A HIGHER ORDER FINITE ELEMENT METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS ON SURFACES

JÖRG GRANDE* AND ARNOLD REUSKEN†

Abstract. A new higher order finite element method for elliptic partial differential equations on a stationary smooth surface Γ is introduced and analyzed. We assume that Γ is characterized as the zero level of a level set function ϕ and *only a finite element approximation ϕ_h (of degree $k \geq 1$) of ϕ is known.* For the discretization of the partial differential equation, finite elements (of degree $m \geq 1$) on a piecewise *linear* approximation of Γ are used. The discretization is lifted to Γ_h, which denotes the zero level of ϕ_h, using a quasi-orthogonal coordinate system that is constructed by applying a gradient recovery technique to ϕ_h.

A complete discretization error analysis is presented in which the error is split into a geometric error, a quadrature error, and a finite element approximation error. The main result is a $H^1(\Gamma)$-error bound of the form $c(h^m + h^{k+1})$. Results of numerical experiments illustrate the higher order convergence of this method.

Key words. Laplace Beltrami equation, surface finite element method, high order, gradient recovery, error analysis

AMS subject classifications. 58J32, 65N15, 65N30, 76D45, 76T99

1. Introduction. In the past decade the study of numerical methods for partial differential equations (PDEs) on surfaces has been a rapidly growing research area. The development of finite element (FE) methods for solving elliptic PDEs on surfaces can be traced back to the paper [9], which considers a piecewise polygonal surface and uses a FE space on a triangulation of this discrete surface. This approach has been further analyzed and extended in several directions, see, e.g., [10, 11] and the references therein. Another approach has been introduced in [5] and builds on the ideas of [2]. The method in that paper applies to cases in which the surface is given implicitly by some level set function and the key idea is to solve the PDE on a narrow band around the surface. Unfitted FE spaces on this narrow band are used for the discretization. Another surface FE method based on an outer (bulk) mesh has been introduced in [17] and further studied in [16, 7]. The main idea of this method is to take the *traces* of unfitted FE spaces that are defined on meshes of an outer domain to discretize the surface PDE, instead of extending the PDE off the surface, as in [2, 5]. Most of these methods mentioned above have been studied both for stationary and evolving surfaces.

In only very few papers *higher order* (i.e. degree at least 2) FE methods for PDEs on (stationary) surfaces are treated [6, 20]. There are, however, higher order methods using finite differences and the closest point method, e.g. [3, 21]. Higher order approximation of the surface curvature is treated in [14]. Isoparametric higher order FE methods for singular integral equations are considered in [15]. Rigorous error analyses for surface PDEs are given only in [6] and [20]. We first outline the main results from [6], then introduce the method treated in this paper, and finally discuss the differences between this method and the one considered in [20].

For a smooth bounded and connected surface $\Gamma \subset \mathbb{R}^3$ we consider the Laplace-Beltrami problem: for given $f \in L^2(\Gamma)$ with $\int_\Gamma f\, ds = 0$ determine $u \in H^1_*(\Gamma) :=$

*Institut für Geometrie und Praktische Mathematik, RWTH-Aachen University, D-52056 Aachen, Germany; email: grande@igpm.rwth-aachen.de

†Institut für Geometrie und Praktische Mathematik, RWTH-Aachen University, D-52056 Aachen, Germany; email: reusken@igpm.rwth-aachen.de

$\{ u \in H^1(\Gamma) \mid \int_\Gamma u \, ds = 0 \}$ such that

$$\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v \, ds = \int_\Gamma f v \, ds \quad \text{for all} \quad v \in H^1(\Gamma). \tag{1.1}$$

The exact surface is approximated by a quasi-uniform shape-regular polyhedral surface $\hat{\Gamma}_h$ having triangular faces, and with vertices on Γ. In [6] it is assumed that Γ is represented as the *zero level of a smooth signed distance function* d. Based on d a parametric mapping, consisting of piecewise polynomial mappings of degree k, is defined on $\hat{\Gamma}_h$, which results in a corresponding discrete surface $\hat{\Gamma}_h^k$. Using the same mapping a standard higher order FE space on $\hat{\Gamma}_h$ is lifted to $\hat{\Gamma}_h^k$. This lifted space on $\hat{\Gamma}_h^k$ is used for the discretization of (1.1). An extensive error analysis of this method is presented in [6], resulting in optimal error bounds. For example, for the $H^1(\Gamma)$-error (where the discrete solution is lifted to Γ) a bound of the form $c(h^m + h^{k+1})$ is proved. Here k is the degree of the polynomials used in the parametrization of $\hat{\Gamma}_h^k$ and m the degree of the polynomials in the FE space on $\hat{\Gamma}_h$. We emphasize that in this method *explicit knowledge of the exact signed distance function to* Γ *is an essential ingredient.*

In many applications the exact signed distance function to the surface Γ is not known. One often encounters situations in which Γ is the zero level of a smooth level set function ϕ (not necessarily a signed distance function) and one *only has a FE approximation of* ϕ *available.* This paper deals with the question: (how) can one develop a higher order FE method in such a setting? We will present a constructive affirmative answer to this question.

We restrict ourselves to the model problem (1.1) with a stationary surface Γ. We assume Γ to be sufficiently smooth. Our approach is fundamentally different from the one in [6], in the sense that we do not need the exact distance function d. Instead, we only(!) need a FE approximation ϕ_h^k of a level set function ϕ, which has Γ as its zero level. The discrete level set function ϕ_h^k comes from a standard FE space on a quasi-uniform triangulation of a bulk domain that contains Γ. In the error analysis we assume that ϕ_h^k satisfies an error bound of the form

$$\|\phi_h^k - \phi\|_{L^\infty(U)} + h\|\phi_h^k - \phi\|_{H^1_\infty(U)} \leq ch^{k+1}, \tag{1.2}$$

where U is a (small) neighborhood of Γ in $\mathbb{R}^3$. The zero level of ϕ_h^k is denoted by Γ_h^k. Note that for $k > 1$, Γ_h^k cannot be easily constructed. From (1.2) it follows that $\text{dist}(\Gamma, \Gamma_h^k) \leq ch^{k+1}$ holds. The method that we introduce is new and is built upon the following key ingredients:

- For $k = 1$ the function $\hat{\phi}_h := \phi_h^1$ is piecewise linear, hence its zero level is piecewise planar. A triangulation $\hat{\Gamma}_h$ can easily be determined. This triangulation is in general *very* shape-*ir*regular. Nevertheless, the trace of an outer FE space or a standard FE space directly on $\hat{\Gamma}_h$ turns out to have optimal approximation properties [17, 18]. Such a FE space on $\hat{\Gamma}_h$ is denoted by $\hat{S}_h$.
- We take $k > 1$. For the parametrization of Γ_h^k we use a *quasi-normal field*, as introduced in [19]. Given ϕ_h^k we apply a *gradient recovery method* to obtain a Lipschitz continuous vector field n_h. It is close to the normal field n corresponding to ϕ. Using this quasi-normal field, there is a unique decomposition $x = p_h(x) + d_h(x) n_h(p_h(x))$ for all x in a neighborhood of Γ_h^k, with $p_h(x) \in \Gamma_h^k$ and $d_h \in \mathbb{R}$ an *approximate* signed distance function. It can be shown that $p_h : \hat{\Gamma}_h \to \Gamma_h^k$ is a *bijection.* This p_h is used for the parametrization of Γ_h^k. For given $x \in \hat{\Gamma}_h$ its image $p_h(x) \in \Gamma_h^k$ can be determined (with high accuracy) using the known field n_h and only few evaluations of ϕ_h^k [12].

- Using the parametrization p_h the FE space $\hat{S}_h$ on $\hat{\Gamma}_h$ is lifted to Γ_h^k and used for a Galerkin type discretization of (1.1), i.e. we take (1.1) with Γ replaced by Γ_h^k, $H^1(\Gamma)$ replaced by the lifted FE space, f suitably extended, and instead of ∇_Γ we use the tangential gradient along Γ_h^k.
- Only evaluations of p_h and its Jacobian Dp_h can be computed. Hence, *quadrature* is needed. The FE space is pulled back to $\hat{\Gamma}_h$, integrals over Γ_h^k are transformed to integrals over $\hat{\Gamma}_h$, and quadrature is applied on triangles in $\hat{\Gamma}_h$. We then (only) need evaluations of p_h, Dp_h, and of the exact normals on $\hat{\Gamma}_h$ and on Γ_h^k. The latter are easily determined using ϕ_h^k.

The method is described more precisely in section 5. The implementation is discussed in section 14.

We discuss the main difference between the method described above and the higher order method treated in [20]. In the latter the higher order trace spaces used are the trace on Γ_h^k (not on $\hat{\Gamma}_h$!) of outer higher order FE spaces. This yields a method, which is theoretically interesting, but not feasible in most practical situations because it is not clear how integrals over Γ_h^k can be determined in a cheap and accurate way. We also note that in [20] a main topic is the effect of replacing ∇_Γ by ∇ in the discretisation of (1.1). This is not addressed in the present paper.

Apart from the new discretization method outlined above, the main contribution of this paper is an error analysis of this method. A key point related to this is the following. Within each triangle T of the "base" triangulation $\hat{\Gamma}_h$ the parametrization p_h is *only Lipschitz*. This low regularity is due to the construction of the quasi-normal field n_h. The bilinear form pulled back to $\hat{\Gamma}_h$ consists of a sum of integrals of the form $\int_T F\nabla_{\hat{\Gamma}_h}\hat{u}_h\cdot\nabla_{\hat{\Gamma}_h}\hat{v}_h\,d\hat{s}_h$ with a function F that has very low smoothness. (It is not even continuous.) Therefore, the analysis of the quadrature error is not straightforward. The lack of smoothness *in the interior of the triangles* is an important reason why the analysis in this paper is (even) more technical than the one in [6]. The structure of the error analysis is outlined in section 6. As a main result, cf. Theorem 13.1, we prove an $H^1(\Gamma)$ error bound (where the discrete solution is lifted to Γ) of the form $c(h^m + h^{k+1})$. Here m is the degree of the polynomials used in the FE space $\hat{S}_h$.

