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Analysis of trace finite element methods for surface partial differential equations

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In this paper we consider two variants of a trace finite element method for solving elliptic partial differential equations on a stationary smooth manifold Γ. A discretization error analysis for both methods in one general framework is presented. Higher-order finite elements are treated and rather general numerical approximations Γ_h of the manifold Γ are allowed. Optimal-order discretization error bounds are derived. Furthermore, the conditioning of the stiffness matrices is studied. It is proved that for one of these two variants the corresponding scaled stiffness matrix has a condition number $\sim h^{-2}$, independently of how Γ_h intersects the outer triangulation.

Keywords: Trace finite elements; error analysis; elliptic surface equation.

1. Introduction

Partial differential equations (PDEs) posed on evolving surfaces arise in many applications. In fluid dynamics, the concentration of surface active agents attached to an interface between two phases of immiscible fluids is governed by a transport-diffusion equation on the interface (Gross & Reusken, 2011). Another example is the diffusion of trans-membrane receptors in the membrane of a deforming and moving cell, which is typically modelled by a parabolic PDE posed on an evolving surface (Alberta *et al.*, 2002).

Recently, several numerical approaches for solving PDEs on surfaces have been introduced. The finite element method of Dziuk & Elliott (2007) for the discretization of a PDE on an evolving surface is based on the *Lagrangian* description of a surface evolution and benefits from a special invariance property of test functions along material trajectories. If one considers the *Eulerian* description of a surface evolution, e.g., based on the level set method (Sethian, 1996), then the surface is usually defined implicitly. In this case, regular surface triangulations and material trajectories of points on the surface are not easily available. Hence, Eulerian numerical techniques for the discretization of PDEs on surfaces have been studied in the literature. In Adalsteinsson & Sethian (2003) and Xu & Zhao (2003) numerical approaches were introduced that are based on extensions of PDEs off a two-dimensional surface to a three-dimensional neighbourhood of the surface. Then one can apply a standard finite element or finite difference discretization to treat the extended equation in $\mathbb{R}^3$. For a discussion of this extension approach we refer the reader to Greer (2008), Dziuk & Elliott (2010) and Chernyshenko & Olshanskii (2013). A related approach was developed in Elliott *et al.* (2011), where advection–diffusion equations are numerically solved on evolving diffuse interfaces.

A different Eulerian technique for the numerical solution of an elliptic PDE posed on a stationary hypersurface in $\mathbb{R}^3$ was introduced in Olshanskii *et al.* (2009). The main idea of this method is to use finite element spaces that are induced by the volume triangulations (tetrahedral decompositions) of a bulk domain in order to discretize a PDE on the embedded surface. This method does not use an extension of the surface PDE. It is instead based on a restriction (trace) of the outer finite element spaces to the (approximated) surface. This leads to discrete problems for which the number of degrees of freedom corresponds to the two-dimensional nature of the surface problem, similarly to the Lagrangian approach. At the same time, the method is essentially Eulerian as the surface is not tracked by a surface mesh and may be defined implicitly as the zero level of a level set function. Optimal discretization error bounds were proved in Olshanskii *et al.* (2009). The approach was further developed, for stationary surfaces, in Demlow & Olshanskii (2012) and Olshanskii *et al.* (2014b), where adaptive and streamline diffusion variants of this trace finite element method were introduced and analysed. In the recent papers Olshanskii & Reusken (2013), Olshanskii *et al.* (2014a), the trace method is extended to an Eulerian finite element method for the discretization of PDEs on evolving surfaces.

Recently, in Ranner (2013) and Deckelnick *et al.* (2013), for this Eulerian trace finite element method the following interesting result was derived. If, in this method with piecewise linears, the *tangential* gradients ∇_Γ used in the bilinear form are replaced by the *full* gradients ∇, the method still has optimal convergence behaviour. For the discretization of the Laplace–Beltrami equation on a stationary smooth surface Γ we thus have the following two variants of the trace method: find $u_h, u_h^\Gamma \in V_{h,m}^\Gamma$ such that

$$\int_{\Gamma_h} \nabla u_h \cdot \nabla v_h \,\mathrm{d}s_h = \int_{\Gamma_h} f_h v_h \,\mathrm{d}s_h \quad \text{for all } v_h \in V_{h,m}^\Gamma, \tag{1.1}$$

$$\int_{\Gamma_h} \nabla_{\Gamma_h} u_h^\Gamma \cdot \nabla_{\Gamma_h} v_h \,\mathrm{d}s_h = \int_{\Gamma_h} f_h v_h \,\mathrm{d}s_h \quad \text{for all } v_h \in V_{h,m}^\Gamma, \tag{1.2}$$

with $V_{h,m}^\Gamma$ a trace finite element space (precise definition given below) with piecewise polynomials of degree m, Γ_h an approximation of Γ and f_h an approximation of the exact data f. Method (1.2) is the original trace finite element method introduced and analysed, for the case $m=1$, in Olshanskii *et al.* (2009). Method (1.1) is introduced and analysed, for the case $m=1$, in Ranner (2013) and Deckelnick *et al.* (2013). In the latter references it is shown that this method has optimal order of convergence for piecewise linear trace elements. Method (1.1) has two advantages compared to (1.2). First, it is more stable in the sense that $\|\nabla v_h\|_{L^2(\Gamma_h)} \geqslant \|\nabla_{\Gamma_h} v_h\|_{L^2(\Gamma_h)}$ holds. This affects the conditioning of the stiffness matrix; cf. discussion below. Second, if Γ_h is given implicitly, the implementation of (1.1) is in general simpler than that of (1.2) because in the former we only have to evaluate functions on Γ_h and we do not need any information about normals on Γ_h. On the other hand, although the two methods have the same order of convergence, the discretization error in u_h is in general larger than in u_h^Γ.

The two main contributions of this paper are the following. First, we present a *discretization error analysis of both methods in one general framework*. We do not restrict to the case $m=1$, but allow arbitrary degree m finite element polynomials. Furthermore, we do not consider a specific construction of Γ_h (e.g., by interpolating Γ or by using level set functions) but only assume that Γ_h satisfies certain accuracy conditions, e.g., $\mathrm{dist}(\Gamma_h,\Gamma) \leqslant ch^{k+1}$ and $\|n - n_h\|_{L^\infty(\Gamma_h)} \leqslant ch^k$ (with n and n_h the normals on Γ, Γ_h). The analysis explains why in general the method (1.1) can be expected to be less accurate than (1.2). Furthermore, the analysis reveals the different roles of the data approximation error (replacing f by f_h), the finite element approximation error (quality of $V_{h,m}$) and the geometric error (approximation of Γ by Γ_h). We derive optimal error bounds both in H^1 and L^2 norms, e.g., $\|u^\mathrm{e} - u_h^{(\Gamma)}\|_{L^2(\Gamma_h)} \leqslant c(h^{m+1} + h^{k+1})$. To our knowledge, neither for (1.1) nor for (1.2) are error bounds for $m \geqslant 2$ known in the literature. In relation to the geometric error, we *assume that the integrals in* (1.1) *and* (1.2) *can be determined exactly*. In practice, for the case of higher-order approximations Γ_h of Γ (i.e., $k \geqslant 2$), this is often not a realistic assumption. If the exact distance function to Γ is known, one can use polynomial approximations Γ_h as presented in Demlow (2009) to satisfy this assumption. If, however, Γ is given implicitly (via a level set function) it is not obvious how to satisfy this assumption. This topic in relation to quadrature errors in the evaluation of the integrals in (1.1) and (1.2) is treated in the recent preprint Grande & Reusken (2014).

The second main contribution is related to *linear algebra aspects*. For this, we restrict to the case $m = 1$. For the trace finite element method, the conditioning properties of the mass and stiffness matrices are different from those of standard finite element discretizations of elliptic problems. This topic is addressed in Olshanskii & Reusken (2010). Only if certain (fairly reasonable) conditions on how the approximate surface Γ_h intersects the outer volume triangulation are fulfilled, do the diagonally scaled mass matrix for $V_{h,1}$ and the diagonally scaled stiffness matrix for (1.2) have condition numbers that behave like h^{-2}. Recently, in Burman *et al.* (2013) a stabilization procedure for the discretization (1.2) was introduced which results in a stiffness matrix with a condition number $\sim h^{-2}$, *independently* of how Γ_h intersects the outer volume triangulation. As mentioned above, discretization (1.1) is more stable than (1.2). In particular, the conditioning of the stiffness matrix corresponding to (1.1) is better than that of (1.2). We prove that, without any stabilization, the stiffness matrix for (1.1), with an appropriate scaling, has a condition number $\sim h^{-2}$, *independently* of how Γ_h intersects the outer volume triangulation. As far as we know, linear algebra aspects related to (1.1) have not been studied in the literature, yet.

We include a section with results of a numerical experiment in which (for $k = 1$) the two methods are compared.

2. Laplace–Beltrami equation and finite element discretizations

As a model problem for an elliptic equation we consider the pure diffusion (i.e., Laplace–Beltrami) equation. We assume that Ω is an open subset in $\mathbb{R}^3$ which contains a connected compact smooth hypersurface Γ without boundary. The (outward-pointing) normal on Γ is denoted by n_Γ. For a sufficiently smooth function $g : \Omega \to \mathbb{R}$ the tangential derivative is defined by

$$\nabla_\Gamma g = (I - n_\Gamma n_\Gamma^{\mathrm{T}})\nabla g. \tag{2.1}$$

By $\Delta_\Gamma = \nabla_\Gamma \cdot \nabla_\Gamma$ we denote the *Laplace–Beltrami operator* on Γ. We consider the Laplace–Beltrami problem in weak form: for given $f \in L^2(\Gamma)$ with $\int_\Gamma f \,\mathrm{d}s = 0$, determine $u \in H^1(\Gamma)$ with $\int_\Gamma u \,\mathrm{d}s = 0$ such that

$$\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v \,\mathrm{d}s = \int_\Gamma fv \,\mathrm{d}s \quad \text{for all } v \in H^1(\Gamma). \tag{2.2}$$

The solution u is unique and satisfies $u \in H^2(\Gamma)$ with $\|u\|_{H^2(\Gamma)} \leqslant c\|f\|_{L^2(\Gamma)}$ and a constant c independent of f; cf. Dziuk (1988).

