirichlet boundary. We consider the following problem:

$$\partial_t u + w\partial_x u - \partial_x(\alpha\partial_x u) = f \quad \text{in} \ \ \Omega_i,\ i = 1, 2,\ t \in [0, T], \tag{4.1}$$

$$[\alpha\partial_x u]_\Gamma = 0, \tag{4.2}$$

$$[\beta u]_\Gamma = 0, \tag{4.3}$$

$$u(\cdot, 0) = u_0 \quad \text{in} \ \ \Omega_i,\ i = 1, 2, \tag{4.4}$$

$$u(-1, t) = u(1, t) \quad \text{for } t \in [0, T]. \tag{4.5}$$

For the end time point we choose $T = 4$. The diffusivities are $(\alpha_1, \alpha_2) = (1, 2)$ and the Henry weights $(\beta_1, \beta_2) = (1.5, 1)$. The interface is determined by the initial interface $\Gamma(0)$ consisting of two points $x_a = -1/3$, $x_b = 1/3$ and a given smooth velocity function $w(t)$, which is constant w.r.t. x (due to incompressibility). At initial time $t = 0$ we define $\Omega_1(0) := [x_a, x_b]$ and $\Omega_2(0) = \Omega \setminus \Omega_1(0)$. Define the shift function $s(t) := \int_0^t w(r)\,dr$. We consider three cases:

1. constant interface, no convection: $w(t) = 0$, $s(t) = 0$,
2. constant vel., Γ^* is planar: $w(t) = \frac{1}{2}$, $s(t) = \frac{1}{2}t$,
3. time dep. vel., Γ^* is curved: $w(t) = \frac{1}{4}\cos(\frac{\pi}{2}t)$, $s(t) = \frac{1}{2\pi}\sin(\frac{1}{2}\pi t)$.

The moving interface is given by $\Gamma(t) = \{x_a + s(t), x_b + s(t)\}$. If t is such that one of these points intersects the boundary $\partial\Omega$, one has to take the periodicity condition into account; cf. Figure 4.1 for an illustration.

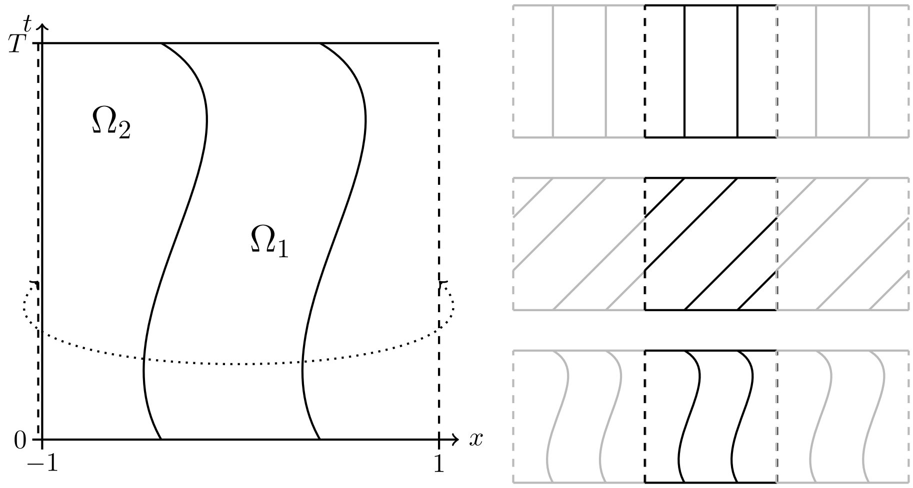

FIG. 4.1. *Sketch of space-time domain and interface (left) and three different interface evolutions (right) where dashed lines indicate periodic boundaries and grey domains periodically shifted counterparts of the original domain.*