2. Preliminaries. Let ϕ be a smooth function with a smooth, bounded and connected zero level set $\Gamma \subset \Omega \subset \mathbb{R}^3$, and let $\Omega_1 = \{x \in \Omega \mid \phi(x) \leq 0\}$ be the enclosed (compact) region. Furthermore U is a (small) open subset of $\mathbb{R}^3$ with $\Gamma \subset U \subset \Omega$. This neighborhood is sufficiently small such that on U we have a local coordinate system

$$x = p(x) + d(x)n(p(x)), \quad x \in U, \tag{2.1}$$

with n the normal vector field on Γ (pointing out of Ω_1), $p : U \to \Gamma$ and d the signed distance function to Γ (negative in Ω_1). For every $x \in U$ the normal field has the unique value $n(x) = n(p(x))$. We assume that $\|\nabla\phi(x)\| \geq c_0 > 0$ for all $x \in U$ holds.

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of regular *quasi-uniform* tetrahedral triangulations on Ω. Furthermore V_h^k denotes a standard FE space on $\mathcal{T}_h$ consisting of continuous piecewise polynomial functions of degree k. Let $\phi_h^k \in V_h^k$ be an approximation of ϕ that satisfies

$$\|\phi_h^k - \phi\|_{L^\infty(U)} + h\|\phi_h^k - \phi\|_{H^1_\infty(U)} \leq ch^{k+1}. \tag{2.2}$$

In the remainder we take a fixed value $k \geq 1$. To simplify notation, we write $\phi_h = \phi_h^k$. The *linear* FE approximation ϕ_h^1 plays a special role and is denoted by $\hat{\phi}_h$. The zero level sets of ϕ_h and $\hat{\phi}_h$ are denoted by Γ_h and $\hat{\Gamma}_h$, respectively. The outward pointing normal fields on Γ_h and $\hat{\Gamma}_h$ are denoted by $\bar{n}_h$ and $\hat{n}_h$, respectively. From (2.2) we obtain, cf. [6]:

$$\begin{aligned} &\operatorname{dist}(\Gamma,\hat{\Gamma}_h) \le ch^2, \quad \operatorname{dist}(\Gamma,\Gamma_h) \le ch^{k+1}, \\ &\|n-\hat{n}_h\|_{L^\infty(\Gamma_h)} \le ch, \quad \|n-\bar{n}_h\|_{L^\infty(\Gamma_h)} \le ch^k. \end{aligned} \tag{2.3}$$

The two eigenvalues of the Hessian $D^2d \in \mathbb{R}^{3\times 3}$ corresponding to the eigenvectors orthogonal to $n(x)$ are denoted by $\kappa_i(x)$, $i=1,2$. These are related to the principal curvatures of Γ by the formula $\kappa_i(x) = \kappa_i(p(x))/\big(1+d(x)\kappa_i(p(x))\big)$. In [6] it is shown that for the surface measures ds_h on Γ_h and ds on Γ we have the relation $\mu_h(x)\,ds_h(x) = ds(p(x))$, with

$$\mu_h(x) = n(x)^T\bar{n}_h(x)\prod_{i=1}^{2}(1-d(x)\kappa_i(x)), \quad x\in\Gamma_h. \tag{2.4}$$

Using this formula and the results in (2.3) one obtains:

$$\begin{aligned} \|\mu_h-1\|_{L^\infty(\Gamma_h)} &\le c\|d\|_{L^\infty(\Gamma_h)} + c\|1-n^T\bar{n}_h\|_{L^\infty(\Gamma_h)} \\ &= c\|d\|_{L^\infty(\Gamma_h)} + c\|n-\bar{n}_h\|^2_{L^\infty(\Gamma_h)} \le ch^{k+1}. \end{aligned} \tag{2.5}$$

Here and in the remainder, c is used to denote different constants, which are all independent of h.

We need an $\mathcal{O}(h)$ neighborhood of Γ_h, denoted by Ω_{Γ_h}, consisting of all tetrahedra with distance to Γ_h smaller than ch, with a given $c>0$. We assume that h is sufficiently small such $\Gamma_h \subset \Omega_{\Gamma_h} \subset U$ and $\hat{\Gamma}_h \subset \Omega_{\Gamma_h}$ hold, cf. (2.3).

3. Quasi-normal field. In this section we define the notion of a quasi-normal field, as introduced in [19]. Such a quasi-normal field is constructed using a (simple) gradient recovery technique. Only this field, and not the gradient recovery technique, is then used in the FE method further on.

A gradient recovery operator is a mapping $G_h : V_h^k \to (V_h^k)^3$, which has to satisfy certain reasonable approximation and stability conditions.

ASSUMPTION 3.1. *Let I_h be the nodal interpolation in the FE space V_h^k. We assume that for ϕ sufficiently smooth the gradient recovery method $G_h : V_h^k \to (V_h^k)^3$ satisfies:*

$$\|G_h(I_h\phi) - \nabla\phi\|_{L^\infty(U)} \le ch^k, \tag{3.1}$$

$$\|G_hv_h\|_{L^\infty(U)} \le c\|v_h\|_{H^1_\infty(U^e)} \quad \textit{for all} \;\; v_h \in V_h^k. \tag{3.2}$$

Here U^e denotes the neighborhood U enlarged with a suitable patch of surrounding elements.

REMARK 1. In the literature gradient recovery techniques are known and often used in error estimators, cf. [1]. In such a setting one usually requires a power $k+1$, instead of k, in (3.1). In [19] the polynomial-preserving recovery (PPR) technique is considered. For the PPR technique, (3.2) and (3.1) with $k+1$ are shown to hold in two dimensions in [22]. To indicate that the conditions (3.1) and (3.2) are mild ones, as an example we describe a very simple gradient recovery technique satisfying Assumption 3.1. It is used in the experiments in section 14. The set of finite element nodes is denoted by N_h. To each finite element node $\xi \in N_h$ we assign the set $\mathcal{T}_\xi$ of all tetrahedra containing ξ. For $\xi \in U^e$ this $\mathcal{T}_\xi$ is chosen such that $T \in \mathcal{T}_\xi \Rightarrow T \subset U^e$. Let $n_\xi := |\mathcal{T}_\xi|$. The gradient recovery is defined by simple local averaging, namely $(Gv_h)(\xi) := \frac{1}{n_\xi} \sum_{T\in\mathcal{T}_\xi} \nabla v_{h|T_\xi}(\xi)$ for all ξ. Let $\phi_h = I_h\phi$ be the nodal interpolation of a smooth function ϕ. From standard interpolation theory we get

$$\max_{\xi\in N_h\cap U^e} \|(G_h\phi_h)(\xi) - \nabla\phi(\xi)\| \leq c \max_{T\in\mathcal{T}_h\cap U^e} \|\nabla\phi_h - \nabla\phi\|_{L^\infty(T)} \leq ch^k \|\phi\|_{H^{k+1}(U^e)}.$$

Hence, using $I_h(G_h\phi_h) = G_h\phi_h$ we get

$$\begin{aligned}
&\|G_h\phi_h - \nabla\phi\|_{L^\infty(U)} \leq \|I_h(G_h\phi_h - \nabla\phi)\|_{L^\infty(U)} + \|I_h(\nabla\phi) - \nabla\phi\|_{L^\infty(U)} \\
&\leq c \max_{\xi\in N_h\cap U^e} \|(G_h\phi_h)(\xi) - \nabla\phi(\xi)\| + ch^k\|\phi\|_{H^{k+1}(U^e)} \leq ch^k\|\phi\|_{H^{k+1}(U^e)},
\end{aligned}$$

and thus the condition in (3.1) is satisfied. With similar arguments, using stability properties of I_h, one can verify that for this simple recovery operator condition (3.2) is satisfied, too. Properties of different gradient recovery techniques with respect to the construction of a quasi-normal field will be analyzed in a forthcoming paper.

Given the gradient recovery operator G_h we apply it to $\phi_h = \phi_h^k$ and define the *quasi-normal field*:

$$n_h(x) = \frac{(G_h\phi_h)(x)}{\|(G_h\phi_h)(x)\|}, \quad x \in \Omega_{\Gamma_h}. \tag{3.3}$$

Note that this field is only Lipschitz continuous; a main point in the analysis is that n_h can be approximated by a smooth vector field (cf. Lemma 7.1). The result (3.5) in the following lemma explains why we call n_h a "quasi-normal field". By $B(x;r)$ we denote the ball with center x and radius r.