We introduce two *trace* finite element methods for the discretization of this equation. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of tetrahedral triangulations of the domain $\Omega \subset \mathbb{R}^3$ that contains Γ. These triangulations are assumed to be regular, consistent and stable (Braess, 2007). Given $\mathcal{T}_h$, we need an approximation Γ_h of Γ. Possible constructions of Γ_h and precise conditions that Γ_h has to satisfy will be discussed further on. For the definition of the method, we assume (only) that Γ_h is a Lipschitz hypersurface without boundary, which is 'close to' Γ. The local triangulation $\mathcal{T}_h^\Gamma \subset \mathcal{T}_h$ is defined by $\mathcal{T}_h^\Gamma = \{T \in \mathcal{T}_h \mid \mathrm{meas}_2(\Gamma_h \cap T) > 0\}$. If $\Gamma_h \cap T$ consists of a face F of T, we include in $\mathcal{T}_h^\Gamma$ only one of the two tetrahedra which have this F as their intersection. The domain formed by the triangulation $\mathcal{T}_h^\Gamma$ is denoted by ω_h. On the local domain

ω_h we define the standard space of H^1-conforming finite elements, with finite elements of degree $m \geqslant 1$:

$$V_{h,m} := \{v_h \in C(\omega_h) \mid v_{h|T} \in \mathcal{P}_m \text{ for all } T \in \mathcal{T}_h^\Gamma\}. \tag{2.3}$$

We also define the corresponding *trace space*:

$$V_{h,m}^\Gamma := \{v_{h|\Gamma_h} \mid v_h \in V_{h,m}\}, \quad V_{h,m}^{\Gamma,0} := \left\{ v_h \in V_{h,m}^\Gamma \;\middle|\; \int_{\Gamma_h} v_h \, \mathrm{d}s_h = 0 \right\}. \tag{2.4}$$

On Γ_h we need an approximation of the data f, denoted by f_h. We assume that $\int_{\Gamma_h} f_h \, \mathrm{d}s_h = 0$ holds. In this paper we consider the following two discretization methods: (i) find $u_h \in V_{h,m}^{\Gamma,0}$ such that

$$\int_{\Gamma_h} \nabla u_h \cdot \nabla v_h \, \mathrm{d}s_h = \int_{\Gamma_h} f_h v_h \, \mathrm{d}s_h \quad \text{for all } v_h \in V_{h,m}^\Gamma \tag{2.5}$$

and (ii) find $u_h^\Gamma \in V_{h,m}^{\Gamma,0}$ such that

$$\int_{\Gamma_h} \nabla_{\Gamma_h} u_h^\Gamma \cdot \nabla_{\Gamma_h} v_h \, \mathrm{d}s_h = \int_{\Gamma_h} f_h v_h \, \mathrm{d}s_h \quad \text{for all } v_h \in V_{h,m}^\Gamma. \tag{2.6}$$

These discrete problems have unique solutions. This follows from $\|\nabla_{\Gamma_h} v_h\|_{L^2(\Gamma_h)} \leqslant \|\nabla v_h\|_{L^2(\Gamma_h)}$ and the fact that $\|\nabla_{\Gamma_h} v_h\|_{L^2(\Gamma_h)} = 0$ implies that v_h is constant on Γ_h.

Remark 2.1 In relation to (2.5) there is the following subtle issue related to nonuniqueness of ∇u_h, ∇v_h. For $v_h \in V_{h,m}^\Gamma$ there may be different $w_h, \tilde{w}_h \in V_{h,m}$ with $v_h = w_{h|\Gamma_h} = \tilde{w}_{h|\Gamma_h}$ (similarly for u_h). Hence one has a choice: $\nabla v_h := \nabla w_h$ or $\nabla v_h := \nabla \tilde{w}_h$. This freedom in the choice of ∇v_h (and ∇u_h) influences neither the unique solvability nor the discretization error (analysis). Below, for the unique discrete solution $u_h \in V_{h,m}^{\Gamma,0}$ *we always use one and the same extension to* $V_{h,m}$, *which is also denoted by* u_h. In the implementation of (2.5) this ambiguity is not noticed, because u_h and v_h are represented as traces of outer finite element functions, e.g., $v_h = \phi_i$, with $\phi_i \in V_{h,m}$ a nodal finite element basis function. Then $\nabla v_h := \nabla \phi_i$ is a natural, unique choice. The nonuniqueness is then 'hidden' in the fact that the set of traces of the outer finite element basis functions ϕ_i form only a frame (in general not a basis) of the trace space $V_{h,m}^\Gamma$.

3. Preliminaries

In the analysis of the methods introduced above we always assume that Γ is sufficiently smooth. We do not specify the required smoothness of Γ. The signed distance function to Γ is denoted by d, with d negative in the interior of Γ. On

$$U_\delta := \{x \in \mathbb{R}^3 \mid |d(x)| < \delta\}, \tag{3.1}$$

with $\delta > 0$ sufficiently small, we define

$$n(x) = \nabla d(x), \quad H(x) = D^2 d(x), \quad P(x) = I - n(x)n(x)^{\mathrm{T}}, \tag{3.2}$$

$$p(x) = x - d(x)n(x), \quad v^{\mathrm{e}}(x) = v(p(x)) \quad \text{for } v \text{ defined on } \Gamma. \tag{3.3}$$

The eigenvalues of $H(x)$ are denoted by $\kappa_1(x), \kappa_2(x)$ and 0. Note that v^{e} is simply the constant extension of v (given on Γ) along the normals n. The tangential derivative can be written as $\nabla_\Gamma g(x) = P(x)\nabla g(x)$

for $x \in \Gamma$. We assume $\delta_0 > 0$ to be sufficiently small such that on U_{δ_0} the decomposition

$$x = p(x) + d(x)n(x)$$

is unique for all $x \in U_{\delta_0}$. In the remainder we only consider U_δ with $0 < \delta \leqslant \delta_0$. In the analysis we use the following formulas from Demlow & Dziuk (2007):

$$\nabla u^{\mathrm{e}}(x) = (I - d(x)H(x))\nabla_\Gamma u(p(x)) \quad \text{a.e on } U_{\delta_0}, \;\; u \in H^1(\Gamma), \tag{3.4}$$

$$\kappa_i(\mathbf{x}) = \frac{\kappa_i(p(x))}{1 + d(x)\kappa_i(p(x))} \quad \text{for } x \in U_{\delta_0}, \;\; i = 1, 2. \tag{3.5}$$

The first one follows from differentiating the relation $u^{\mathrm{e}}(x) = u(p(x))$ and using $\nabla p(x) = P(x) - d(x)H(x)$. Using the result (3.5) one obtains that if $\delta \in (0, \delta_0]$ satisfies

$$5\delta < \left(\max_{i=1,2} \|\kappa_i\|_{L^\infty(\Gamma)} \right)^{-1}, \tag{3.6}$$

then

$$\|d\|_{L^\infty(U_\delta)} \max_{i=1,2} \|\kappa_i\|_{L^\infty(U_\delta)} \leqslant \tfrac{1}{4} \tag{3.7}$$

holds. In the following lemma Sobolev norms on U_δ of the normal extension u^{e} are related to corresponding norms on Γ. Such results are known in the literature, e.g., Dziuk (1988), Demlow & Dziuk (2007). For completeness we include a proof. Note that these results only involve Γ and its neighbourhood U_δ. The approximate surface Γ_h does not play a role.

LEMMA 3.1 Let (3.6) be satisfied. For all $u \in H^m(\Gamma)$ the following holds:

$$\|D^\mu u^{\mathrm{e}}\|_{L^2(U_\delta)} \leqslant c\sqrt{\delta}\|u\|_{H^m(\Gamma)}, \quad |\mu| = m \geqslant 0, \tag{3.8}$$

with a constant c independent of δ and u.

Proof. Define

$$\mu(x) := (1 - d(x)\kappa_1(x))(1 - d(x)\kappa_2(x)), \quad x \in U_\delta.$$

From Demlow & Dziuk (2007, (2.20), (2.23)) we have

$$\mu(x)\mathrm{d}x = \mathrm{d}r\,\mathrm{d}s(p(x)), \quad x \in U_\delta,$$

where dx is the volume measure in U_δ, ds is the surface measure on Γ and r is the local coordinate at $x \in \Gamma$ in the direction $n(p(x)) = n(x)$. Using (3.7) we obtain

$$\tfrac{9}{16} \leqslant \mu(x) \leqslant \tfrac{25}{16} \quad \text{for all } x \in U_\delta. \tag{3.9}$$

Using the local coordinate representation $x = (p(x), r)$, for $x \in U_\delta$, we have

$$\begin{aligned}\int_{U_\delta} u^{\mathrm{e}}(x)^2 \mu(x)\,\mathrm{d}x &= \int_{-\delta}^{\delta} \int_\Gamma [u^{\mathrm{e}}(p(x), r)]^2\,\mathrm{d}s(p(x))\,\mathrm{d}r \\ &= \int_{-\delta}^{\delta} \int_\Gamma [u(p(x), 0)]^2\,\mathrm{d}s(p(x))\,\mathrm{d}r = 2\delta\|u\|^2_{L^2(\Gamma)}.\end{aligned}$$

Combining this with (3.9) yields the result for $m=0$. Using (3.4) we obtain

$$\int_{U_\delta} [\nabla u^{\mathrm{e}}(x)]^2 \mu(x)\,\mathrm{d}x = \int_{-\delta}^{\delta}\int_{\Gamma} [(I - d(x)H(x))\nabla_\Gamma u(p(x))]^2\,\mathrm{d}s(p(x))\,\mathrm{d}r.$$

In combination with $\|I - dH\|_{L^\infty(U_\delta)} \leqslant c$ we obtain the result for $m=1$. For $m \geqslant 2$ the same argument can be applied repeatedly if we differentiate (3.4) and use the chain rule. □

In the remainder we assume that δ_0 is sufficiently small such that it satisfies (3.6).

4. Approximation error bounds

From $\|\nabla_{\Gamma_h} v(x)\| \leqslant \|\nabla v(x)\|$ it follows that

$$\min_{v_h \in V_{h,m}^\Gamma} (\|u^{\mathrm{e}} - v_h\|_{L^2(\Gamma_h)} + h\|\nabla_{\Gamma_h}(u^{\mathrm{e}} - v_h)\|_{L^2(\Gamma_h)}) \tag{4.1}$$

$$\leqslant \min_{v_h \in V_{h,m}^\Gamma} (\|u^{\mathrm{e}} - v_h\|_{L^2(\Gamma_h)} + h\|\nabla(u^{\mathrm{e}} - v_h)\|_{L^2(\Gamma_h)}) \tag{4.2}$$

holds. In this section we derive bounds for the approximation error on the right-hand side. The analysis is simpler than the one presented in Olshanskii *et al.* (2009). This is due to Lemma 4.3, which was not used in Olshanskii *et al.* (2009). Furthermore, in Olshanskii *et al.* (2009) only $m=1$ (linear finite elements) is treated, whereas below we treat $m \geqslant 1$.

For the derivation of an optimal approximation error bound we need some mild assumptions on the family of approximate surfaces $\{\Gamma_h\}_{h>0}$, in particular on how Γ_h is related to the triangulation $\mathcal{T}_h$. In Remark 4.2 we discuss a few standard cases in which these assumptions are satisfied. The closed connected Lipschitz manifold Γ_h can be partitioned as follows:

$$\Gamma_h = \cup_{T \in \mathcal{T}_h^\Gamma} \Gamma_T, \quad \Gamma_T := \Gamma_h \cap T.$$

The unit normal (pointing outwards from the interior of Γ_h) is denoted by $n_h(x)$, and is defined almost everywhere (a.e.) on Γ_h.

ASSUMPTION 4.1 (A1) We assume that there is a constant c_0 independent of h such that for the local domain ω_h, we have

$$\omega_h \subset U_\delta \quad \text{with } \delta = c_0 h \leqslant \delta_0. \tag{4.3}$$

(A2) We assume that for each $T \in \mathcal{T}_h^\Gamma$ the local surface section Γ_T consists of connected parts $\Gamma_T^{(i)}$, $i = 1, \dots p$, such that $\partial \Gamma_T^{(i)} \cap \partial T$ is a simple closed curve and $\|n_h(x) - n_h(y)\| \leqslant c_1 h$ holds for $x, y \in \Gamma_T^{(i)}$. The number p and constant c_1 are uniformly bounded w.r.t. h and $T \in \mathcal{T}_h$.