We construct a reference solution by using Lagrangian coordinates. First consider $\Omega = \mathbb{R}$ (no boundary). For $\hat{x} \in \mathbb{R}$ the characteristic through $\hat{x}$ is given by $(\hat{x} + s(t), t)$. Let $\hat{u}(\hat{x}, t)$ be a function that satisfies $\partial_t \hat{u} - \partial_{\hat{x}}(\alpha_i \partial_{\hat{x}} \hat{u}) = f$ on $\Omega_i(0)$, $t \in [0, T]$, and the interface conditions (4.2) and (4.3) at $\Gamma(0)$. Note that $\hat{u}$ does not depend on the velocity function w. The function $u(x, t) := \hat{u}(x - s(t), t)$ then satisfies (4.1), (4.2), (4.3). We choose the following reference solution (in $(\hat{x}, t)$ coordinates):

$$\hat{u}_{\text{ref}}(\hat{x}, t) = sin(k\pi t) \cdot \begin{cases} (a\hat{x} + b\hat{x}^3), & \hat{x} \text{ in } \Omega_1(0), \\ sin(\pi\hat{x}), & \hat{x} \text{ in } \Omega_2(0). \end{cases} \tag{4.6}$$

We set $k = 2\frac{1}{8}$ and the constants a and b are chosen such that the interface conditions (4.2) and (4.3) at $\Gamma(0)$ are fulfilled. As a reference solution in our experiments below we use $u_{\text{ref}}(x, t) := \hat{u}_{\text{ref}}(x - s(t), t)$, where we corrected for periodicity. A corresponding right-hand side f is used as data in (4.1). The reference solution for the three velocity functions given above is displayed in Figure 4.2.

4.2. Discretization. We use an equidistant mesh in space and time with n_s intervals in space and n_t intervals in time and apply uniform refinements in space and time. The corresponding mesh sizes are denoted by h_s (space) and h_t (time). The polynomial degrees in space and time are denoted by p_s and p_t, respectively. We present results for $p_s = 1$ and $p_t = 0, 1, 2$. We recall that in spatial direction we use continuous (piecewise linear) finite elements, whereas in time direction the functions are allowed to be discontinuous at the mesh points. For the stabilization parameter in the Nitsche method we take the value $\lambda = 2$. We use sufficiently accurate quadrature rules such that quadrature errors are negligible. This is a delicate assumption, in particular, if we have a curved space-time interface as in the third example.

4.3. Numerical results. We present results that illustrate the convergence behavior of the Nitsche XFEM-DG method for the three velocity functions introduced

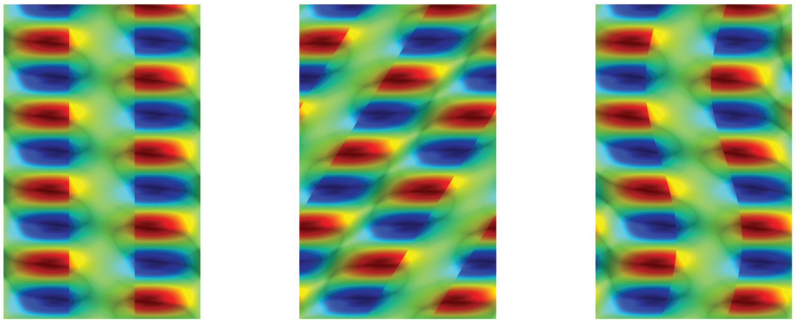

FIG. 4.2. *Solutions of the test cases with* $w(t) = 0$ *(left),* $w(t) = 0.5$ *(middle), and* $w(t) = \frac{1}{4}\cos(\frac{1}{2}\pi t)$ *(right).*

above. Already for this one-dimensional case it is interesting to study how convergence depends on the parameters p_s, p_t, n_s, n_t. Here we present only results for a small subset of such experiments. As already mentioned above, an elaborate numerical study of properties of this method will be presented in a forthcoming paper. For the first case (stationary interface) we (only) study the error dependence on h_t, for a fixed h_s sufficiently small. We fix $p_s = 1$ and vary $p_t \in \{0, 1, 2\}$. For the second case (constant nonzero velocity) we fix $p_s = p_t = 1$ and vary h_s, h_t. Finally, for the case with a curved space-time interface we consider only $p_t = p_s = 1$ and for a fixed small mesh size h_s we study the error dependence on h_t.