LEMMA 3.1. *Let Assumption 3.1 be satisfied. Let $r_x > 0$ (depending on x) be small enough such that $B(x,r_x) \subset U$ for all $x \in \Gamma_h$. There exist constants c and $h_0 > 0$ such that for all $h \leq h_0$ and all $x \in \Gamma_h$ the following holds:*

$$\|n_h(x) - n_h(y)\| \leq c\|x-y\|, \quad \textit{for all} \;\; y \in B(x;r_x), \tag{3.4}$$

$$|\langle n_h(x), x-y\rangle| \leq ch^k\|x-y\| + c\|x-y\|^2, \quad \textit{for all} \;\; y \in \Gamma_h \cap B(x;r_x). \tag{3.5}$$

Proof. Given in [19]. In that paper the power $k+1$ instead of k is assumed in (3.1), but this stronger assumption is not needed in the proof of this lemma. □

The quasi-normal field can be used to define a local coordinate system similar to (2.1). Given n_h we define the map $E : \Gamma_h \times \mathbb{R} \to \mathbb{R}^3$, $E(z,t) := z + tn_h(z)$. In Lemma 3.1 and Theorem 3.2 in [19] it is proved that from (3.4) and (3.5) it follows that for $\epsilon > 0$ sufficiently small, this mapping is a bijection between $B_{\Gamma_h,\epsilon} := \Gamma_h \times (-\epsilon,\epsilon)$ and $E(B_{\Gamma_h,\epsilon}) =: U_{\Gamma_h} \subset \mathbb{R}^3$. We assume that ϵ and h are sufficiently small such that $U_{\Gamma_h} \subset U$, cf. (2.1), and $\Gamma \subset U_{\Gamma_h}$, $\hat{\Gamma}_h \subset U_{\Gamma_h}$. There is a unique decomposition

$$x = p_h(x) + d_h(x)n_h(p_h(x)), \quad x \in U_{\Gamma_h}, \tag{3.6}$$

with the *skew* projection $p_h : U_{\Gamma_h} \to \Gamma_h$ and d_h an *approximate* signed distance function to Γ_h , $|d_h(x)| = \|x - p_h(x)\|$. This decomposition resembles the one in (2.1). In the latter, however, one needs the exact level set function ϕ (to compute $n(p(x))$), whereas (3.6) is based on the quasi-normal field, which can be determined from the FE approximation ϕ_h. Furthermore, $d_h(x) = 0$ iff $x \in \Gamma_h$ holds, and we have the useful formula $d_h(x) = \langle x - p_h(x), n_h(p_h(x))\rangle$.

4. Parametrization of $\mathbf{\Gamma_h}$. We use $\hat{\Gamma}_h$ (the zero level of the piecewise linear function $\hat{\phi}_h$) and the quasi-normal field n_h for a computable parametrization of Γ_h (the zero level of the higher order FE function ϕ_h). From the assumptions above, it follows that

$$p_h|_{\hat{\Gamma}_h} : \hat{\Gamma}_h \to \Gamma_h \quad \text{is a bijection.}$$

Note that this bijection is (only) Lipschitz. The Lipschitz manifold $\hat{\Gamma}_h$ consists of triangles and convex quadrilaterals. Each quadrilateral is subdivided into two triangles. The resulting triangular mesh of $\hat{\Gamma}_h$ is denoted by $\mathcal{F}_h$, i.e.,

$$\hat{\Gamma}_h = \cup\{T \mid T \in \mathcal{F}_h\}. \tag{4.1}$$

The family $\{\mathcal{F}_h\}_{h>0}$ may be quite shape-*irregular*, but this does not cause problems, cf. remark 2 below. The mapping p_h is used for the parametrization of Γ_h. We need a transformation formula between integrals over $T \in \hat{\Gamma}_h$ and over $p_h(T)$, which is derived in the following lemma.

LEMMA 4.1. *For $T \in \mathcal{F}_h$, let $H \subset \mathbb{R}^3$ be the plane containing T, and let $\tilde{x} \mapsto U\tilde{x} + u$ be a parametrization $\mathbb{R}^2 \to H$ with an orthogonal matrix $U \in \mathbb{R}^{3\times 2}$. Then, for any measurable function $g\colon p_h(T) \to \mathbb{R}$ the transformation formula*

$$\int_{p_h(T)} g(y)\, d\sigma(y) = \int_T g(p_h(x))\, \hat{\mu}_h(x) d\sigma(x) \tag{4.2}$$

holds, with $\hat{\mu}_h(x) = \sqrt{\det(U^T Dp_h(x)^T Dp_h(x) U)}$.

Proof. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be an injective Lipschitz-mapping, and let $T \subseteq \mathbb{R}^2$ be Lebesgue-measurable. We recall the transformation rule

$$\int_{F(T)} g(y)\, d\sigma(y) = \int_T g(F(x))\, \mu(x) d\sigma(x), \quad \mu(x) = \sqrt{\det(DF(x)^T DF(x))}.$$

We apply this formula to the parametrization $x = F(\tilde{x}) = U\tilde{x} + u$ of H. The surface measure on H is

$$d\sigma(x) = \sqrt{\det(U^T U)}\, d\tilde{x} = d\tilde{x}, \tag{4.3}$$

because U is orthogonal. We also apply this formula to the parametrization $y = F(x) := p_h(x) = p_h(U\tilde{x} + u)$. The surface measure on this set is

$$d\sigma(y) = \sqrt{\det(U^T Dp_h(x)^T Dp_h(x) U)}\, d\tilde{x} = \sqrt{\det(U^T Dp_h(x)^T Dp_h(x) U)}\, d\sigma(x),$$

by the result in (4.3). ∎

5. Finite element discretization. We introduce the FE discretization of the Laplace-Beltrami equation (1.1). Our method has some similarity with the one presented in [6], but an essential difference is that we (only) need the FE approximations ϕ_h and $\hat{\phi}_h$ of ϕ. From ϕ_h the quasi-normal field n_h can be determined.

Let $\hat{S}_h$ be a FE space of piecewise polynomials of degree $m \geq 1$ on the triangulation $\mathcal{F}_h$ of $\hat{\Gamma}_h$, cf. (4.1):

$$\hat{S}_h = \{\, \hat{v}_h \in C(\hat{\Gamma}_h) \mid \hat{v}_{h|T} \in P_m \quad \text{for all } T \in \mathcal{F}_h \,\}. \tag{5.1}$$

REMARK 2. We briefly discuss two possible choices for the space $\hat{S}_h$. A first possibility is to use a trace space as introduced and analyzed in [17]. Such a space is constructed by taking the trace of a standard outer FE space, e.g. the space V_h^m used for the approximation of the level set function, cf. section 2. Its (optimal) approximation properties depend on the shape-regularity of $\{\mathcal{T}_h\}_{h>0}$ not on the shape-regularity of the family $\{\mathcal{F}_h\}_{h>0}$.

A second possibility is to define standard polynomial spaces directly on the triangulation $\mathcal{F}_h$. Although this triangulation is in general very shape irregular, it has a maximal angle property in three dimensions: in [18] it is shown, that if in the construction of $\mathcal{F}_h$ the quadrilaterals are subdivided in two triangles in a suitable way, the maximal inner angles in the resulting triangulation are uniformly bounded away from π. Hence, standard FE spaces on such a triangulation have optimal approximation quality, cf. [18] for more information.

We lift the space $\hat{S}_h$ to Γ_h by using the bijection $p_h : \hat{\Gamma}_h \to \Gamma_h$:

$$S_h := \{ v_h = \hat{v}_h \circ (p_h|_{\hat{\Gamma}_h})^{-1} \mid \hat{v}_h \in \hat{S}_h \}. \tag{5.2}$$

For the discretization we need a (sufficiently accurate) extension of the data f on Γ to Γ_h. This extension is denoted by f_h and is such that $\int_{\Gamma_h} f_h \, ds = 0$ holds.

REMARK 3. One possible choice for the extension f_h is $f_h = f^e - \frac{1}{|\Gamma_h|}\int_{\Gamma_h} f^e \, ds_h$, where f^e denotes the constant extension along the exact normals on Γ. This, however, is not feasible, since in our setting it is not reasonable to assume that the normals to Γ are known. Another possibility arises if we assume that f is a (smooth) function that is defined in a neighborhood U of Γ. As extension we may then take:

$$f_h(x) := f(x) - c_f \quad \text{for} \quad x \in \Gamma_h \quad \text{with } c_f := \frac{1}{|\Gamma_h|}\int_{\Gamma_h} f \, ds_h. \tag{5.3}$$

In the remainder we restrict to the latter choice of the extension. For f we assume the smoothness property $f \in H^1_\infty(U)$.

The discrete problem is as follows: Determine $u_h \in S_h$ with $\int_{\Gamma_h} u_h \, ds_h = 0$ such that

$$\begin{aligned} &a(u_h, v_h) = l(v_h) \quad \text{for all} \quad v_h \in S_h, \\ &a(u_h, v_h) := \int_{\Gamma_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} v_h \, ds_h, \quad l(v_h) := \int_{\Gamma_h} f_h v_h \, ds_h. \end{aligned} \tag{5.4}$$

For the implementation of this method we pull the discretization back to $\hat{\Gamma}_h$ and apply quadrature on the triangulation $\mathcal{F}_h$ of $\hat{\Gamma}_h$. We first treat the pull back procedure. For this we derive a relation between the tangential gradient on Γ_h and the tangential gradient on $\hat{\Gamma}_h$. For this we need several projections, defined as follows, with $\hat{n}_h(x)$ the exact normal on $\hat{\Gamma}_h$:

$$\hat{\mathbf{Q}}(x) = \mathbf{I} - \frac{1}{\hat{\alpha}(x)}\hat{n}_h(x)\bar{n}_h(y)^T, \quad \hat{\alpha}(x) = \hat{n}_h(x)^T\bar{n}_h(y), \quad y = p_h(x), \quad x \in \hat{\Gamma}_h, \tag{5.5}$$

$$\hat{\mathbf{P}}(x) = \mathbf{I} - \hat{n}_h(x)\hat{n}_h(x)^T, \quad x \in \hat{\Gamma}_h, \tag{5.6}$$

$$\bar{\mathbf{P}}(y) := \mathbf{I} - \bar{n}_h(y)\bar{n}_h(y)^T, \quad y \in \Gamma_h, \tag{5.7}$$

Note that $\hat{\mathbf{Q}}(x)$, $x \in \hat{\Gamma}_h$, is an oblique projection which maps into the tangential space $\bar{n}_h(y)^\perp$. The following commutation relations hold:

$$\hat{\mathbf{Q}}(x)\bar{\mathbf{P}}(y) = \bar{\mathbf{P}}(y), \;\; \bar{\mathbf{P}}(y)\hat{\mathbf{Q}}(x) = \hat{\mathbf{Q}}(x), \;\; \hat{\mathbf{P}}(x)\hat{\mathbf{Q}}(x) = \hat{\mathbf{P}}(x), \;\; \hat{\mathbf{Q}}(x)\hat{\mathbf{P}}(x) = \hat{\mathbf{Q}}(x). \tag{5.8}$$