REMARK 4.2 Condition (A1) essentially means that $\mathrm{dist}(\Gamma_h, \Gamma) \leqslant c_0 h$ holds, which is a very mild condition on the accuracy of Γ_h as an approximation of Γ. The condition ensures that the local triangulation $\mathcal{T}_h^\Gamma$ has sufficient resolution for representing the surface Γ approximately. Condition (A2) allows multiple intersections (namely p) of Γ_h with one tetrahedron $T \in \mathcal{T}_h^\Gamma$. An illustration for the two-dimensional case is shown in Fig. 1. We discuss three situations in which Assumption 4.1 is satisfied. For the case $\Gamma_h = \Gamma$ and with h sufficiently small, the conditions in Assumption 4.1 hold. If Γ_h is a shape-regular triangulation, consisting of triangles with diameter $\mathcal{O}(h)$ and vertices on Γ, then for h sufficiently small the conditions are satisfied. Finally,

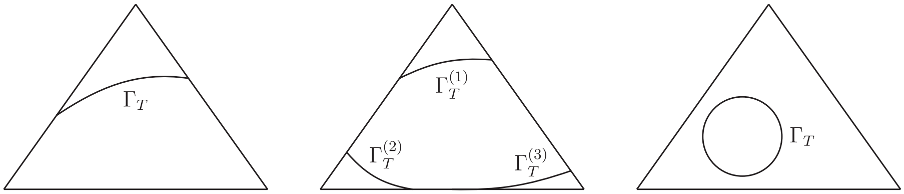

FIG. 1. Illustration of local surface sections $\Gamma_T^{(i)}$; cf. Assumption 4.1 (A2). The left-hand picture is the generic case ($p=1$); the middle picture has $p=3$ intersections; the situation in the right-hand picture is not allowed.

consider the case in which Γ is the zero level of a smooth level set function ϕ and ϕ_h is a finite element approximation of ϕ, on the triangulation $\mathcal{T}_h$. Let Γ_h be the zero level of ϕ_h. If $\|\phi-\phi_h\|_{L^\infty(\omega_h)} + h\|\nabla(\phi-\phi_h)\|_{L^\infty(\omega_h)} \leqslant ch^2$ holds, then the conditions are satisfied, provided h is sufficiently small.

A (slightly) simplified version of the following lemma is presented in Hansbo & Hansbo (2002, 2004).

LEMMA 4.3 Let Assumption 4.1 (A2) be satisfied. There exist constants c, $h_0>0$, independent of how Γ_h intersects $\mathcal{T}_h^\Gamma$, and with c independent of h, such that for $h \leqslant h_0$ the following holds. For all $T \in \Gamma_h^\Gamma$ and all $v \in H^1(T)$,

$$\|v\|_{L^2(\Gamma_T)}^2 \leqslant c\big(h_T^{-1}\|v\|_{L^2(T)}^2 + h_T\|\nabla v\|_{L^2(T)}^2\big), \tag{4.4}$$

with $h_T := \mathrm{diam}(T)$.

Proof. Since $\mathcal{T}_h^\Gamma$ is a shape-regular triangulation, there is a constant independent of h such that

$$\|v\|_{L^2(\partial T)}^2 \leqslant c(h_T^{-1}\|v\|_{L^2(T)}^2 + h_T\|\nabla v\|_{L^2(T)}^2) \quad \text{for all } v \in H^1(T) \tag{4.5}$$

holds; cf. Brenner & Scott (2002). Take $T \in \mathcal{T}_h^\Gamma$ and let $\tilde{\Gamma}_T = \Gamma_T^{(i)}$ be one of the parts of Γ_T as described in (A2). If $\tilde{\Gamma}_T$ coincides with a face of T, the result (4.4) immediately follows from (4.5). If this is not the case, the local surface section $\tilde{\Gamma}_T$ divides T into two disjoint subdomains T_1, T_2, with $T_1 \cup T_2 = T$ and $\mathrm{meas}_3(T_i) > 0$ for $i=1,2$. From (A2) it follows that for $i=1$ or $i=2$ we have $\partial\tilde{\Gamma}_T \subset \partial T_i$ and $(\partial T_i \setminus \tilde{\Gamma}_T) \subset \partial T$. We assume that this holds for $i=1$. Take $x_0 \in \tilde{\Gamma}_T$ such that $n_h(x_0)$ exists; we assume that $n_h(x_0)$ is outward pointing from T_1 (otherwise we change the sign). We choose an orthogonal coordinate system $z=(z_1,z_2,z_3)$ with origin at x_0 and the third basis vector equal to $n_h(x_0)$. The entries of the normal vector $n_h(y)$, $y \in \tilde{\Gamma}_T$ in the z-coordinate system are defined by $n_h(y) = (n_h^1(y), n_h^2(y), n_h^3(y))$. Hence, $n_h(x_0) = (0,0,1)$. From Assumption 4.1 (A2) we obtain

$$|n_h^3(y) - 1| \leqslant \|n_h(y) - n_h(x_0)\| \leqslant c_1 h \quad \text{for } y \in \tilde{\Gamma}_T.$$

Thus there is a constant c such that, for h sufficiently small, $1 \leqslant n_h^3(y)^{-1} \leqslant c$ holds a.e. on $\tilde{\Gamma}_T$. For $v \in H^1(T)$, we obtain

$$2\int_{T_1} v\frac{\partial v}{\partial z_3}\,\mathrm{dz} = \int_{T_1} \mathrm{div}_z \begin{pmatrix}0\\0\\v^2\end{pmatrix}\mathrm{dz} = \int_{\partial T_1} n_{T_1}\cdot\begin{pmatrix}0\\0\\v^2\end{pmatrix}\mathrm{dz}$$
$$= \int_{\tilde{\Gamma}_T} n_h^3 v^2\,\mathrm{dz} + \int_{\partial T_1\setminus\tilde{\Gamma}_T} n_{T_1}^3 v^2\,\mathrm{dz}.$$

Using $n_h^3(y)^{-1} \leqslant c$ we obtain

$$\int_{\tilde{\Gamma}_T} v^2\,\mathrm{dz} \leqslant c\left(\int_{T_1} v\frac{\partial v}{\partial z_3}\,\mathrm{dz} - \int_{\partial T_1\setminus\tilde{\Gamma}_T} n_T^3 v^2\,\mathrm{dz}\right)$$
$$\leqslant c\big(\|v\|_{L^2(T)}\|\nabla v\|_{L^2(T)} + \|v\|_{L^2(\partial T)}^2\big) \leqslant c\big(h_T^{-1}\|v\|_{L^2(T)}^2 + h_T\|\nabla v\|_{L^2(T)}^2\big),$$

where in the last inequality we used (4.5). Summing over the parts $\Gamma_T^{(i)}$, $i = 1,\ldots,p$ and using that p is uniformly bounded, we obtain estimate (4.4). □

In the remainder we assume that h is sufficiently small. In particular, $h \leqslant h_0$ as in Lemma 4.3 is assumed to be satisfied. As an easy.

THEOREM 4.4 Let Assumption 4.1 be satisfied. Let $I_h : C(\omega_h) \to V_{h,m}$ be the nodal interpolation. There exists a constant c, independent of h and of how Γ_h intersects $\mathcal{T}_h^\Gamma$, such that

$$\min_{v_h \in V_{h,m}^\Gamma}\big(\|u^e - v_h\|_{L^2(\Gamma_h)} + h\|\nabla(u^e - v_h)\|_{L^2(\Gamma_h)}\big)$$
$$\leqslant \|u^e - I_h u^e\|_{L^2(\Gamma_h)} + h\|\nabla(u^e - I_h u^e)\|_{L^2(\Gamma_h)} \leqslant ch^{m+1}\|u\|_{H^{m+1}(\Gamma)} \tag{4.6}$$

for all $u \in H^{m+1}(\Gamma)$ holds.

Proof. From $u \in H^{m+1}(\Gamma)$ it follows that $u^e \in H^{m+1}(\omega_h)$. Using Lemma 4.3 and standard error bounds for the nodal interpolation I_h we obtain, with $v_h := I_h u^e \in V_{h,m}$,

$$\|u^e - v_h\|_{L^2(\Gamma_h)}^2 + h^2\|\nabla(u^e - v_h)\|_{L^2(\Gamma_h)}^2$$
$$= \sum_{T\in\mathcal{T}_h^\Gamma}\big(\|u^e - v_h\|_{L^2(\Gamma_T)}^2 + h^2\|\nabla(u^e - v_h)\|_{L^2(\Gamma_T)}^2\big)$$
$$\leqslant c\sum_{T\in\mathcal{T}_h^\Gamma}\big(h_T^{-1}\|u^e - v_h\|_{L^2(T)}^2 + (h^2 h_T^{-1} + h_T)\|\nabla(u^e - v_h)\|_{L^2(T)}^2$$
$$+ h^2 h_T\|\nabla^2(u^e - v_h)\|_{L^2(T)}^2\big)$$
$$\leqslant c\sum_{T\in\mathcal{T}_h^\Gamma} h^{2m+1}\|u^e\|_{H^{m+1}(T)}^2 = ch^{2m+1}\|u^e\|_{H^{m+1}(\omega_h)}^2.$$

Using Assumption 4.1 (A1) and (3.8) with $\delta = c_0 h$ we obtain $\|u^e\|_{H^{m+1}(\omega_h)}^2 \leqslant ch\|u\|_{H^{m+1}(\Gamma)}^2$, which completes the proof. □

From the result in this theorem we conclude that for the trace space $V_{h,m}^{\Gamma}$ we have optimal approximation error bounds under (very) mild conditions on the approximate surface Γ_h. If Γ and the exact solution u are sufficiently smooth, we obtain an h^{m+1} bound as in (4.6) (for finite elements of degree m), provided Assumption 4.1 is satisfied. The latter essentially only requires the accuracy estimate $\mathrm{dist}(\Gamma_h, \Gamma) \leqslant ch$ for the approximate surface.

5. Finite element error bounds

In this section we prove optimal discretization error bounds both in the $H^1(\Gamma_h)$ and the $L^2(\Gamma_h)$ norm. For the discrete problem (2.5) such bounds for $m = 1$ (piecewise linear finite elements) are derived in Deckelnick *et al.* (2013). For the discrete problem (2.6) these error bounds for $m = 1$ are derived in Olshanskii *et al.* (2009). In both references it is assumed that Γ_h is a piecewise planar approximation of Γ with $\mathrm{dist}(\Gamma_h, \Gamma) \leqslant ch^2$. In this section we consider a more general setting with $m \geqslant 1$ and more general approximate surfaces Γ_h. Furthermore, we present the error analysis of the two discretizations in one unified setting, which reveals the main (theoretical) differences between the two methods.

In the analysis we need one further assumption, which quantifies the quality of Γ_h as an approximation of Γ ('geometric error').

ASSUMPTION 5.1 We assume that $\Gamma_h \subset U_{\delta_0}$ is a Lipschitz surface without boundary and that the projection $p : \Gamma_h \to \Gamma$ is a bijection. The corresponding unit normal field is defined a.e. on Γ_h and denoted by n_h. We assume that the following holds, for a $k \geqslant 1$:

$$\|d\|_{L^\infty(\Gamma_h)} \leqslant ch^{k+1}, \tag{5.1}$$

$$\|n - n_h\|_{L^\infty(\Gamma_h)} \leqslant ch^k. \tag{5.2}$$

These are the key assumptions we need in the analysis below. There is one further assumption we introduce. On each Γ_h there holds a Poincaré inequality with a constant $c = c(h)$. We assume that this constant is uniform w.r.t. h, i.e., we assume that there exists c, independent of h, such that

$$\|v\|_{L^2(\Gamma_h)} \leqslant c\|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} \quad \text{for all } v \in H^1(\Gamma_h)/\mathbb{R}. \tag{5.3}$$

REMARK 5.2 We discuss cases in which assumptions (5.1) and (5.2) are satisfied. Clearly, if $\Gamma_h = \Gamma$, there is no geometric error, i.e., these assumptions are fulfilled with $k = \infty$. Consider the case in which Γ is the zero level of a smooth level set function ϕ and ϕ_h is a finite element approximation of ϕ, on the triangulation $\mathcal{T}_h$. Let Γ_h be the zero level of ϕ_h. If $\|\phi - \phi_h\|_{L^\infty(\omega_h)} + h\|\nabla(\phi - \phi_h)\|_{L^\infty(\omega_h)} \leqslant ch^{k+1}$ holds, then conditions (5.1) and (5.2) are satisfied. In Demlow (2009) a method for constructing polynomial approximations to Γ is presented that satisfies conditions (5.1) and (5.2) (cf. Demlow, 2009, Proposition 2.3). In that method the exact distance function to Γ is needed.