Stationary interface. This case is relatively easy and can be treated efficiently by other methods (e.g., method of lines with an XFEM space discretization [19]) as well. As pointed out in Remark 3.1, for the case with a stationary interface, we expect that the $L^2(\Omega)$-error at time T can be bounded by

$$\|u_h - u_{\mathrm{ref}}\|_{0,T} \leq c\big(h_s^{p_s+1} + h_t^{2p_t+1}\big). \tag{4.7}$$

We consider $p_s = 1$ and a mesh with a sufficiently fine spatial resolution ($n_s = 1024$) and a relatively large time step such that the temporal error dominates the total error. In Table 4.1 and Figure 4.3 the errors for $p_t = 0, 1, 2$ are given. The observed convergence behavior is consistent with the $\mathcal{O}(h_t^{2p_t+1})$ bound in (4.7).

TABLE 4.1
Error $\|u_h - u_{\mathrm{ref}}\|_{0,T}$ *for* $n_s = 1024$*,* $p_t \in \{0, 1, 2\}$*, and a stationary interface.*

$p_t = 0$							
$n_s \backslash n_t$	16	32	64	128	256	512	1024
1024	3.671e-01	1.729e-01	8.167e-02	3.917e-02	1.909e-02	9.411e-03	4.671e-03
eoc(t)	–	1.09	1.08	1.06	1.04	1.02	1.01

$p_t = 1$						
$n_s \backslash n_t$	16	32	64	128	256	512
1024	6.541e-02	1.129e-02	1.789e-03	2.425e-04	3.074e-05	2.477e-06
eoc(t)	–	2.53	2.72	2.82	2.98	3.63

$p_t = 2$					
$n_s \backslash n_t$	4	8	16	32	64
1024	1.726e-01	7.737e-02	3.888e-03	1.420e-04	2.312e-06
eoc(t)	–	1.16	4.31	4.78	5.94

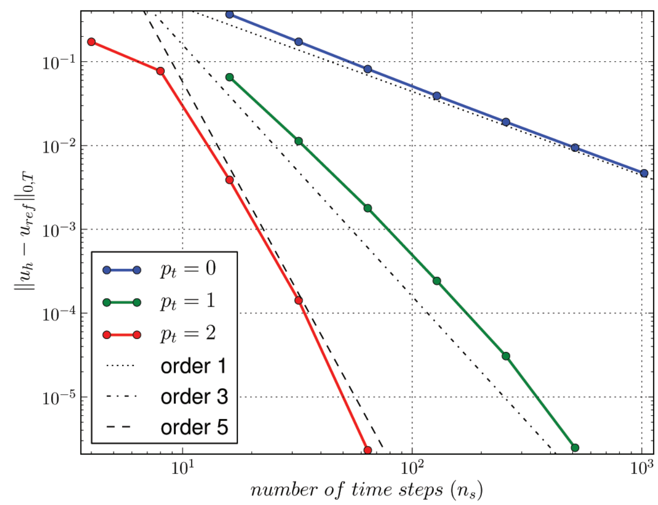

FIG. 4.3. *Error* $\|u_h - u_{\mathrm{ref}}\|_{0,T}$ *for* $n_s = 1024$, $p_t \in \{0, 1, 2\}$, *and a stationary interface.*

Constant velocity. We consider the case of a moving interface with velocity function $w(t) = 1/2$. We restrict ourselves to $p_t = p_s = 1$ and investigate both the temporal and the spatial error convergence. The numerical results for $n_s \in \{8, 16, \ldots, 1024\}$ and $n_t \in \{16, 32, \ldots, 512\}$ are displayed in Table 4.2. If we consider the finest resolution in space or time we can assume that the corresponding other part of the total error is dominating. For the smallest time step ($n_t = 512$) the estimated order of convergence w.r.t. h_s (*eoc*(s)) is given in the last column in Table 4.2. For the smallest spatial mesh size ($n_s = 1024$) the estimated order of convergence w.r.t. h_t (*eoc*(t)) is given in the last row. Note that the error at the ($n_s = 1024, n_t = 512$) entry is dominated by the temporal error and therefore the estimated order of convergence *eoc(s)* reduces to 1.15. The results in Table 4.2 are visualized in Figure 4.4 where for each fixed spatial (left) or temporal (right) mesh size an error graph is plotted. The results indicate an $\mathcal{O}(h_t^3) + \mathcal{O}(h_s^2)$ error behavior.