LEMMA 5.1. *For $\hat{v}_h \in \hat{S}_h$, let $v_h = \hat{v}_h \circ (p_h|_{\hat{\Gamma}_h})^{-1} \in S_h$. For the tangential gradients the following relations hold for almost all $x \in \hat{\Gamma}_h$:*

$$
\begin{aligned}
&\nabla_{\hat{\Gamma}_h} \hat{v}_h(x) = \hat{\mathbf{P}}(x) Dp_h(x)^T \nabla_{\Gamma_h} v_h(y) = W(x) \nabla_{\Gamma_h} v_h(y) \quad \textit{with} \;\; y = p_h(x), \\
&\quad W(x) := \mathbf{I} - \hat{\mathbf{Q}}(x) + \hat{\mathbf{P}}(x) Dp_h(x)^T,
\end{aligned}
\tag{5.9}
$$

Proof. As $p_h|_{\hat{\Gamma}_h} : \hat{\Gamma}_h \to \Gamma_h$ is a bijection, we have $\hat{v}_h(x) = v_h(p_h(x))$. This relation and the ones below hold for almost all $x \in \hat{\Gamma}_h$. We apply the tangential gradient on $\hat{\Gamma}_h$ to both sides of this relation. This yields the first relation in (5.9). From

$$
\begin{aligned}
\hat{\mathbf{P}}(x) Dp_h(x)^T \nabla_{\Gamma_h} v_h(y) &= \hat{\mathbf{P}}(x) Dp_h(x)^T \bar{\mathbf{P}}(y) \nabla_{\Gamma_h} v_h(y), \\
\hat{\mathbf{P}}(x) Dp_h(x)^T \bar{\mathbf{P}}(y) &= W(x) \bar{\mathbf{P}}(y),
\end{aligned}
$$

we obtain the second relation in (5.9). ∎

From Lemma 9.1 below it follows that for h sufficiently small the matrix W is invertible. We assume that this condition on h is satisfied, i.e. W is invertible. We introduce the symmetric positive definite matrix function

$$
T_h(x) := W(x) W(x)^T. \tag{5.10}
$$

Using the transformation formulas in (4.2) and (5.9) we obtain the following "pulled back" *equivalent formulation of the discrete problem* (5.4): Determine $\hat{u}_h \in \hat{S}_h$ with $\int_{\hat{\Gamma}_h} \hat{u}_h \hat{\mu}_h \, d\hat{s}_h = 0$ such that

$$
\begin{aligned}
\int_{\hat{\Gamma}_h} F \nabla_{\hat{\Gamma}_h} \hat{u}_h \cdot \nabla_{\hat{\Gamma}_h} \hat{v}_h \, d\hat{s}_h &= \int_{\hat{\Gamma}_h} (f_h \circ p_h) \hat{v}_h \, \hat{\mu}_h \, d\hat{s}_h \quad \text{for all} \;\; \hat{v}_h \in \hat{S}_h, \\
F(\hat{x}) &:= T_h(\hat{x})^{-1} \hat{\mu}_h(\hat{x}), \quad \hat{x} \in \hat{\Gamma}_h.
\end{aligned}
\tag{5.11}
$$

Clearly, for the implementation of this discretization we need quadrature.

We introduce quadrature along the same lines as in [4]. Let $\tilde{T}$ be the unit triangle in $\mathbb{R}^2$, $T \in \mathcal{F}_h$ and $M_T : \tilde{T} \to T$ an affine mapping $M_T \tilde{x} = B_T \tilde{x} + b_T = x$, $\tilde{x} \in \tilde{T}$, $x \in T$. We consider a quadrature rule on $\tilde{T}$ of the form $Q_{\tilde{T}}(\tilde{\phi}) = \sum_{l=1}^{L} \tilde{\omega}_l \tilde{\phi}(\tilde{\xi}_l)$ with *strictly positive* weights ω_l and quadrature nodes $\tilde{\xi}_l \in \tilde{T}$. This induces a quadrature rule on T:

$$
Q_T(\hat{\phi}) := \sum_{l=1}^{L} \hat{\omega}_l \hat{\phi}(\hat{\xi}_l), \quad \hat{\omega}_l = |T| \tilde{\omega}_l, \; \hat{\xi}_l = M_T(\tilde{\xi}_l). \tag{5.12}
$$

Note that, although not explicit in the notation, $\hat{\omega}_l$, $\hat{\xi}_l$ depend on T. We apply quadrature to the discrete problem (5.11) as follows. First we consider the approximation of the bilinear form $a(u_h, v_h)$. Using the correspondence $v_h \circ p_h = \hat{v}_h$, we can represent $a(u_h, v_h)$ as follows, cf. (5.11):

$$
a(u_h, v_h) = \int_{\hat{\Gamma}_h} F \nabla_{\hat{\Gamma}_h} \hat{u}_h \cdot \nabla_{\hat{\Gamma}_h} \hat{v}_h \, d\hat{s}_h = \sum_{T \in \mathcal{F}_h} \sum_{i,j=1}^{3} \int_T F_{ij} \partial_i^\Gamma \hat{u}_h \partial_j^\Gamma \hat{v}_h \, d\hat{s}_h, \tag{5.13}
$$

with ∂_i^Γ the ith component of the vector $\nabla_{\hat{\Gamma}_h} = \hat{\mathbf{P}} \nabla$. Quadrature results in an approximate bilinear form, given by

$$
a_h(u_h, v_h) = \sum_{T \in \mathcal{F}_h} Q_T(F \nabla_{\hat{\Gamma}_h} \hat{u}_h \cdot \nabla_{\hat{\Gamma}_h} \hat{v}_h). \tag{5.14}
$$

For the right hand-side functional $l(v_h) = \int_{\Gamma_h} f_h v_h \, ds_h = \int_{\hat{\Gamma}_h} (f \circ p_h - c_f) \hat{v}_h \hat{\mu}_h \, d\hat{s}_h$ we have the approximation

$$
\begin{aligned}
l_h(v_h) &= \sum_{T \in \mathcal{F}_h} Q_T\big(f_h^q \hat{\mu}_h \hat{v}_h\big), \quad \text{for all } v_h \in S_h, \ \hat{v}_h = v_h \circ p_h, \\
f_h^q &:= f \circ p_h - c_f^q, \quad c_f^q := \frac{1}{A} \sum_{T \in \mathcal{F}_h} Q_T(f \circ p_h \, \hat{\mu}_h), \quad A := \sum_{T \in \mathcal{F}_h} Q_T(\hat{\mu}_h).
\end{aligned} \tag{5.15}
$$

The constant shift c_f^q is taken such that the consistency condition $l_h(1) = 0$ is satisfied. The *final discrete problem*, i.e., after quadrature, *is as follows*: Determine $u_h^q \in S_h$ with $\sum_{T \in \mathcal{F}_h} Q_T(\hat{u}_h^q \hat{\mu}_h) = 0$ such that

$$
a_h(u_h^q, v_h) = l_h(v_h) \quad \text{for all} \quad v_h \in S_h. \tag{5.16}
$$

Using Lemma 9.3 it follows that this final discrete problem has a unique solution.

6. Outline of the analysis. In the sections 7–12 we present an error analysis of the discrete problem (5.16). The analysis is rather technical and contains ingredients that are not standard in the literature. We outline the structure and main ideas of the analysis. Central in the analysis is the Strang Lemma 9.4, in which the discretization error is bounded by three different error components, namely an approximation error, a geometric error and a quadrature error. In the sections 10–12 bounds for these three components are derived. In the sections 7 and 8 properties of the quasi-normal field n_h and the skew projection p_h are derived. An important result is Lemma 8.2, stating that by using suitable projections the error bound of order $\mathcal{O}(h^k)$ in (8.8) can be improved to $\mathcal{O}(h^{k+1})$ in (8.9), (8.10).
In section 12 the error due to quadrature is analyzed. As far as we know, this error has not been considered in other papers with error analyses of FE methods for surface PDEs. Treating the quadrature issue is essential for the analysis of our method. The reason for this is that the discrete problem *before* quadrature (5.11) contains an integrand that is *not smooth inside the triangles* $T \in \mathcal{F}_h$. The non-smoothness is caused by the use of the quasi-normal field, which is only Lipschitz. Due to the nonsmooth integrand, standard analyses of quadrature errors as in e.g. [4], do not yield satisfactory bounds. The analysis of the quadrature error is based on the following idea. Consider an integral $\int_T F g_1 \cdot g_2 \, d\hat{s}_h$, with vector functions g_i that are smooth on T and a matrix function F that is not necessarily smooth on T. Assume that F^s is a smooth approximation of F. For the quadrature error we use the splitting

$$
\begin{aligned}
E_T(F g_1 \cdot g_2) &:= \int_T F g_1 \cdot g_2 \, d\hat{s}_h - Q_T(F g_1 \cdot g_2) \\
&= \int_T (F - F^s) g_1 \cdot g_2 \, d\hat{s}_h + E_T(F^s g_1 \cdot g_2) + Q_T\big((F^s - F) g_1 \cdot g_2\big).
\end{aligned}
$$

The error terms $\int_T (F - F^s) g_1 \cdot g_2 \, d\hat{s}_h$ and $Q_T((F^s - F) g_1 \cdot g_2)$ can be controlled by suitable bounds for $F - F^s$ (as in Corollary 12.3). Since $F^s g_1 \cdot g_2$ is smooth the term $E_T(F^s g_1 \cdot g_2)$ can be bounded using standard quadrature error analysis. A smooth approximation of the (matrix) function F is derived and analyzed in section 12.1.