REMARK 5.3 It can be shown that under reasonable assumptions on Γ_h, the uniform Poincaré inequality (5.3) holds. We sketch a proof. Let $v \in H^1(\Gamma_h)/\mathbb{R}$ be given. We define its lift to Γ by $v^\ell(p(x)) = v(x)$ for $x \in \Gamma_h$. We use a transformation formula $\int_\Gamma (v^\ell)^2 \,\mathrm{d}s = \int_{\Gamma_h} v^2 \mu_h \,\mathrm{d}s_h$, with μ_h as given in, e.g., Demlow & Dziuk (2007). This μ_h satisfies (under reasonable assumptions on Γ_h) $\|1 - \mu_h\|_{L^\infty(\Gamma_h)} \leqslant ch^2$. Define $c_v := (1/|\Gamma|) \int_\Gamma v^\ell \,\mathrm{d}s$. Using $\int_{\Gamma_h} v \,\mathrm{d}s_h = 0$ and the bound for $\|1 - \mu_h\|_{L^\infty(\Gamma_h)}$ we get $|c_v| \leqslant ch^2 \|v\|_{L^2(\Gamma_h)}$.

For $\hat{v}^\ell := v^\ell - c_v$ the Poincaré inequality on Γ holds. Combining these results we obtain

$$\begin{aligned}\|v\|_{L^2(\Gamma_h)} &\leqslant c\|v^\ell\|_{L^2(\Gamma)} \leqslant c(\|\hat{v}^\ell\|_{L^2(\Gamma)} + |c_v|)\\ &\leqslant c\|\nabla_\Gamma v^\ell\|_{L^2(\Gamma)} + ch^2\|v\|_{L^2(\Gamma_h)} \leqslant c\|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} + ch^2\|v\|_{L^2(\Gamma_h)},\end{aligned}$$

from which, for h sufficiently small, the uniform estimate (5.3) follows.

We define the following projections:

$$P_h(x) = I - n_h(x)n_h(x)^{\mathrm{T}}, \quad \tilde{P}_h(x) = I - n_h(x)n(x)^{\mathrm{T}}/(n_h(x)^{\mathrm{T}}n(x)), \quad x \in \Gamma_h.$$

We collect a few results from Demlow & Dziuk (2007). The surface gradient of $u \in H^1(\Gamma)$ can be represented in terms of $\nabla_{\Gamma_h} u^{\mathrm{e}}$ as follows:

$$\nabla_\Gamma u(p(x)) = (I - d(x)H(x))^{-1}\tilde{P}_h(x)\nabla_{\Gamma_h} u^{\mathrm{e}}(x) \quad \text{a.e. on } \Gamma_h. \tag{5.4}$$

For $x \in \Gamma_h$ define

$$\mu_h(x) = (1 - d(x)\kappa_1(x))(1 - d(x)\kappa_1(x))n(x)^{\mathrm{T}}n_h(x).$$

The integral transformation formula

$$\mu_h(x)\,\mathrm{d}s_h(x) = \mathrm{d}s(p(x)), \quad x \in \Gamma_h \tag{5.5}$$

holds, where $\mathrm{d}s_h(x)$ and $\mathrm{d}s(p(x))$ are the surface measures on Γ_h and Γ, respectively. From $\|n(x) - n_h(x)\|^2 = 2(1 - n(x)^{\mathrm{T}}n_h(x))$ and Assumption 5.1 we obtain

$$\|1 - \mu_h\|_{L^\infty(\Gamma_h)} \leqslant ch^{k+1}, \tag{5.6}$$

with a constant c independent of h. Using relations (5.4) and (5.5) we obtain

$$\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v\,\mathrm{d}s = \int_{\Gamma_h} A_h \nabla_{\Gamma_h} u^{\mathrm{e}} \cdot \nabla_{\Gamma_h} v^{\mathrm{e}}\,\mathrm{d}s_h \quad \text{for all } u, v \in H^1(\Gamma), \tag{5.7}$$

$$\text{with } A_h(x) = \mu_h(x)\tilde{P}_h(x)(I - d(x)H(x))^{-2}\tilde{P}_h(x). \tag{5.8}$$

We introduce a compact notation for the bilinear forms used in (2.5) and (2.6):

$$a_h(u_h, v_h) := \int_{\Gamma_h} \nabla u_h \cdot \nabla v_h\,\mathrm{d}s_h, \quad a_h^\Gamma(u_h, v_h) := \int_{\Gamma_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} v_h\,\mathrm{d}s_h.$$

Furthermore, for the data error we introduce the notation

$$\delta_f := f_h - \mu_h f^{\mathrm{e}}.$$

We now derive approximate Galerkin orthogonality relations for the discrete problems.

LEMMA 5.4 Let u be the solution of the Laplace–Beltrami equation (2.2) and $u_h, u_h^\Gamma \in V_{h,m}^\Gamma$ be the solutions of the discrete problems (2.5) and (2.6), respectively. Define $\hat{A}_h := P_h A_h P_h$, with A_h as in (5.8).

The following hold:

$$\begin{aligned} &a_h(u^{\mathrm{e}} - u_h, v_h) = F_h(v_h) \quad \text{for all } v_h \in V_{h,m}, \\ &\quad \text{with } F_h(v_h) := \int_{\Gamma_h} (I - \hat{A}_h)\nabla u^{\mathrm{e}} \cdot \nabla v_h \,\mathrm{d}s_h - \int_{\Gamma_h} \delta_f v_h \,\mathrm{d}s_h; \end{aligned} \tag{5.9}$$

$$\begin{aligned} &a_h^{\Gamma}(u^{\mathrm{e}} - u_h^{\Gamma}, v_h) = F_h^{\Gamma}(v_h) \quad \text{for all } v_h \in V_{h,m}, \\ &\quad \text{with } F_h^{\Gamma}(v_h) := \int_{\Gamma_h} (P_h - \hat{A}_h)\nabla u^{\mathrm{e}} \cdot \nabla v_h \,\mathrm{d}s_h - \int_{\Gamma_h} \delta_f v_h \,\mathrm{d}s_h. \end{aligned} \tag{5.10}$$

Proof. Take $v_h \in V_{h,m}$. The function $v_{h|\Gamma_h}$ can be lifted on Γ by defining $v_h^l(p(x)) := v_h(x)$, $x \in \Gamma_h$. From the definition of the discrete problem (2.5) and the transformation rule (5.7) we obtain

$$\begin{aligned} a_h(u_h, v_h) &= \int_{\Gamma_h} f_h v_h \,\mathrm{d}s_h = \int_{\Gamma} f v_h^l \,\mathrm{d}s + \int_{\Gamma_h} \delta_f v_h \,\mathrm{d}s_h \\ &= \int_{\Gamma} \nabla_\Gamma u \cdot \nabla_\Gamma v_h^l \,\mathrm{d}s + \int_{\Gamma_h} \delta_f v_h \,\mathrm{d}s_h = \int_{\Gamma_h} A_h \nabla_{\Gamma_h} u^{\mathrm{e}} \cdot \nabla_{\Gamma_h} v_h \,\mathrm{d}s + \int_{\Gamma_h} \delta_f v_h \,\mathrm{d}s_h \\ &= \int_{\Gamma_h} \hat{A}_h \nabla u^{\mathrm{e}} \cdot \nabla v_h \,\mathrm{d}s + \int_{\Gamma_h} \delta_f v_h \,\mathrm{d}s_h, \end{aligned}$$

where in the last equality we used that $\nabla_{\Gamma_h} v_h = P_h \nabla v_h$. Combining this with $a_h(u^{\mathrm{e}}, v_h) = \int_{\Gamma_h} \nabla u^{\mathrm{e}} \cdot \nabla v_h \,\mathrm{d}s_h$ we get the result in (5.9). Similar arguments can be used to derive (5.10):

$$\begin{aligned} a_h^{\Gamma}(u_h^{\Gamma}, v_h) &= \int_{\Gamma_h} \nabla_{\Gamma_h} u_h^{\Gamma} \cdot \nabla_{\Gamma_h} v_h \,\mathrm{d}\mathbf{s}_h = \int_{\Gamma} f v_h^l \,\mathrm{d}s + \int_{\Gamma_h} \delta_f v_h \,\mathrm{d}s_h \\ &= \int_{\Gamma_h} \hat{A}_h \nabla u^{\mathrm{e}} \cdot \nabla v_h \,\mathrm{d}s + \int_{\Gamma_h} \delta_f v_h \,\mathrm{d}s_h. \end{aligned}$$

We combine this with $a_h^{\Gamma}(u^{\mathrm{e}}, v_h) = \int_{\Gamma_h} P_h \nabla u^{\mathrm{e}} \cdot \nabla v_h \,\mathrm{d}s_h$ and thus obtain (5.10). □

Note that the only difference between the perturbation terms F_h and F_h^{Γ} in (5.9) and (5.10) is in the matrices $I - \hat{A}_h$ and $P_h - \hat{A}_h$. We derive bounds for the perturbation terms F_h and F_h^{Γ}. We need some additional notation, namely $H^2(\Gamma)^{\mathrm{e}} := \{v^{\mathrm{e}} \mid v \in H^2(\Gamma)\}$.

LEMMA 5.5 Let Assumption 5.1 be fulfilled and assume that the data error satisfies $\|\delta_f\|_{L^2(\Gamma_h)} \leqslant ch^{k+s}\|f\|_{L^2(\Gamma)}$ for an $s \in [0, 1]$. The following hold, with constants c independent of h:

$$|F_h(v)| \leqslant ch^k \|f\|_{L^2(\Gamma)} \left(\|v\|_{L^2(\Gamma_h)} + \|\nabla v\|_{L^2(\Gamma_h)} \right) \quad \text{for all } v \in V_{h,m} + H^2(\Gamma)^{\mathrm{e}}, \tag{5.11}$$

$$|F_h^{\Gamma}(v)| \leqslant ch^{k+s} \|f\|_{L^2(\Gamma)} \left(\|v\|_{L^2(\Gamma_h)} + \|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} \right) \quad \text{for all } v \in V_{h,m} + H^2(\Gamma)^{\mathrm{e}}, \tag{5.12}$$

$$|F_h(v^{\mathrm{e}})| \leqslant ch^{k+s} \|f\|_{L^2(\Gamma)} \left(\|v\|_{L^2(\Gamma)} + \|\nabla_\Gamma v\|_{L^2(\Gamma)} \right) \quad \text{for all } v \in H^2(\Gamma). \tag{5.13}$$

Proof. For the second term in $F_h(v)$ and $F_h^{\Gamma}(v)$ we obtain

$$\left| \int_{\Gamma_h} \delta_f v \,\mathrm{d}s_h \right| \leqslant \|\delta_f\|_{L^2(\Gamma_h)} \|v\|_{L^2(\Gamma_h)} \leqslant ch^{k+s} \|f\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma_h)}. \tag{5.14}$$