TABLE 4.2
Error $\|u_h - u_{\mathrm{ref}}\|_{0,T}$ *for several spatial* and *temporal discretizations.*

$p_t = 1$							
$n_s \backslash n_t$	16	32	64	128	256	512	*eoc(s)*
8	9.064e-02	5.854e-02	5.684e-02	5.728e-02	5.736e-02	5.737e-02	–
16	6.654e-02	1.077e-02	1.067e-02	1.179e-02	1.194e-02	1.196e-02	2.26
32	7.019e-02	1.062e-02	3.082e-03	3.488e-03	3.645e-03	3.673e-03	1.70
64	7.185e-02	1.206e-02	1.417e-03	6.608e-04	8.368e-04	8.670e-04	2.08
128	7.222e-02	1.250e-02	1.810e-03	1.732e-04	2.006e-04	2.234e-04	1.95
256	7.229e-02	1.262e-02	1.941e-03	2.531e-04	3.031e-05	5.107e-05	2.13
512	7.230e-02	1.265e-02	1.974e-03	2.867e-04	3.290e-05	9.463e-06	2.43
1024	7.229e-02	1.266e-02	2.069e-03	2.958e-04	4.139e-05	4.278e-06	1.15
eoc(t)	–	2.51	2.67	2.74	2.83	3.27	–

Nonconstant velocity. In this last example we consider a space-time interface which is curved. The velocity is given by $w(t) = \frac{1}{4}\cos(\frac{\pi}{2}t)$. Different from the

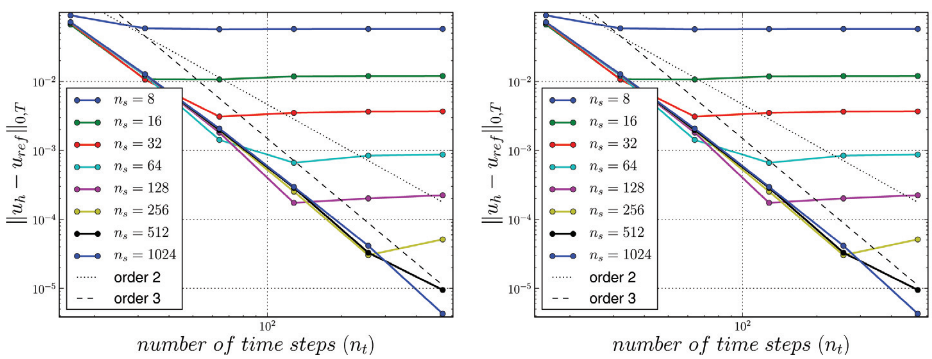

FIG. 4.4. *Error $\|u_h - u_{\mathrm{ref}}\|_{0,T}$ for a different number of elements (n_s) plotted over number of time steps(n_t) (left) and the same error for a different number of time steps (n_t) plotted over the number of elements (n_s) (right).*

previous two examples, we have to allow quadrature errors. This quadrature is based on a piecewise linear (in time) approximation of the interface. The resolution of this approximation is sufficiently high such that quadrature errors are negligible. Similar to the procedure before we use $p_t = p_s = 1$ and a small mesh size h_s ($n_s = 1024$) such that the error is dominated by the temporal error. In Table 4.3 we give the resulting error $\|u_h - u_{\mathrm{ref}}\|_{0,T}$ for different time steps h_t. Similar to the results in the preceding sections we observe a third order convergence w.r.t. h_t.

We conclude with the following observation. If one compares the errors in the three examples for $n_s = 1024$, $p_t = p_s = 1$ one sees that these are of the same size. Hence, at least in this one-dimensional test example, the method yields for the case with a curved space-time interface ($w(t) = \frac{1}{4}\cos(\frac{1}{2}\pi t)$) a similar discretization accuracy as for the case with a stationary interface ($w(t) = 0$). In other words, the accuracy of this Nitsche XFEM-DG method is not very sensitive w.r.t. the location of the space-time interface.

TABLE 4.3
Error $\|u_h - u_{\mathrm{ref}}\|_{0,T}$ for a different number of time steps.

$p_t = 1$						
$n_s \backslash n_t$	16	32	64	128	256	512
1024	6.646e-02	1.155e-02	1.768e-03	2.532e-04	3.265e-05	2.532e-06
eoc(t)	–	2.52	2.71	2.80	2.96	3.69

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