In the analysis different (skew) projections play a key role. For these projections we use boldface notation. For the readers convenience we summarize these projections and the normal fields that are used:

$$
\begin{aligned}
& n : U \to \mathbb{R}^3 \ \ \text{(exact normal on } \Gamma), \quad n_h : \Omega_{\Gamma_h} \to \mathbb{R}^3 \ \ \text{(quasi-normal field)}, \\
& \hat{n}_h : \hat{\Gamma}_h \to \mathbb{R}^3 \ \ \text{(exact normal on } \hat{\Gamma}_h), \quad \bar{n}_h : \Gamma_h \to \mathbb{R}^3 \ \ \text{(exact normal on } \Gamma_h), \\
& \mathbf{P}(x) = \mathbf{I} - n(x)n(x)^T, \quad x \in U, \qquad (6.1) \\
& \hat{\mathbf{P}}(x) = \mathbf{I} - \hat{n}_h(x)\hat{n}_h(x)^T, \quad x \in \hat{\Gamma}_h, \quad \bar{\mathbf{P}}(y) := \mathbf{I} - \bar{n}_h(y)\bar{n}_h(y)^T, \quad y \in \Gamma_h. \qquad (6.2) \\
& \hat{\mathbf{Q}}(x) = \mathbf{I} - \frac{1}{\hat{\alpha}(x)}\hat{n}_h(x)\bar{n}_h(y)^T, \quad \hat{\alpha}(x) = \hat{n}_h(x)^T\bar{n}_h(y), \quad y = p_h(x), \quad x \in \hat{\Gamma}_h, \qquad (6.3) \\
& \mathbf{Q}(x) = \mathbf{I} - \frac{1}{\alpha(x)}n_h(x)\bar{n}_h(x)^T, \quad \alpha(x) := n_h(x)^T\bar{n}_h(x), \quad x \in \Gamma_h. \qquad (6.4)
\end{aligned}
$$

We use the following notation in many proofs below: For any $x \in \hat{\Gamma}_h$ (or $x \in U$):

$$y := p_h(x) \in \Gamma_h, \quad z := p(y) = p \circ p_h(x) \in \Gamma, \quad \text{and} \quad \zeta := p(x) \in \Gamma.$$

7. Properties of n_h and p_h. In this section we derive some properties of the quasi-normal field n_h and the skew projection p_h onto Γ_h that we need in the analysis further on. We start with a lemma in which it is shown that Dn_h is close to a smooth (matrix) function.

LEMMA 7.1. *The following holds for all sufficiently small h:*

$$\|n_h - \frac{\nabla\phi}{\|\nabla\phi\|}\|_{L^\infty(\Omega_{\Gamma_h})} \leq ch^k, \qquad (7.1)$$

$$\|n_h - n\|_{L^\infty(\Gamma_h)} \leq ch^k, \qquad (7.2)$$

$$\|Dn_h - D(\frac{\nabla\phi}{\|\nabla\phi\|})\|_{L^\infty(\Omega_{\Gamma_h})} \leq ch^{k-1}. \qquad (7.3)$$

Proof. For the gradient recovery operator applied to the FE approximation ϕ_h of the level set function ϕ we write $G_h = G_h\phi_h$. Using (3.1),(3.2),(2.2) and standard interpolation error results we get

$$
\begin{aligned}
\|G_h - \nabla\phi\|_{L^\infty(U)} &\leq \|G_h(\phi_h - I_h\phi)\|_{L^\infty(U)} + \|G_h(I_h\phi) - \nabla\phi\|_{L^\infty(U)} \\
&\leq c\|\phi_h - I_h\phi\|_{H^1_\infty(U^e)} + ch^k \leq ch^k.
\end{aligned} \qquad (7.4)
$$

From this and $n_h = \|G_h\|^{-1}G_h$ the result in (7.1) follows. Using this result we get, for $x \in \Omega_{\Gamma_h}$:

$$\|n_h(x) - n(x)\| \leq ch^k + \|n(x) - \frac{\nabla\phi(x)}{\|\nabla\phi(x)\|}\| = ch^k + \|\frac{\nabla\phi(p(x))}{\|\nabla\phi(p(x))\|} - \frac{\nabla\phi(x)}{\|\nabla\phi(x)\|}\|.$$

For $x \in \Gamma_h$ we have $\|x - p(x)\| \leq ch^{k+1}$ and thus $\|\nabla\phi(x) - \nabla\phi(p(x))\| \leq ch^{k+1}$. Hence we get the result (7.2). For the derivatives we have

$$
\begin{aligned}
Dn_h = D(\frac{G_h}{\|G_h\|}) &= \frac{1}{\|G_h\|}\big(I - \frac{1}{\|G_h\|^2}G_hG_h^T\big)DG_h \\
D(\frac{\nabla\phi}{\|\nabla\phi\|}) &= \frac{1}{\|\nabla\phi\|}\big(I - \frac{1}{\|\nabla\phi\|^2}\nabla\phi\nabla\phi^T\big)D^2\phi.
\end{aligned} \qquad (7.5)
$$

Using an inverse inequality and the result in (7.4) we obtain

$$\begin{aligned}\|G_h-\nabla\phi\|_{H^1_\infty(U)} &\le \|G_h-I_h(\nabla\phi)\|_{H^1_\infty(U)}+\|I_h(\nabla\phi)-\nabla\phi\|_{H^1_\infty(U)}\\ &\le ch^{-1}\|G_h-I_h(\nabla\phi)\|_{L^\infty(U)}+ch^k\\ &\le ch^{-1}\big(\|G_h-\nabla\phi\|_{L^\infty(U)}+\|\nabla\phi-I_h(\nabla\phi)\|_{L^\infty(U)}\big)+ch^k\le ch^{k-1}.\end{aligned}$$

From this it follows that $\|DG_h-D^2\phi\|_{L^\infty(U)}\le ch^{k-1}$ holds. Using this and the result in (7.4) in combination with the formulas (7.5) proves the result in (7.3). □

The next lemma quantifies how well p_h approximates p and d_h approximates d.

LEMMA 7.2. *For h sufficiently small the following holds:*

$$\|p_h-p\|_{L^\infty(\Omega_{\Gamma_h})}\le ch^{k+1},\quad \|p\circ p_h-p\|_{L^\infty(\Omega_{\Gamma_h})}\le ch^{k+1}, \tag{7.6}$$

$$\|d_h-d\|_{L^\infty(\Omega_{\Gamma_h})}\le ch^{k+1}. \tag{7.7}$$

Proof. We give an outline of the proof. Details are given in [13]. Take $x\in\Omega_{\Gamma_h}$ and $q\in\Gamma_h$ such that $p(x)=p(q)$. Let $\bar d\in\mathbb{R}$ be such that $\bar d\, n(q)=x-q$. The relation

$$\|p_h(x)-q\|^2=\bar d\langle n(q),p_h(x)-q\rangle-d_h(x)\langle n_h(p_h(x)),p_h(x)-q\rangle \tag{7.8}$$

holds. Using (7.2), (3.5), $|\bar d|\le ch$ and $|d_h(x)|\le ch$ one can derive bounds of the form $ch^{k+1}\|p_h(x)-q\|+ch\|p_h(x)-q\|^2$ for both terms on the right hand-side in (7.8). Hence, for h sufficiently small, $\|p_h(x)-q\|\le ch^{k+1}$ holds. Using $\|q-p(x)\|\le ch^{k+1}$ and a triangle inequality we obtain the first estimate in (7.6). The second result in (7.6) follows from a triangle inequality. For the result in (7.7) we use the representation

$$\begin{aligned}d_h(x)-d(x)&=\langle p(x)-p_h(x),n(p(x))\rangle+\langle x-p_h(x),n_h(p_h(x))-n(p_h(x))\rangle\\ &\quad+\langle x-p_h(x),n(p_h(x))-n(p(x))\rangle.\end{aligned}$$

Each of the terms on the right hand-side can be bounded by ch^{k+1}. □

8. Properties of the Jacobian Dp_h. The Jacobian Dp_h plays a key role in the discretization (5.11) (cf. definition of W and $\hat\mu_h$). In this section we derive properties of this Jacobian that we need in our analysis. First we consider the Jacobian of the *exact* projection p onto Γ given in (2.1). Differentiating the relation (2.1) and using $n(x)=n(p(x))$ we get, for $x\in U$,

$$\big(\mathbf{I}+d(x)H(p(x))\big)Dp(x)=\mathbf{P}(x),\quad \mathbf{P}(x)=\mathbf{I}-n(x)n(x)^T,H(y)=Dn(y). \tag{8.1}$$

This formula has equivalent representations due to $\mathbf{P}(x)=\mathbf{P}(p(x))$ and $H(p(x))=\mathbf{P}(x)H(p(x))=H(p(x))\mathbf{P}(x)$. We derive a formula for Dp_h, cf. Lemma 8.1 below. It turns out that we need a skew projection as a substitute for the projection $\mathbf{P}$ in (8.1). This skew projection $\mathbf{Q}$ is the one given in (6.4). The following relations hold, with $\bar{\mathbf{P}}$ as in (6.2):

$$\mathbf{Q}\bar{\mathbf{P}}=\bar{\mathbf{P}},\quad \bar{\mathbf{P}}\mathbf{Q}=\mathbf{Q}. \tag{8.2}$$

LEMMA 8.1. *For a. e. $x\in U$, the following relations hold with $y=p_h(x)\in\Gamma_h$,*

$$\big(\mathbf{I}+d_h(x)\mathbf{Q}(y)Dn_h(y)\big)Dp_h(x)=\mathbf{Q}(y), \tag{8.3}$$

$$\mathbf{Q}(y)Dp_h(x)=Dp_h(x)=Dp_h(x)\mathbf{Q}(y). \tag{8.4}$$

Proof. Let $\bar{d}_h$ be the exact signed distance function to Γ_h. Differentiating $\bar{d}_h(p_h(x)) = 0$, which holds for a.e. $x \in U$, yields

$$\bar{n}_h(y)^T Dp_h(x) = 0. \tag{8.5}$$

Applying this to the differential of $p_h = id - d_h \cdot n_h \circ p_h$,

$$Dp_h = \mathbf{I} - d_h Dn_h(y) Dp_h - n_h(y) \nabla d_h^T, \tag{8.6}$$

yields $0 = \bar{n}_h(y)^T - d_h \bar{n}_h(y)^T Dn_h(y) Dp_h - \bar{n}_h(y)^T n_h(y) \nabla d_h^T$. Hence,

$$\nabla d_h^T = \frac{1}{\alpha} \left(\bar{n}_h(y)^T - d_h \bar{n}_h(y)^T Dn_h(y) Dp_h \right).$$