Below, we delete the argument $x \in \Gamma_h$ in the notation. Using (3.4), $P(p(x)) = P(x)$ and $HP = PH$, we obtain

$$(I - \hat{A}_h)\nabla u^{\mathrm{e}} = (I - \hat{A}_h)P(I - dH)\nabla_\Gamma u(p(x)). \tag{5.15}$$

We combine $\hat{A}_h = P_h A_h P_h$ with the definition of A_h and with (5.1), (5.6), $P_h\tilde{P}_h = P_h$ and obtain

$$\|\hat{A}_h - P_h\|_{L^\infty(\Gamma_h)} \leqslant ch^{k+1}. \tag{5.16}$$

Hence, using (5.2) yields

$$\begin{aligned} \|(I - \hat{A}_h)P\|_{L^\infty(\Gamma_h)} &\leqslant \|(I - P_h)P\|_{L^\infty(\Gamma_h)} + ch^{k+1} \\ &\leqslant \|P - P_h\|_{L^\infty(\Gamma_h)} + ch^{k+1} \leqslant ch^k. \end{aligned} \tag{5.17}$$

With the result in (5.15) we thus obtain

$$\begin{aligned} \left| \int_{\Gamma_h} (I - \hat{A}_h)\nabla u^{\mathrm{e}} \cdot \nabla v \, \mathrm{d}s_h \right| &\leqslant ch^k \|\nabla_\Gamma u(p(\cdot))\|_{L^2(\Gamma_h)} \|\nabla v\|_{L^2(\Gamma_h)} \\ &\leqslant ch^k \|\nabla_\Gamma u\|_{L^2(\Gamma)} \|\nabla v\|_{L^2(\Gamma_h)} \leqslant ch^k \|f\|_{L^2(\Gamma)} \|\nabla v\|_{L^2(\Gamma_h)}, \end{aligned} \tag{5.18}$$

and combining this with the result in (5.14) completes the proof of (5.11). In the definition of $F_h^\Gamma(v)$ we have the matrix $P_h - \hat{A}_h$, instead of $I - \hat{A}_h$. For the former we have (cf. (5.16)) $\|\hat{A}_h - P_h\|_{L^\infty(\Gamma_h)} \leqslant ch^{k+1}$. Furthermore, we have

$$(P_h - \hat{A}_h)\nabla u^{\mathrm{e}} \cdot \nabla v = P_h(P_h - \hat{A}_h)\nabla u^{\mathrm{e}} \cdot \nabla v = (P_h - \hat{A}_h)\nabla u^{\mathrm{e}} \cdot \nabla_{\Gamma_h} v.$$

Similarly to (5.18), we obtain

$$\begin{aligned} \left| \int_{\Gamma_h} (P_h - \hat{A}_h)\nabla u^{\mathrm{e}} \cdot \nabla v \, \mathrm{d}s_h \right| &\leqslant ch^{k+1} \|\nabla_\Gamma u(p(\cdot))\|_{L^2(\Gamma_h)} \|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} \\ &\leqslant ch^{k+1} \|f\|_{L^2(\Gamma)} \|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)}, \end{aligned}$$

and combining this with the result in (5.14) we get the bound (5.12). We finally consider the estimate (5.13). We use (3.4) and thus obtain (cf. (5.15))

$$(I - \hat{A}_h)\nabla u^{\mathrm{e}} \cdot \nabla v^{\mathrm{e}} = [(I - dH)P(I - \hat{A}_h)P(I - dH)]\nabla_\Gamma u(p(x)) \cdot \nabla_\Gamma v(p(x)).$$

For the matrix in the square brackets we have (cf. (5.16)),

$$\|(I - dH)P(I - \hat{A}_h)P(I - dH)\|_{L^\infty(\Gamma_h)} \leqslant \|P(I - P_h)P\|_{L^\infty(\Gamma_h)} + ch^{k+1}.$$

Using $P(I - P_h)P = Pn_h n_h^{\mathrm{T}} P = (P - P_h)n_h n_h^{\mathrm{T}}(P - P_h)$ and (5.2) we obtain $\|P(I - P_h)P\|_{L^\infty(\Gamma_h)} \leqslant ch^{2k}$. From this it follows that the norm of the matrix in the square brackets is bounded by ch^{k+1}. Using similar arguments as in the derivation of (5.12) above we then obtain the bound (5.13). □

REMARK 5.6 We comment on the data error $\|\delta_f\|_{L^2(\Gamma_h)}$, with $\delta_f = f_h - \mu_h f^{\mathrm{e}}$. For the choice $f_h = f^{\mathrm{e}} - (1/|\Gamma_h|)\int_{\Gamma_h} f^{\mathrm{e}} \, \mathrm{d}s_h$, which in practice often *cannot* be realized, we obtain, using (5.6), the data error bound $\|\delta_f\|_{L^2(\Gamma_h)} \leqslant ch^{k+1}\|f\|_{L^2(\Gamma)}$. For this data error bound we only need $f \in L^2(\Gamma)$, i.e., we avoid higher-order regularity assumptions on f. Another, more feasible, possibility arises if we assume f to be defined the neighbourhood U_{δ_0} of Γ. As an extension one can then use

$$f_h(x) = f(x) - c_f, \quad c_f := \frac{1}{|\Gamma_h|} \int_{\Gamma_h} f \, \mathrm{d}s_h. \tag{5.19}$$

Using $\int_\Gamma f \, \mathrm{d}s = 0$, (5.1), (5.6) and a Taylor expansion we obtain $|c_f| \leqslant ch^{k+1} \|f\|_{H^1_\infty(U_{\delta_0})}$ and $\|f - \mu_h f^{\mathrm{e}}\|_{L^2(\Gamma_h)} \leqslant ch^{k+1} \|f\|_{H^1_\infty(U_{\delta_0})}$. Hence, we get data error bound $\|\delta_f\|_{L^2(\Gamma_h)} \leqslant \hat{c}h^{k+1} \|f\|_{L^2(\Gamma)}$ with $\hat{c} = \hat{c}(f) = c\|f\|_{H^1_\infty(U_{\delta_0})} \|f\|^{-1}_{L^2(\Gamma)}$ and a constant c independent of f. Thus in problems with smooth data, $f \in H^1_\infty(U_{\delta_0})$, the extension defined in (5.19) satisfies the condition on the data error in Lemma 5.5 with $s = 1$. In less regular situations, $s < 1$ may be more realistic.

Note that for $s > 0$ the bound on F_h^Γ in (5.12) is of higher order in h than the one for F_h in (5.11). This difference is reflected in the discretization error bounds derived below. For $F_h(v^{\mathrm{e}})$ the higher-order bound in (5.13) is obtained by using the special structure of v^{e} (namely constant in the normal direction). The latter bound is used in the proof of the L^2 error bound in Theorem 5.8.

THEOREM 5.7 Let Assumptions 4.1 and 5.1 be fulfilled. Assume that the data error satisfies $\|\delta_f\|_{L^2(\Gamma_h)} \leqslant ch^{k+s} \|f\|_{L^2(\Gamma)}$ for an $s \in [0, 1]$. Let u_h and u_h^Γ be the solutions of the discrete problems (2.5) and (2.6), respectively. The following error bounds hold, with a constant c independent of h and f:

$$\|\nabla(u^{\mathrm{e}} - u_h)\|_{L^2(\Gamma_h)} \leqslant c\left(h^m \|u\|_{H^{m+1}(\Gamma)} + h^k \|f\|_{L^2(\Gamma)}\right), \tag{5.20}$$

$$\|\nabla_{\Gamma_h}(u^{\mathrm{e}} - u_h^\Gamma)\|_{L^2(\Gamma_h)} \leqslant c\left(h^m \|u\|_{H^{m+1}(\Gamma)} + h^{k+s} \|f\|_{L^2(\Gamma)}\right). \tag{5.21}$$

Proof. Define $e_h := u^{\mathrm{e}} - u_h$ and $\psi_h := I_h u^{\mathrm{e}} \in V_{h,m}$. We consider the splitting

$$\|\nabla e_h\|^2_{L^2(\Gamma_h)} = a_h(e_h, e_h) = a_h(e_h, u^{\mathrm{e}} - \psi_h) + F_h(\psi_h - u^{\mathrm{e}}) + F_h(e_h).$$

For the first two terms on the right-hand side we use (5.11) and the interpolation error bounds of Theorem 4.4 and thus obtain

$$\begin{aligned} a_h(e_h, u^{\mathrm{e}} - \psi_h) + F_h(\psi_h - u^{\mathrm{e}}) &\leqslant c(\|\nabla e_h\|_{L^2(\Gamma_h)} + h^k \|f\|_{L^2(\Gamma_h)}) h^m \|u\|_{H^{m+1}(\Gamma)} \\ &\leqslant \tfrac{1}{4} \|\nabla e_h\|^2_{L^2(\Gamma_h)} + ch^{2m} \|u\|^2_{H^{m+1}(\Gamma)} + ch^{2k} \|f\|^2_{L^2(\Gamma_h)}. \end{aligned} \tag{5.22}$$

For the third term we need the Poincaré inequality (5.3). Define $c_u = \int_{\Gamma_h} u^{\mathrm{e}} \, \mathrm{d}s_h$. Using $\int_\Gamma u \, \mathrm{d}s = 0$ and (5.6) we obtain $|c_u| \leqslant ch^{k+1} \|u\|_{L^2(\Gamma)} \leqslant ch^{k+1} \|f\|_{L^2(\Gamma)}$. Note that $\int_{\Gamma_h} e_h - c_u \, \mathrm{d}s_h = 0$ holds, hence with the Poincaré inequality we obtain

$$\begin{aligned} \|e_h\|_{L^2(\Gamma_h)} &\leqslant \|e_h - c_u\|_{L^2(\Gamma_h)} + ch^{k+1} \|f\|_{L^2(\Gamma)} \\ &\leqslant c\|\nabla_{\Gamma_h} e_h\|_{L^2(\Gamma_h)} + ch^{k+1} \|f\|_{L^2(\Gamma)} \leqslant c\|\nabla e_h\|_{L^2(\Gamma_h)} + ch^{k+1} \|f\|_{L^2(\Gamma)}, \end{aligned}$$

and using this in estimate (5.11) yields

$$F_h(e_h) \leqslant ch^k \|f\|_{L^2(\Gamma)} (\|\nabla e_h\|_{L^2(\Gamma_h)} + h^{k+1} \|f\|_{L^2(\Gamma)}) \leqslant \tfrac{1}{4} \|\nabla e_h\|^2_{L^2(\Gamma_h)} + ch^{2k} \|f\|^2_{L^2(\Gamma)}.$$

Combining this with the result in (5.22) proves the bound in (5.20). The result in (5.21) follows with very similar arguments. Define $e_h^\Gamma = u^{\mathrm{e}} - u_h^\Gamma$, and consider the splitting

$$\|\nabla_{\Gamma_h} e_h^\Gamma\|^2_{L^2(\Gamma_h)} = a_h^\Gamma(e_h^\Gamma, e_h^\Gamma) = a_h^\Gamma(e_h^\Gamma, u^{\mathrm{e}} - \psi_h) + F_h^\Gamma(\psi_h - u^{\mathrm{e}}) + F_h^\Gamma(e_h^\Gamma).$$

Note that $\|\nabla_{\Gamma_h}(u^{\mathrm{e}} - \psi_h)\|_{L^2(\Gamma_h)} \leqslant \|\nabla(u^{\mathrm{e}} - \psi_h)\|_{L^2(\Gamma_h)}$, and hence for bounding the interpolation error we can use Theorem 4.4. We can repeat the arguments used above. Since in the bound for $F_h^\Gamma(v)$ in (5.12) we have a term h^{k+s} (instead of h^k) we get the factor h^{k+s} in the bound (5.21). □

The result in this theorem yields optimal H^1-error bounds for both methods, and also for the case of higher-order finite elements ($m \geqslant 2$). Of course, this optimal bound is obtained only if the approximation error term, which is of order h^m, is not dominated by the geometric error term, which is of order h^k and h^{k+s}, respectively. Assume $s = 1$; cf. Remark 5.6. For the case $k = m$ (which typically holds in case of linear finite elements, i.e., $m = 1$), we see that for the method in (2.5) the geometric error is of the same order as the approximation error, whereas for the method (2.6) the geometric error is of higher order. For the method in (2.6) and $m \geqslant 2$ we get the optimal order of convergence h^m even if we only have $k = m - 1$. The method (2.5) does not have this property.