Inserting this into (8.6) and rearranging completes the proof of (8.3). The equation $\mathbf{Q}(y) Dp_h = Dp_h$ follows immediately from (8.5) and the definition of $\mathbf{Q}$. The equation $Dp_h(x) = Dp_h(x)\mathbf{Q}(y)$ follows from (8.3). ▯

Below, we frequently use that $\mathbf{I} + M$, $M \in \mathbb{R}^{n\times n}$, is invertible if $\rho(M) < 1$ and that

$$(\mathbf{I} + M)^{-1} = \mathbf{I} - (\mathbf{I} + M)^{-1} M, \quad M \in \mathbb{R}^{n\times n}, \quad \rho(M) < 1. \tag{8.7}$$

LEMMA 8.2. *For sufficiently small h, the following holds, with projections $\mathbf{P}$, $\hat{\mathbf{P}}$, $\bar{\mathbf{P}}$ defined in* (6.1),(6.2), (6.2)*:*

$$\|Dp - Dp_h\|_{L^\infty(\hat{\Gamma}_h)} \le ch^k, \tag{8.8}$$

$$\|\mathbf{P}\big(Dp - Dp_h\big)\hat{\mathbf{P}}\|_{L^\infty(\hat{\Gamma}_h)} \le ch^{k+1}, \tag{8.9}$$

$$\|(\bar{\mathbf{P}} \circ p_h)\big(Dp - Dp_h\big)\hat{\mathbf{P}}\|_{L^\infty(\hat{\Gamma}_h)} \le ch^{k+1}. \tag{8.10}$$

Proof. Let $x \in \hat{\Gamma}_h$ be arbitrary and $\zeta = p(x) \in \Gamma$. Let $\tilde{n} = \nabla\phi/\|\nabla\phi\|$ which is defined on U. As $\tilde{n} \equiv n$ on Γ, we have $p(x) = x - d(x)\tilde{n}(\zeta)$. Differentiating this relation we obtain the following representation for the Jacobian Dp:

$$Dp(x) = \big(\mathbf{I} + d(x)\mathbf{P}(\zeta)D\tilde{n}(\zeta)\big)^{-1}\mathbf{P}(\zeta) = (\mathbf{I} + B_1)^{-1}\mathbf{P}(\zeta), \quad B_1 = d(x)\mathbf{P}(\zeta)D\tilde{n}(\zeta).$$

From (8.3) we get, with $y = p_h(x) \in \Gamma_h$:

$$Dp_h(x) = \big(\mathbf{I}+d_h(x)\mathbf{Q}(y)Dn_h(y)\big)^{-1}\mathbf{Q}(y) = (\mathbf{I}+B_2)^{-1}\mathbf{Q}(y),\ B_2 = d_h(x)\mathbf{Q}(y)Dn_h(y).$$

Using (7.6) we get $\|\zeta - y\| \le ch^{k+1}$. Define $R_i := (\mathbf{I} + B_i)^{-1}B_i$, hence, by (8.7), $(\mathbf{I} + B_i)^{-1} = \mathbf{I} - R_i$. From $|d_h(x)| \le ch^2$, $|d(x)| \le ch^2$ and the definition of B_i we get $\|R_i\| \le ch^2$. From the definitions we obtain

$$Dp(x) - Dp_h(x) = \mathbf{P}(\zeta) - \mathbf{Q}(y) + R_2\mathbf{Q}(y) - R_1\mathbf{P}(\zeta), \tag{8.11}$$

$$\begin{aligned}\mathbf{P}(\zeta) - \mathbf{Q}(y) = {} & \big(\frac{1}{\alpha(y)} - 1\big)n_h(y)\bar{n}_h(y)^T \\ & + \big(n_h(y) - n(\zeta)\big)\bar{n}_h(y)^T + n(\zeta)\big(\bar{n}_h(y) - n(\zeta)\big)^T.\end{aligned} \tag{8.12}$$

Using (7.2) and (2.3) we get $\|n_h(y) - n(y)\| \leq ch^k$, $\|\bar{n}_h(y) - n(y)\| \leq ch^k$, and combining this with $|\alpha(y)-1| = \frac{1}{2}\|n_h(y)-\bar{n}_h(y)\|^2$, the smoothness of n and $\|\zeta - y\| \leq ch^{k+1}$, we obtain

$$\Big|\frac{1}{\alpha(y)} - 1\Big| \leq ch^{2k}, \tag{8.13}$$

$$\|n_h(y) - n(\zeta)\| \leq \|n_h(y) - n(y)\| + \|n(y) - n(\zeta)\| \leq ch^k, \tag{8.14}$$

$$\|\bar{n}_h(y) - n(\zeta)\| \leq \|\bar{n}_h(y) - n(y)\| + \|n(y) - n(\zeta)\| \leq ch^k. \tag{8.15}$$

From these results and (8.12) we get

$$\|\mathbf{Q}(y) - \mathbf{P}(\zeta)\| \leq ch^k. \tag{8.16}$$

Using the smoothness of $\tilde{n}$ and the result in (7.3) we obtain:

$$\|D\tilde{n}(\zeta) - Dn_h(y)\| \leq \|D\tilde{n}(\zeta) - D\tilde{n}(y)\| + \|D\tilde{n}(y) - Dn_h(y)\| \leq ch^{k-1}.$$

Combining this with $|d(x)| \leq ch^2$ and $|d(x) - d_h(x)| \leq ch^{k+1}$, cf. (7.7), yields

$$\begin{aligned} \|R_1 - R_2\| &= \|(\mathbf{I} + B_1)^{-1} - (\mathbf{I} + B_2)^{-1}\| \leq c\|B_1 - B_2\| \\ &\leq c|d(x)|\|\mathbf{P}(\zeta) - \mathbf{Q}(y)\| + c|d(x)|\|D\tilde{n}(\zeta) - Dn_h(y)\| + c|d(x) - d_h(x)| \leq ch^{k+1}. \end{aligned} \tag{8.17}$$

Combining this and (8.16) with the result in (8.11) proves the result in (8.8).
Let $\tilde{\mathbf{P}}$ denote either $\mathbf{P}(x)$ or $\bar{\mathbf{P}}(y)$. For obtaining the additional h factor in (8.9)-(8.10) we only have to treat the term

$$\tilde{\mathbf{P}}\Big(\big(n_h(y) - n(\zeta)\big)\bar{n}_h(y)^T - n(\zeta)\big(\bar{n}_h(y) - n(\zeta)\big)^T\Big)\hat{\mathbf{P}}(x),$$

because all other terms have bounds ch^{k+1}, as derived above. The additional h factor comes from the terms $\|\bar{n}_h(y)^T\hat{\mathbf{P}}(x)\|$ and $\|\tilde{\mathbf{P}}n(\zeta)\|$. For the former we have

$$\|\bar{n}_h(y)^T\hat{\mathbf{P}}(x)\| = \|\hat{\mathbf{P}}(x)\big(\bar{n}_h(y) - \hat{n}_h(x)\big)\| \leq ch. \tag{8.18}$$

For the latter, if $\tilde{\mathbf{P}} = \mathbf{P}(x)$ then $\tilde{\mathbf{P}}n(\zeta) = \tilde{\mathbf{P}}n(x) = 0$ holds, and if $\tilde{\mathbf{P}} = \bar{\mathbf{P}}(y)$ we obtain from (8.15):

$$\|\tilde{\mathbf{P}}n(\zeta)\| = \|\bar{\mathbf{P}}(y)\big(n(\zeta) - \bar{n}_h(y)\big)\| \leq \|n(\zeta) - \bar{n}_h(y)\| \leq ch^k.$$

This completes the proof. ◻

The estimates in lemma 8.2 play a key role in the error analysis of our method. In Section 14 we give results of a numerical experiment which show that the bounds in the estimates are sharp. In particular, for obtaining the h^{k+1} bounds the projections in the terms on the left hand-side are essential.

9. Strang Lemma. In this section we derive a Strang lemma. In the analysis we will need the constant extension of a function w on Γ along the normals n to a function w^e on U given by

$$w^e(x) := w \circ p(x) \quad \text{for all } x \in U. \tag{9.1}$$

We also use the *lift* of a function defined on Γ_h (or on $\hat{\Gamma}_h$) to a function defined on Γ along the normals n. More precisely, for a function w defined on Γ_h (or on $\hat{\Gamma}_h$), its lift w^ℓ to Γ is given by

$$w^\ell \circ p(x) = w(x) \quad \text{for all } x \in \Gamma_h \text{ (or } x \in \hat{\Gamma}_h\text{)}. \tag{9.2}$$

The lifted FE space is denoted by $S_h^\ell := \{\, v_h^\ell \mid v_h \in S_h \,\}$. We need the following matrix function on Γ:

$$A_\Gamma(p(x)) = \frac{1}{\mu_h(x)}\mathbf{P}(x)[I - d(x)H(x)]\bar{\mathbf{P}}(x)[I - d(x)H(x)]\mathbf{P}(x), \quad x \in \Gamma_h, \tag{9.3}$$

with μ_h as in (2.4) and the projections as in (6.1), (6.2). From [6, formula (2.14)] we have the integral identity

$$\int_{\Gamma_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} v_h \, ds_h = \int_\Gamma A_\Gamma \nabla_\Gamma u_h^\ell \cdot \nabla_\Gamma v_h^\ell \, ds.$$