We apply a duality argument to obtain an $L^2(\Gamma_h)$-error bound. In this analysis the estimate (5.13) is used.

Theorem 5.8 Let Assumptions 4.1 and 5.1 be fulfilled. Assume that the data error satisfies $\|\delta_f\|_{L^2(\Gamma_h)} \leqslant ch^{k+s}\|f\|_{L^2(\Gamma)}$ for an $s \in [0, 1]$. Let u_h and u_h^Γ be the solutions of the discrete problems (2.5) and (2.6), respectively. The following error bounds hold, with a constant c independent of h and f:

$$\|u^{\mathrm{e}} - u_h\|_{L^2(\Gamma_h)} \leqslant c\big(h^{m+1}\|u\|_{H^{m+1}(\Gamma)} + h^{k+s}\|f\|_{L^2(\Gamma)}\big), \tag{5.23}$$

$$\|u^{\mathrm{e}} - u_h^\Gamma\|_{L^2(\Gamma_h)} \leqslant c\big(h^{m+1}\|u\|_{H^{m+1}(\Gamma)} + h^{k+s}\|f\|_{L^2(\Gamma)}\big). \tag{5.24}$$

Proof. Define $e_h := u^{\mathrm{e}} - u_h$ and let e_h^l be the lift of $e_{h|\Gamma_h}$ on Γ and $c_e := \int_\Gamma e_h^l \,\mathrm{d}s$. Consider the problem: Find $w \in H^1(\Gamma)$ with $\int_\Gamma w \,\mathrm{d}s = 0$ such that

$$\int_\Gamma \nabla_\Gamma w \cdot \nabla_\Gamma v \,\mathrm{d}s = \int_\Gamma (e_h^l - c_e) v \,\mathrm{d}s \quad \text{for all } v \in H^1(\Gamma). \tag{5.25}$$

The solution w satisfies $w \in H^2(\Gamma)$ and $\|w\|_{H^2(\Gamma)} \leqslant c\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}}$ with $\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}} := \|e_h^l - c_e\|_{L^2(\Gamma)}$. We take $\psi_h = I_h w^{\mathrm{e}} \in V_{h,m}$ and with $\hat{A}_h = P_h A_h P_h$ we obtain

$$\begin{aligned}
\|e_h^l\|^2_{L^2(\Gamma)/\mathbb{R}} &= \int_\Gamma \nabla_\Gamma w \cdot \nabla_\Gamma (e_h^l - c_e) \,\mathrm{d}s = \int_\Gamma \nabla_\Gamma w \nabla_\Gamma e_h^l \,\mathrm{d}s \\
&= \int_{\Gamma_h} A_h \nabla_{\Gamma_h} e_h \cdot \nabla_{\Gamma_h} w^{\mathrm{e}} \,\mathrm{d}s_h = \int_{\Gamma_h} \nabla e_h \cdot \nabla w^{\mathrm{e}} \,\mathrm{d}s_h + \int_{\Gamma_h} \nabla e_h \cdot (\hat{A}_h - I) \nabla w^{\mathrm{e}} \,\mathrm{d}s_h \\
&= a_h(e_h, w^{\mathrm{e}} - \psi_h) + F_h(\psi_h - w^{\mathrm{e}}) + F_h(w^{\mathrm{e}}) + \int_{\Gamma_h} \nabla e_h \cdot (\hat{A}_h - I) \nabla w^{\mathrm{e}} \,\mathrm{d}s_h.
\end{aligned} \tag{5.26}$$

We consider the four terms in (5.26). Using the interpolation error bound in Theorem 4.4 (with $m = 1$), the error bound in Theorem 5.7 and $\|w\|_{H^2(\Gamma)} \leqslant c\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}}$ we obtain

$$a_h(e_h, w^{\mathrm{e}} - \psi_h) \leqslant c(h^{m+1}\|u\|_{H^{m+1}(\Gamma)} + h^{k+1}\|f\|_{L^2(\Gamma)})\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}}.$$

For the second term we use (5.11) and the interpolation error bound (with $m=1$), which yields

$$F_h(\psi_h - w^{\mathrm{e}}) \leqslant ch^{k+1}\|f\|_{L^2(\Gamma)}\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}}.$$

For the third term we use (5.13) and obtain

$$F_h(w^{\mathrm{e}}) \leqslant ch^{k+s}\|f\|_{L^2(\Gamma)}\|w\|_{H^1(\Gamma)} \leqslant ch^{k+s}\|f\|_{L^2(\Gamma)}\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}}. \tag{5.27}$$

For the last term we use (5.15) (with u replaced by w) and (5.17), which yields

$$\begin{aligned}\int_{\Gamma_h} \nabla e_h \cdot (\hat{A}_h - I)\nabla w^{\mathrm{e}}\,\mathrm{d}s_h &\leqslant ch^k\|\nabla e_h\|_{L^2(\Gamma_h)}\|\nabla_\Gamma w\|_{L^2(\Gamma)}\\ &\leqslant c(h^{m+k}\|u\|_{H^{m+1}(\Gamma)} + h^{2k}\|f\|_{L^2(\Gamma)})\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}}.\end{aligned}$$

Using these bounds in (5.26) yields

$$\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}} \leqslant c\big(h^{m+1}\|u\|_{H^{m+1}(\Gamma)} + h^{k+s}\|f\|_{L^2(\Gamma)}\big). \tag{5.28}$$

Now note that

$$|c_e| = \left|\int_\Gamma u - u_h^l\,\mathrm{d}s\right| = \left|\int_\Gamma u_h^l\,\mathrm{d}s\right| = \left|\int_{\Gamma_h} (\mu_h - 1)u_h\,\mathrm{d}s_h\right| \leqslant ch^{k+1}\|f\|_{L^2(\Gamma)},$$

and thus

$$\|e_h\|_{L^2(\Gamma_h)} \leqslant c\|e_h^l\|_{L^2(\Gamma)} \leqslant c\|e_h^l\|_{L^2(\Gamma)/\mathbb{R}} + ch^{k+1}\|f\|_{L^2(\Gamma)},$$

and combining this with (5.28) completes the proof of (5.23). The result (5.24) can be proved with very similar arguments. A proof of (5.24) for $m=k=1$ is given in Olshanskii *et al.* (2009). □

Note that the bounds in (5.23) and (5.24) are the same. We have an optimal error if $k+s \geqslant m+1$ holds. If we have an optimal data approximation error, i.e., $s=1$ (cf. Remark 5.6), we need $k \geqslant m$ to obtain an optimal L^2-error bound of order h^{m+1}. Inspection of the proof above shows that the factor h^{k+s} in (5.23) originates (only) from the estimate (5.27). All other geometric error terms are of order h^{k+1}. In the proof of (5.24) the term $F_h^\Gamma(w^{\mathrm{e}})$ has to be bounded. For this the estimate (5.12) is used. Inspection of the proof of the latter estimate reveals that the factor h^{k+s} in the bound in (5.12) cannot be improved if we use the special choice $v = w^{\mathrm{e}}$. Thus in both error bounds, (5.23) and (5.24), we get the same geometric error term of order h^{k+s}.

6. Conditioning of the stiffness matrix

In this section we address linear algebra aspects of the discretizations in (2.5) and (2.6). The discrete solution is determined by using the standard nodal basis of the (outer) finite element space $V_{h,m}$. This nodal basis and the corresponding nodes are denoted by $\{\phi_i\}_{1\leqslant i\leqslant N}$ and $\{x_i\}_{1\leqslant i\leqslant N}$, respectively. Hence, $V_{h,m} = \mathrm{span}\{(\phi_i)_{|\omega_h} \mid 1 \leqslant i \leqslant N\}$ and $\dim(V_{h,m}) = N$. By construction we have $\mathrm{span}\{(\phi_i)_{|\Gamma_h} \mid 1 \leqslant i \leqslant N\} = V_{h,m}^\Gamma$. In relation to the linear algebra, a key point is that in general the $\{(\phi_i)_{|\Gamma_h}\}_{1\leqslant i\leqslant N}$ are not independent, hence these do *not* form a basis of the trace space $V_{h,m}^{\Gamma}$. This can be illustrated by simple examples; cf. Olshanskii & Reusken (2010). The representation $v_h = \sum_{i=1}^{N} V_i \phi_i$, $v_h \in V_{h,m}$ induces the isomorphism $v_h \to V := (V_i)_{1 \leqslant i \leqslant N} \in \mathbb{R}^N$. The vector corresponding to $w_h \in V_{h,m}$ is denoted by W. We introduce the mass and stiffness matrices:

$$\langle MV, W \rangle = \int_{\Gamma_h} v_h w_h \, \mathrm{d}s_h \quad \text{for all } v_h, w_h \in V_{h,m}, \tag{6.1}$$

$$\langle AV, W \rangle = \int_{\Gamma_h} \nabla v_h \cdot \nabla w_h \, \mathrm{d}s_h \quad \text{for all } v_h, w_h \in V_{h,m}, \tag{6.2}$$

$$\langle A_\Gamma V, W \rangle = \int_{\Gamma_h} \nabla_{\Gamma_h} v_h \cdot \nabla_{\Gamma_h} w_h \, \mathrm{d}s_h \quad \text{for all } v_h, w_h \in V_{h,m}. \tag{6.3}$$

If $\{(\phi_i)_{|\Gamma_h}\}_{1 \leqslant i \leqslant N}$ are dependent, there exists $V \in \mathbb{R}^N$, $V \neq 0$ such that $v_h = \sum_{i=1}^{N} V_i \phi_i = 0$ on Γ_h. This implies $\langle MV, V \rangle = 0$ and thus M is singular. Furthermore, $v_{h|\Gamma_h} = 0$ implies $(\nabla_{\Gamma_h} v_h)_{|\Gamma_h} = 0$ and thus $\langle A_\Gamma V, V \rangle = 0$, hence A_Γ is singular. This indicates that the conditioning properties of the mass matrix M and the stiffness matrix A_Γ are different from those of standard finite element discretizations of elliptic problems. In Olshanskii & Reusken (2010) this conditioning issue is studied. We outline a few important results. In numerical experiments it is observed that for the Laplace–Beltrami equation discretized with linear trace finite elements on an approximate surface Γ_h that is obtained as the zero level of a piecewise linear level set function, the mass matrix M has one zero eigenvalue (within machine accuracy) and the stiffness matrix A_Γ has two zero eigenvalues. The effective condition number is defined as the quotient of the largest and the smallest nonzero eigenvalue. Typically, both the *diagonally scaled* mass matrix $D_M^{-1}M$ and the *diagonally scaled* stiffness matrix $D_{A_\Gamma}^{-1}A_\Gamma$ have effective condition numbers that behave like h^{-2}. In Olshanskii & Reusken (2010) a rather technical analysis is presented that gives a theoretical explanation of these conditioning properties. The analysis is only for the two-dimensional case (i.e., Γ is a curve) and uses technical assumptions related to how Γ_h intersects the local triangulation $\mathcal{T}_h^{\Gamma}$. An example of such an assumption is that the relative size of the set of vertices in $\mathcal{T}_h^{\Gamma}$, having a certain maximal distance to Γ_h, gets smaller if this distance gets smaller (for precise statements we refer the reader to Olshanskii & Reusken, 2010). In the recent paper Burman *et al.* (2013), a stabilization technique for (2.6) is introduced, which improves the conditioning properties of A_Γ.

The discretization (2.5) is more stable than (2.6) in the sense that $\|\nabla v_h\|_{L^2(\Gamma_h)} \geqslant \|\nabla_{\Gamma_h} v_h\|_{L^2(\Gamma_h)}$ holds. In relation to this, note that $v_{h|\Gamma_h} = 0$ does *not* necessarily imply that $(\nabla v_h)_{|\Gamma_h} = 0$ holds. Based on this, one might expect a better conditioning of the matrix A compared to A_Γ. This is indeed observed in numerical experiments; cf. Section 7. In this section we derive conditioning properties of the stiffness matrix A for the case $m = 1$, i.e., linear finite elements. Using an elementary analysis it is shown that a suitably scaled A has a condition number that behaves like h^{-2} (on the space orthogonal to the constant), *independently of how Γ_h intersects $\mathcal{T}_h^{\Gamma}$*. Such a robustness property w.r.t. the geometry does not hold for the scaled stiffness matrix A_Γ. As a simple corollary we obtain a conditioning result for a shifted mass matrix; cf. Theorem 6.4.

In the analysis we use (global) inverse estimates. Therefore, in the remainder of this section we assume that the following holds.

ASSUMPTION 6.1 The local triangulation $\mathcal{T}_h^{\Gamma}$ is quasi-uniform.

We consider $m=1$ and use the notation $V_h := V_{h,1}$. In Remark 6.5 we comment on $m \geqslant 2$. For a node x_i, let $\mathcal{T}(x_i)$ be the set of all tetrahedra $T \in \mathcal{T}_h^\Gamma$ that contain x_i. Define

$$\delta_T := \frac{|\Gamma_T|}{|T|}h, \quad T \in \mathcal{T}_h^\Gamma, \quad d_i := \sum_{T \in \mathcal{T}(x_i)} \delta_T, \quad 1 \leqslant i \leqslant N.$$

(Recall $\Gamma_T = \Gamma_h \cap T$.) We introduce a weighted $L^2(\omega_h)$ norm,

$$\|v\|_{\delta,\omega_h}^2 = \sum_{T \in \mathcal{T}_h^\Gamma} \delta_T \int_T v^2 \,\mathrm{d}x,$$

and a related scaled vector norm,

$$\|V\|_D^2 := \langle DV, V\rangle, \quad D := \mathrm{diag}(d_i).$$

From the fact that $\nabla\phi_i \cdot \nabla\phi_i$ is constant on each $T \in \mathcal{T}(x_i)$ with value $\sim h^{-2}$ it follows that D is uniformly spectrally equivalent to $\mathrm{diag}(A)$. Hence, the scaling with D that is used below can be replaced by a scaling with $\mathrm{diag}(A)$. *The constants used in the lemma and theorems below are independent of h and of how Γ_h intersects the local triangulation $\mathcal{T}_h^\Gamma$.*

LEMMA 6.2 There are constants $c_1 > 0$ and c_2 such that

$$c_1 h^3 \|V\|_D^2 \leqslant \|v_h\|_{\delta,\omega_h}^2 \leqslant c_2 h^3 \|V\|_D^2 \quad \text{for all } v_h \in V_h.$$

Proof. We use the compact notation $\sim$ to represent inequalities in both directions with constants independent of h and of how Γ_h intersects ω_h. The set of vertices of T is denoted by $\mathcal{V}(T)$. For $v_h \in V_h$ we have $V_i = v_h(x_i)$ and using the quasi-uniformity of $\mathcal{T}_h^\Gamma$ we obtain

$$\int_T v_h^2 \,\mathrm{d}x \sim |T| \sum_{x_i \in \mathcal{V}(T)} v_h(x_i)^2 \sim h^3 \sum_{x_i \in \mathcal{V}(T)} V_i^2.$$

This implies

$$\begin{aligned}\|v_h\|_{\delta,\omega_h}^2 &= \sum_{T \in \mathcal{T}_h^\Gamma} \delta_T \int_T v_h^2 \,\mathrm{d}x \sim h^3 \sum_{T \in \mathcal{T}_h^\Gamma} \delta_T \sum_{x_i \in \mathcal{V}(T)} V_i^2 \\ &= h^3 \sum_{i=1}^N \left(\sum_{T \in \mathcal{T}(x_i)} \delta_T \right) V_i^2 = h^3 \sum_{i=1}^N d_i V_i^2 = h^3 \|V\|_D^2,\end{aligned}$$

and thus the result holds. □

THEOREM 6.3 There are constants $c_1 > 0$ and c_2 such that

$$c_1 h^2 \|V\|_D^2 \leqslant \langle AV, V\rangle \leqslant c_2 \|V\|_D^2 \quad \text{for all } V \in \mathbb{R}^N \text{ with } \int_{\Gamma_h} v_h \,\mathrm{d}s = 0.$$

Proof. We first consider the upper bound. Note that, using an inverse inequality on T we obtain

$$\langle AV,V\rangle = \int_{\Gamma_h} \nabla v_h \cdot \nabla v_h \,\mathrm{d}s \leqslant \sum_{T\in\mathcal{T}_h^\Gamma} |\Gamma_T|\,\|\nabla v_h\|^2_{L^\infty(T)} \leqslant c \sum_{T\in\mathcal{T}_h^\Gamma} h^{-2}\frac{|\Gamma_T|}{|T|}\|v_h\|^2_{L^2(T)}$$
$$= ch^{-3}\sum_{T\in\mathcal{T}_h^\Gamma} \delta_T \int_T v_h^2\,\mathrm{d}x = ch^{-3}\|v_h\|^2_{\delta,\omega_h} \leqslant c\|V\|_D^2,$$

where in the last inequality we used Lemma 6.2.

We now consider the lower bound. Using Lemma 6.2 we obtain

$$h^2\|V\|_D^2 \leqslant ch^{-1}\|v_h\|^2_{\delta,\omega_h} = ch^{-1}\sum_{T\in\mathcal{T}_h^\Gamma}\delta_T\int_T v_h^2\,\mathrm{d}x. \tag{6.4}$$

Take a $T\in\mathcal{T}_h^\Gamma$. Let $\xi\in T,\ \eta\in\Gamma_T$ be such that $|v_h(x)|\leqslant|v_h(\xi)|$ for all $x\in T$ and $|v_h(\eta)|\leqslant|v_h(x)|$ for all $x\in\Gamma_T$. From $v_h(\xi)=v_h(\eta)+(\xi-\eta)\cdot\nabla v_{h|T}$ it follows that $|v_h(\xi)|^2\leqslant c(|v_h(\eta)|^2+h^2\|\nabla v_{h|T}\|^2)$. Using this we obtain

$$h^{-1}\delta_T\int_T v_h^2\,\mathrm{d}x \leqslant |\Gamma_T|\,|v_h(\xi)|^2 \leqslant c(|\Gamma_T||v_h(\eta)|^2 + |\Gamma_T|h^2\|\nabla v_{h|T}\|^2)$$
$$\leqslant c\left(\int_{\Gamma_T} v_h^2\,\mathrm{d}s + h^2\int_{\Gamma_T}\|\nabla v_h\|^2\,\mathrm{d}s\right).$$

Summing over $T\in\mathcal{T}_h^\Gamma$ and using the Poincaré inequality (5.3) (which holds for v_h with $\int_{\Gamma_h} v_h\,\mathrm{d}s_h=0$), we obtain

$$h^{-1}\sum_{T\in\mathcal{T}_h^\Gamma}\delta_T\int_T v_h^2\,\mathrm{d}x \leqslant c\left(\int_{\Gamma_h} v_h^2\,\mathrm{d}s + h^2\int_{\Gamma_h}\|\nabla v_h\|^2\,\mathrm{d}s\right)$$
$$\leqslant c\left(\int_{\Gamma_h}\|\nabla_{\Gamma_h} v_h\|^2\,\mathrm{d}s + h^2\int_{\Gamma_h}\|\nabla v_h\|^2\,\mathrm{d}s\right) \leqslant c\int_{\Gamma_h}\|\nabla v_h\|^2\,\mathrm{d}s = c\langle AV,V\rangle. \tag{6.5}$$

Combining this with the result in (6.4) completes the proof. □

As an immediate consequence of this theorem we obtain for the spectral condition number in the space orthogonal to the one-dimensional kernel (corresponding to the constant function), denoted by $\kappa_*(\cdot)$,

$$\kappa_*(D^{-1}A) \leqslant ch^{-2}. \tag{6.6}$$

We derive a result for a shifted mass matrix.

THEOREM 6.4 There are constants $c_1, c_2 > 0$ such that for all $\alpha \geqslant 0$,

$$c_1(\min\{2h^2,\alpha\}+\alpha h^2)\|V\|_D^2 \leqslant \langle MV,V\rangle + \alpha\langle AV,V\rangle \leqslant c_2(h^2+\alpha)\|V\|_D^2$$

for all $V\in\mathbb{R}^N$ with $\int_{\Gamma_h} v_h\,\mathrm{d}s = 0$.

Proof. First we consider the upper bound. Note that

$$\langle MV, V\rangle = \int_{\Gamma_h} v_h^2 \, ds \leqslant \sum_{T\in\mathcal{T}_h^\Gamma} |\Gamma_T| \|v_h\|^2_{L^\infty(T)} \leqslant ch^{-1} \sum_{T\in\mathcal{T}_h^\Gamma} \delta_T \int_T v_h^2 \, ds$$
$$= ch^{-1}\|v_h\|^2_{\delta,\omega_h} \leqslant ch^2\|V\|_D^2.$$

In combination with Theorem 6.3 this yields the upper bound. For the lower bound we use the result in (6.5) and in Lemma 6.2 and thus obtain

$$\langle MV, V\rangle + h^2\langle AV, V\rangle = \int_{\Gamma_h} v_h^2 \, ds + h^2 \int_{\Gamma_h} \|\nabla v_h\|^2 \, ds \geqslant ch^{-1}\|v_h\|^2_{\delta,\omega_h} \geqslant c_0 h^2 \langle DV, V\rangle,$$

with a constant $c_0 > 0$. Using spectral inequalities for symmetric positive definite matrices we have $M + h^2 A \geqslant c_0 h^2 D$. From this and the lower bound in Theorem 6.3 we obtain

$$M + \alpha A = M + \left(\frac{\alpha}{2h^2}\right) h^2 A + \frac{1}{2}\alpha A \geqslant \min\left\{1, \frac{\alpha}{2h^2}\right\}(M + h^2 A) + \frac{1}{2}\alpha A$$
$$\geqslant c_0 \min\left\{1, \frac{\alpha}{2h^2}\right\} h^2 D + c_1 \alpha h^2 D \geqslant c(\min\{2h^2, \alpha\} + \alpha h^2)D,$$

which proves the lower bound. □

Thus, we have the following bound for the spectral condition number:

$$\kappa_*(D^{-1}(M + \alpha A)) \leqslant c\frac{h^2 + \alpha}{\min\{2h^2, \alpha\} + \alpha h^2}. \tag{6.7}$$

For $\alpha \to \infty$ we get the same bound as in (6.6). For $\alpha \downarrow 0$ the bound tends to infinity. This cannot be avoided, since the mass matrix M can be singular (also in the space orthogonal to the constant), as explained above. More interesting is the case $\alpha \geqslant ch^2$ with $c > 0$. Then the bound takes the form $ch^{-2}(\alpha/(1+\alpha))$. Hence, if $\alpha \sim h^2$, we get a uniform (i.e., independent of h) condition number bound and if $\alpha \sim h$, we get a bound of the form ch^{-1}. The case $\alpha \sim h^2$ is typical if a time-dependent surface diffusion problem is considered in which, for the time discretization, an implicit Euler method is used with $\Delta t \sim h^2$. If, for such a time-dependent problem one uses Crank–Nicolson with $\Delta t \sim h$, this results in a linear system with $\alpha \sim h$.