Using this we obtain that if u_h solves (5.4), then the lifted function $u_h^\ell \in S_h^\ell$ satisfies

$$\int_\Gamma A_\Gamma \nabla_\Gamma u_h^\ell \cdot \nabla_\Gamma v_h \, ds = \int_\Gamma \frac{1}{\mu_h^\ell} f_h^\ell v_h \, ds \quad \text{for all} \;\; v_h \in S_h^\ell. \tag{9.4}$$

We also need the following estimates, which follow from (2.11), (2.12) in [6] for the isoparametric approximation of the interface considered therein:

$$\begin{aligned} &\|\nabla_\gamma v^e\|_{L^2(\gamma)} \le c\|\nabla_\Gamma v\|_{L^2(\Gamma)}, \quad v \in H^1(\Gamma), \gamma \in \{\hat{\Gamma}_h, \Gamma_h\}, \\ &\|\nabla_\Gamma v^\ell\|_{L^2(\Gamma)} \le c\|\nabla_\gamma v\|_{L^2(\gamma)}, \quad v \in H^1(\gamma), \gamma \in \{\hat{\Gamma}_h, \Gamma_h\}. \end{aligned} \tag{9.5}$$

We only sketch the proof. Let w be a smooth function on U. We have $\nabla_\gamma w^e(x) = \mathbf{P}_\gamma(x) Dp(x)^T \nabla_\Gamma w(p(x))$, $x \in \gamma$, and thus $\|\nabla_\gamma w^e(x)\| \le \|Dp(x)\| \|\nabla_\Gamma w(p(x))\|$. We also have $\nabla_\gamma w(x) = \mathbf{P}_\gamma(x) Dp(p(x))^T \nabla_\Gamma u^\ell(p(x))$, $x \in \gamma$. Consider $\xi \in \mathbb{R}^3$ with $\mathbf{P}\xi = \xi$. For sufficiently small h we have $\|\mathbf{P}_\gamma \mathbf{P}\xi\| \ge \frac{1}{2}\|\xi\|$. Together with (8.1), we get $\|\nabla_\Gamma w^\ell(p(x))\| \le c\|\nabla_\gamma w(x)\|$. Using the transformation rule and (8.1) yield (9.5) for the smooth function w. One concludes with a standard density argument.

We first derive ellipticity of the bilinear form $a_h(\cdot,\cdot)$ in (5.14). For this we need that the matrix F, cf. (5.11), is positive definite. To derive this result, in the next lemma we first consider the matrix W.

Lemma 9.1. *For h sufficiently small the following holds:*

$$\|W - \mathbf{I}\|_{L^\infty(\hat{\Gamma}_h)} \le ch. \tag{9.6}$$

Proof. Take $x \in \hat{\Gamma}_h$. We recall the definition $W(x) = \mathbf{I} - \hat{\mathbf{Q}}(x) + \hat{\mathbf{P}}(x) Dp_h(x)^T$. We drop the x-dependence in the notation. From (8.3) we get, due to $|d_h(x)| \le ch^2$, for h sufficiently small, $Dp_h(x) = \big(\mathbf{I} + d_h(x)\mathbf{Q}(y)Dn_h(y)\big)^{-1}\mathbf{Q}(y) = \mathbf{Q}(y) + \mathcal{O}(h^2)$, with $y = p_h(x)$. Hence, $W = \mathbf{I} - \hat{\mathbf{Q}} + \hat{\mathbf{P}}\mathbf{Q}(y)^T + \mathcal{O}(h^2)$ holds. Using $\hat{\mathbf{Q}} = \hat{\mathbf{Q}}\hat{\mathbf{P}}$ we get $\|W - \mathbf{I}\| \le \|(\mathbf{I} - \hat{\mathbf{Q}})\hat{\mathbf{P}}\| + \|\hat{\mathbf{P}}(\mathbf{Q}(y)^T - \mathbf{I})\| + ch^2$. Using the definitions of the projection one easily derives $\|(\mathbf{I} - \hat{\mathbf{Q}})\hat{\mathbf{P}}\| \le ch$ and $\|\hat{\mathbf{P}}(\mathbf{Q}(y)^T - \mathbf{I})\| \le ch$. ◻

Corollary 9.2. *For h sufficiently small the matrix $F(x)$ in* (5.11)*, $x \in \hat{\Gamma}_h$, is uniformly symmetric positive definite, i.e. there is a constant $\lambda_{\min}(F) > 0$ such that*

$$z^T F(x) z \ge \lambda_{\min}(F)\|z\|^2 \quad \textit{for almost all} \;\; x \in \hat{\Gamma}_h \quad \textit{and all} \;\; z \in \mathbb{R}^3.$$

Proof. From lemma 9.1 and (5.10), (5.11), we obtain for all sufficiently small h and arbitrary $z \in \mathbb{R}^3$ that

$$z^T F(x) z = \hat{\mu}_h(x) \|W(x)^{-1} z\|^2 \geq \frac{\hat{\mu}_h(x)}{2} \|z\|^2, \quad x \in \hat{\Gamma}_h.$$

We recall the definition $\hat{\mu}_h(x) = \sqrt{\det(U^T Dp_h(x)^T Dp_h(x) U)}$. The matrix U depends on the triangle $T \in \mathcal{F}_h$ and satisfies $\hat{\mathbf{P}}(x)U = U$, cf. Lemma 4.1. For h sufficiently small we have $Dp_h(x) = \mathbf{Q}(y) + \mathcal{O}(h^2)$, $y = p_h(x)$. With $\zeta = p(x)$, we have $\mathbf{P}(\zeta) = \mathbf{P}(x)$. From (2.3) and $n(x)n(x)^T - \hat{n}_h(x)\hat{n}_h(x)^T = n(x)\big(n(x) - \hat{n}_h(x)\big)^T + \big(n(x) - \hat{n}_h(x)\big)\hat{n}_h(x)^T$ it follows that

$$\|\mathbf{P} - \hat{\mathbf{P}}\|_{L^\infty(\hat{\Gamma}_h)} \leq ch. \tag{9.7}$$

Using this and the result in (8.16), we get

$$\|\mathbf{Q}(y) - \hat{\mathbf{P}}(x)\| \leq \|\mathbf{Q}(y) - \mathbf{P}(\zeta)\| + \|\mathbf{P}(\zeta) - \mathbf{P}(x)\| + \|\mathbf{P}(x) - \hat{\mathbf{P}}(x)\| \leq ch.$$

This yields $Dp_h(x) = \hat{\mathbf{P}}(x) + \mathcal{O}(h)$ and consequently

$$U^T Dp_h(x)^T Dp_h(x) U = U^T \hat{\mathbf{P}}(x) \hat{\mathbf{P}}(x) U + \mathcal{O}(h) = U^T U + \mathcal{O}(h) = \mathbf{I} + \mathcal{O}(h).$$

Thus, for h sufficiently small, we have that $\hat{\mu}_h(x) = 1 + \mathcal{O}(h)$ ($x \in \hat{\Gamma}_h$) is uniformly (in x) bounded from below by a strictly positive constant. □

Using the result of the previous corollary we can derive ellipticity of $a_h(\cdot,\cdot)$.

LEMMA 9.3. *Assume that the quadrature rule $Q_{\tilde{T}}$ is exact for all polynomials of degree $2m-2$. There exists a constants $\gamma > 0$ and $h_0 > 0$ such that for all $h \leq h_0$*

$$a_h(v_h, v_h) \geq \gamma \|\nabla_{\Gamma_h} v_h\|^2_{L^2(\Gamma_h)} \quad \textit{for all} \ \ v_h \in S_h. \tag{9.8}$$

Proof. Let h be sufficiently small such that the matrix F is uniformly positive definite, cf. Corollary 9.2, and $c_0 := \lambda_{\min}(F) > 0$, independent of h. Using the fact that the quadrature rule $Q_{\tilde{T}}$ on $\tilde{T}$ is exact for all polynomials of degree $2m-2$ and that the weights are strictly positive, we get

$$a_h(v_h, v_h) \geq c_0 \sum_{T \in \mathcal{F}_h} Q_T\big(\|\nabla_{\hat{\Gamma}_h} \hat{v}_h\|^2\big) = c_0 \|\nabla_{\hat{\Gamma}_h} \hat{v}_h\|^2_{L^2(\hat{\Gamma}_h)} \geq \gamma \|\nabla_{\Gamma_h} v_h\|^2_{L^2(\Gamma_h)}$$

with $\gamma > 0$ due to (5.9) and (9.6). □

Based on this ellipticity property we apply standard arguments to derive the following variant of the Strang Lemma.

THEOREM 9.4. *Assume h is sufficiently small such that $a_h(\cdot,\cdot)$ has the ellipticity property* (9.3). *Define the data extension error $\tilde{E}_f := \|f - \frac{1}{\mu_h^\ell} f_h^\ell\|_{L^2(\Gamma)}$. For the solution u_h^q of* (5.16) *the following error bound holds:*

$$\begin{aligned} &\|\nabla_\Gamma(u - (u_h^q)^\ell)\|_{L^2(\Gamma)} \\ &\leq c \min_{v_h \in S_h} \Big[\|\nabla_\Gamma(u - v_h^\ell)\|_{L^2(\Gamma)} + \|(I - A_\Gamma)\mathbf{P}\|_{L^\infty(\Gamma)} \|\nabla_{\Gamma_h} v_h\|_{L^2(\Gamma_h)} \\ &+ \sup_{w_h \in S_h/\mathbb{R}} \frac{a(v_h, w_h) - a_h(v_h, w_h)}{\|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)}}\Big] + c \sup_{w_h \in S_h/\mathbb{R}} \frac{l(w_h) - l_h(w_h)}{\|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)}} + \tilde{E}_f. \end{aligned} \tag{9.9}$$