REMARK 6.5 We comment on the case of higher-order finite elements, i.e., $m \geqslant 2$. Inspection of the proofs shows that the estimates in Lemma 6.2 and the upper bound in Theorem 6.3 also hold if $m \geqslant 2$ is considered. The lower bound in Theorem 6.3, however, does not hold in general. This becomes clear from the following example. Consider a two-dimensional setting with a domain $\omega = [0,1] \times [-1,1]$ which is subdivided into a few triangles with vertices only on $y = -1$ or $y = 1$. We take $\Gamma_h = [0,1] \times \{0\}$. We choose the P_2 finite element function $v(x,y) = \alpha y^2 + (x - \frac{1}{2})$ with $\alpha \gg 1$. For this function we have $\|v\|_{L^2(\omega)} \sim \alpha$, $\int_{\Gamma_h} v \, ds = 0$, $\nabla v = (1, 2\alpha y)^{\mathrm{T}}$ and thus $\|\nabla v\|_{L^2(\Gamma_h)} = 1$. The lower bound in Theorem 6.3 scales with $h^2\|V\|_D^2 \sim \|v\|^2_{L^2(\omega)} \sim \alpha^2$, whereas $\langle AV, V\rangle = \|\nabla v\|^2_{L^2(\Gamma_h)} = 1$ holds. Hence, we conclude that the first inequality $c_1 h^2 \|V\|_D^2 \leqslant \langle AV, V\rangle$ in Theorem 6.3 cannot hold for this simple example. The key point in this example is that we cannot control the values of v_h on ω by its values and gradient on Γ_h.

7. Numerical experiments

In this section we present the results of a numerical experiment. As a test problem we consider the Laplace–Beltrami equation on the unit sphere:

$$-\Delta_\Gamma u = f \quad \text{on } \Gamma,$$

with $\Gamma = \{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\}$ and $\Omega = (-2,2)^3$.

The source term f is taken such that the solution is given by

$$u(x) = \frac{1}{\|x\|^3}(3x_1^2 x_2 - x_2^3), \quad x = (x_1, x_2, x_3) \in \Omega.$$

Using the representation of u in spherical coordinates one can verify that u is an eigenfunction of $-\Delta_\Gamma$:

$$u(r,\phi,\theta) = \sin(3\phi)\sin^3\theta, \quad -\Delta_\Gamma u = 12u =: f(r,\phi,\theta). \tag{7.1}$$

The right-hand side f satisfies the compatibility condition $\int_\Gamma f \,\mathrm{d}s = 0$, likewise does u. Note that u and f are constant along normals at Γ.

A family $\{\mathcal{T}_l\}_{l \geqslant 0}$ of tetrahedral triangulations of Ω is constructed as follows. We triangulate Ω by starting with a uniform subdivision into 48 tetrahedra with mesh size $h_0 = \sqrt{3}$. Then we apply an adaptive red–green refinement algorithm (implemented in the software package DROPS: DROPS package) in which in each refinement step the tetrahedra that contain Γ are refined such that on level $l = 1, 2, \ldots,$ we have

$$h_T \leqslant \sqrt{3}\, 2^{-l} =: h_l \ \text{ for all } T \in \mathcal{T}_l \quad \text{with } T \cap \Gamma \neq \emptyset.$$

The family $\{\mathcal{T}_l\}_{l \geqslant 0}$ is consistent and shape regular. The local triangulation $\mathcal{T}_h^\Gamma$ is quasi-uniform. The interface Γ is the zero level of $\varphi(x) := \|x\|^2 - 1$. Let $\varphi_h := I_h(\varphi)$ where I_h is the standard nodal interpolation operator on $\mathcal{T}_l$, which maps into the space of piecewise linears $V_{h,1}$. The discrete interface is given by $\Gamma_{h_l} := \{x \in \Omega \mid \varphi_h(x) = 0\}$. In the experiments below we only consider this interface approximation, which satisfies the conditions in Assumption 5.1 with $k = 1$. Note that this discrete interface triangulation is very shape *irregular*. For the extension f_h of f we take the constant extension of f along the normals at Γ, i.e., we take $f_h(r,\phi,\theta) = f(1,\phi,\theta) + c_h$, with $f(r,\phi,\theta)$ as in (7.1) and c_h such that $\int_{\Gamma_h} f_h \,\mathrm{d}s_h = 0$. For the computation of the integrals $\int_T f_h \phi_h \,\mathrm{d}s_h$ we use a quadrature rule that is exact up to order five. The extension u^{e} of u is given by $u^{\mathrm{e}}(r,\phi,\theta) := u(1,\phi,\theta) = u(r,\phi,\theta)$ (since u is constant along n_Γ).

We consider the discrete problems (2.5) and (2.6) with solutions u_h and u_h^Γ, respectively.

Experiment 7.1 (Discretization errors for $k = m = 1$.) For the discretization of u we use the finite element space $V_{h,1}^\Gamma$, i.e., the trace of piecewise linear finite element functions. The discretization errors in the $L^2(\Gamma_h)$ norm are given in Table 1.

These results clearly show the h^2 behaviour as predicted by the analysis; cf. Theorem 5.8. We also observe that the discretization error for u_h^Γ is about a factor 2 smaller than for u_h.

Experiment 7.2 (Discretization errors for $k = 1$, $m = 2$.) We keep the piecewise planar approximation Γ_h of Γ (i.e., $k = 1$), but for the discretization of u we now use the finite element space $V_{h,2}^\Gamma$, i.e., the trace of piecewise quadratic finite element functions ($m = 2$). For the two methods the H^1 discretization errors are given in Table 2. We use the notation $e_h := u^{\mathrm{e}} - u_h$, $e_h^\Gamma := u^{\mathrm{e}} - u_h^\Gamma$.

TABLE 1 *Discretization errors and error reduction, $k = m = 1$*

Level l	$\|u^e - u_h\|_{L^2(\Gamma_h)}$	Factor	$\|u^e - u_h^\Gamma\|_{L^2(\Gamma_h)}$	Factor
1	0.6276	—	0.4418	—
2	0.1983	3.16	0.1149	3.85
3	0.05299	3.74	0.02965	3.87
4	0.01348	3.93	0.007298	4.06
5	0.003387	3.98	0.001865	3.91
6	0.0008476	4.00	0.0004629	4.03
7	0.0002120	4.00	0.0001158	4.00

TABLE 2 *Discretization errors and error reduction, $k = 1$, $m = 2$*

l	$\|\nabla_{\Gamma_h} e_h^\Gamma\|_{L^2(\Gamma_h)}$	Factor	$\|\nabla_{\Gamma_h} e_h\|_{L^2(\Gamma_h)}$	Factor	$\|\nabla e_h\|_{L^2(\Gamma_h)}$	Factor
1	0.6891	—	0.8489	—	0.9894	—
2	0.1636	4.21	0.2281	3.72	0.3099	3.19
3	0.04219	3.88	0.08265	2.76	0.1341	2.31
4	0.01054	4.00	0.03552	2.33	0.06046	2.22
5	0.002689	3.92	0.01700	2.09	0.03019	2.00
6	0.0006685	4.02	0.008409	2.02	0.01493	2.02
7	0.0001673	4.00	0.004194	2.01	0.007455	2.00

TABLE 3 *Conditioning of scaled stiffness matrix $\tilde{A}$*

Level l	m	Factor	λ_1	λ_2	λ_m	λ_m/λ_2	Factor	# iter
1	100	—	0	0.021	0.51	24.0	—	11
2	448	4.48	0	0.0053	0.52	98.9	4.1	18
3	1864	4.16	0	0.0013	0.54	412	4.2	33
4	7552	4.05	0	0.00033	0.54	1667	4.0	59

These results show an $\mathcal{O}(h^2)$ convergence for u_h^Γ and an $\mathcal{O}(h)$ convergence for u_h, as predicted by the analysis; cf. Theorem 5.7. Note that for the discrete solution u_h^Γ we benefit from the quadratic finite elements ($m = 2$), whereas for the discrete solution u_h this is not the case. The L^2 errors $\|e_h^\Gamma\|_{L^2(\Gamma_h)}$ and $\|e_h\|_{L^2(\Gamma_h)}$ (not shown), behave like $\mathcal{O}(h^2)$, consistent with the error bounds in Theorem 5.8.

EXPERIMENT 7.3 (Conditioning of mass and stiffness matrices for $k = m = 1$.) We now consider the conditioning of the stiffness matrices and shifted mass matrix corresponding to the discrete problem in Experiment 7.1. The matrices M, A, A_Γ and D are as defined in Section 6. Define $D_{A_\Gamma} := \operatorname{diag}(A_\Gamma)$ and the scaled matrices

$$\tilde{A} := D^{-1/2} A D^{-1/2}, \quad \tilde{A}_\Gamma := D_{A_\Gamma}^{-1/2} A_\Gamma D_{A_\Gamma}^{-1/2}, \quad \tilde{M}_\alpha := D^{-1/2}(M + \alpha A) D^{-1/2}.$$

The discrete problems are solved using a standard CG method with symmetric SOR preconditioner applied to the discrete problems with stiffness matrices A and A_Γ. We use a relative tolerance of 10^{-6}. In the tables below, m gives the number of unknowns (dimension of the matrices). In Tables 3 and 4, '# iter' gives the number of preconditioned CG iterations needed to solve the system (with accuracy 10^{-6}).

TABLE 4 *Conditioning of scaled stiffness matrix* $\tilde{A}_\Gamma$

Level l	m	Factor	λ_1	λ_2	λ_3	λ_m	λ_m/λ_3	Factor	# iter
1	100	—	0	0	0.055	2.16	39.2	—	13
2	448	4.48	0	0	0.014	2.19	154	3.9	25
3	1864	4.16	0	0	0.0033	2.34	710	4.6	49
4	7552	4.05	0	0	0.00077	2.43	3150	4.4	98

TABLE 5 *Conditioning of scaled shifted mass matrix* $\tilde{M}_\alpha$

Level l	m	$\kappa_*(\tilde{M}_\alpha),\ \alpha = h_l^2$	$\kappa_*(\tilde{M}_\alpha),\ \alpha = h_l$
1	100	12.7	25.0
2	448	13.1	50.7
3	1864	13.5	104
4	7552	13.6	209

Furthermore, for different refinement levels we computed the largest and smallest eigenvalues of the scaled matrices.

We observe that, as predicted by the theory, the effective condition number for $\tilde{A}$ behaves like $\sim h^{-2}$ and that this condition number is smaller than for $\tilde{A}_\Gamma$. The better conditioning of $\tilde{A}$ is also reflected in the results for # iter. Also note that $\tilde{A}$ has only one zero eigenvalue (corresponding to the constant function), whereas $\tilde{A}_\Gamma$ has two zero eigenvalues. In Table 5 we present results for the spectral condition number $\kappa_*(\tilde{M}_\alpha)$ for the cases $\alpha = h_l^2$ and $\alpha = h_l$. The results are in very good agreement with the bound derived in (6.7).

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