Proof. Take an arbitrary $v_h \in S_h$. We start with a triangle inequality and (9.5):

$$\begin{aligned}\|\nabla_\Gamma(u-(u_h^q)^\ell)\|_{L^2(\Gamma)} &\le \|\nabla_\Gamma(u-v_h^\ell)\|_{L^2(\Gamma)} + \|\nabla_\Gamma(v_h^\ell-(u_h^q)^\ell)\|_{L^2(\Gamma)}\\ &\le \|\nabla_\Gamma(u-v_h^\ell)\|_{L^2(\Gamma)} + c\|\nabla_{\Gamma_h}(v_h-u_h^q)\|_{L^2(\Gamma_h)}.\end{aligned}$$

We derive a bound for $\|\nabla_{\Gamma_h} e_h\|_{L^2(\Gamma_h)}$, $e_h := u_h^q - v_h$. Let c_1 be a constant such that $\tilde{e}_h := e_h + c_1$ satisfies $\int_{\Gamma_h} \tilde{e}_h \, ds_h = 0$. For arbitrary constants c, there holds $\nabla_{\Gamma_h} c \equiv 0$. In particular, by (5.14), we get the consistency property $a_h(c, \tilde{e}_h) = 0$. Using this, (9.8), and the definition of the discrete problems (5.4), (5.16), we obtain

$$\begin{aligned}&\|\nabla_{\Gamma_h} e_h\|^2_{L^2(\Gamma_h)} = \|\nabla_{\Gamma_h}\tilde{e}_h\|^2_{L^2(\Gamma_h)} \le \gamma^{-1}\, a_h(e_h,\tilde{e}_h)\\ &= \gamma^{-1}\big(a(u_h-v_h,e_h) + a(v_h,\tilde{e}_h) - a_h(v_h,\tilde{e}_h) + l_h(\tilde{e}_h) - l(\tilde{e}_h)\big).\end{aligned} \tag{9.10}$$

We will derive the bound

$$\begin{aligned}a(u_h-v_h,e_h) \le c\big(&\|\nabla_\Gamma(u-v_h^\ell)\|_{L^2(\Gamma)} + \|(I-A_\Gamma)\mathbf{P}\|_{L^\infty(\Gamma)}\|\nabla_{\Gamma_h} v_h\|_{L^2(\Gamma_h)}\\ &+ \tilde{E}_f\big)\|\nabla_{\Gamma_h} e_h\|_{L^2(\Gamma_h)},\end{aligned} \tag{9.11}$$

and combining this with the relation (9.10) and the triangle inequality above proves the result (9.9). To derive (9.11) we note, cf. (9.4), that for all $w_h \in S_h^\ell$ we have

$$\begin{aligned}\int_\Gamma A_\Gamma \nabla_\Gamma u_h^\ell \cdot \nabla_\Gamma w_h \, ds &= \int_\Gamma \frac{1}{\mu_h^\ell} f_h^\ell w_h \, ds = \int_\Gamma \big(\frac{1}{\mu_h^\ell} f_h^\ell - f\big) w_h \, ds + \int_\Gamma f w_h \, ds\\ &= \int_\Gamma \big(\frac{1}{\mu_h^\ell} f_h^\ell - f\big) w_h \, ds + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma w_h \, ds.\end{aligned}$$

Let $\bar{c}$ be a constant that is chosen below and $\bar{e}_h := e_h - \bar{c}$. From the previous equation,

$$\begin{aligned}&a(u_h-v_h,e_h) = a(u_h-v_h,\bar{e}_h) = \int_{\Gamma_h} \nabla_{\Gamma_h}(u_h-v_h)\cdot\nabla_{\Gamma_h}\bar{e}_h \, ds_h\\ &= \int_\Gamma A_\Gamma \nabla_\Gamma(u_h^\ell - v_h^\ell)\cdot\nabla_\Gamma \bar{e}_h^\ell \, ds\\ &= \int_\Gamma \nabla_\Gamma(u-v_h^\ell)\cdot\nabla_\Gamma e_h^\ell \, ds + \int_\Gamma \big(\frac{1}{\mu_h^\ell} f_h^\ell - f\big)\bar{e}_h^\ell \, ds + \int_\Gamma (I-A_\Gamma)\mathbf{P}\nabla_\Gamma v_h^\ell \cdot \nabla_\Gamma e_h^\ell \, ds.\end{aligned} \tag{9.12}$$

holds. Now $\bar{c}$ is chosen as $\bar{c} := \frac{1}{|\Gamma|}\int_{\Gamma_h} \mu_h e_h \, ds_h$ such that we have $\int_\Gamma \bar{e}_h^\ell \, ds = 0$. Hence, the Poincare inequality $\|\bar{e}_h^\ell\|_{L^2(\Gamma)} \le c\|\nabla_\Gamma e_h^\ell\|_{L^2(\Gamma)}$ holds. Using this, the Cauchy-Schwarz inequality and $\|\nabla_\Gamma e_h^\ell\|_{L^2(\Gamma)} \le c\|\nabla_{\Gamma_h} e_h\|_{L^2(\Gamma_h)}$, cf. (9.5), in (9.12), we get the estimate (9.11). ∎

Thus, the total error is split into a *geometric error* (approximation of Γ by Γ_h), an *approximation error* (results from using the FE space) and a *quadrature error*.

10. Approximation error. For the analysis of the approximation error we assume the following approximation quality of the FE space $\hat{S}_h$ on $\hat{\Gamma}_h$, cf. (5.1): there is an interpolation operator $I_h : H^{m+1}(\Gamma) \to \hat{S}_h$ such that for $s = 0, \ldots, m$:

$$\sum_{T\in\mathcal{F}_h} \|w^e - I_h w\|^2_{H^s(T)} \le c h^{2(m+1-s)}\|w\|^2_{H^{m+1}(\Gamma)} \quad \text{for all } w \in H^{m+1}(\Gamma). \tag{10.1}$$

Such an approximation property holds for the two possible choices for $\hat{S}_h$ mentioned in Remark 2, cf. Theorem 4.2 in [20] and [18]. The estimate (10.1) for $s=1$ implies

$$\|\nabla_{\hat{\Gamma}_h}(w^e - I_h w)\|_{L^2(\hat{\Gamma}_h)} \leq ch^m \|w\|_{H^{m+1}(\Gamma)} \quad \text{for all} \ \ w \in H^{m+1}(\Gamma). \tag{10.2}$$

In the analysis we use the spaces $\hat{S}_h$ (on $\hat{\Gamma}_h$), cf. (5.1), S_h (on Γ_h), cf. (5.2), and the lifted space S_h^ℓ (on Γ), cf. (9.2). The analysis requires smoothness of the solution:

ASSUMPTION 10.1. *The solution u of* (1.1) *satisfies* $u \in H^{m+1}(\Gamma) \cap H^2_\infty(\Gamma)$.

In the analysis below we use the following test function $u_{h,*} \in S_h$ to prove an upper bound for the minimum over $v_h \in S_h$ in (9.9):

$$u_{h,*} := \hat{u}_{h,*} \circ (p_h|_{\hat{\Gamma}_h})^{-1} \in S_h, \quad \hat{u}_{h,*} := I_h u \in \hat{S}_h. \tag{10.3}$$

In the lifting procedure both $p_h : \hat{\Gamma}_h \to \Gamma_h$ and $p : U \to \Gamma$ play a role and we have to control error terms of the form $\nabla_\Gamma u(p \circ p_h(x)) - \nabla_\Gamma u(p(x))$, $x \in \hat{\Gamma}_h$. For this we need the regularity assumption $u \in H^2_\infty(\Gamma)$ in Assumption 10.1.

THEOREM 10.1. *Let $m \geq 1$ be such that* (10.2) *and assumption 10.1 are fulfilled. For h sufficiently small the following holds, with $u_{h,*}$ as in* (10.3)*:*

$$\min_{v_h \in S_h} \|\nabla_\Gamma(u - v_h^\ell)\|_{L^2(\Gamma)} \leq \|\nabla_\Gamma(u - u_{h,*}^\ell)\|_{L^2(\Gamma)} \leq ch^m \|u\|_{H^{m+1}(\Gamma)} + ch^{k+1}\|u\|_{H^2_\infty(\Gamma)}$$

Proof. The test functions in (10.3) satisfy

$$u_{h,*} \circ p_h(x) = \hat{u}_{h,*}(x), \quad u_{h,*}^\ell \circ p(y) = u_{h,*}(y), \quad y := p_h(x) \in \Gamma_h,\ x \in \hat{\Gamma}_h,$$

cf. (5.2), (9.2). Using $\hat{u}_* := u^e \in H^1(\hat{\Gamma}_h)$, we define $u_* \in H^1(\Gamma_h)$, and $\tilde{u} \in H^1(\Gamma)$ by

$$u_* \circ p_h(x) = \hat{u}_*(x), \quad \tilde{u} \circ p(y) = u_*(y), \quad y := p_h(x) \in \Gamma_h,\ x \in \hat{\Gamma}_h.$$

Note that $\tilde{u} = u_*^\ell$ on Γ holds. From (5.9) it follows that $\|\nabla_{\Gamma_h}(\hat{v} \circ p_h^{-1})\|_{L^2(\Gamma_h)} \leq c\|\nabla_{\hat{\Gamma}_h}\hat{v}\|_{L^2(\hat{\Gamma}_h)}$ for all $\hat{v} \in H^1(\hat{\Gamma}_h)$ holds. Using this and (9.5) we get

$$\begin{aligned}\|\nabla_\Gamma(\tilde{u} - u_{h,*}^\ell)\|_{L^2(\Gamma)} &= \|\nabla_\Gamma(u_* - u_{h,*})^\ell\|_{L^2(\Gamma)} \leq c\|\nabla_{\Gamma_h}(u_* - u_{h,*})\|_{L^2(\Gamma_h)} \\ &\leq c\|\nabla_{\hat{\Gamma}_h}(u^e - \hat{u}_{h,*})\|_{L^2(\hat{\Gamma}_h)}.\end{aligned}$$

Hence, with the triangle inequality and (10.2) we obtain

$$\begin{aligned}\|\nabla_\Gamma(u - u_{h,*}^\ell)\|_{L^2(\Gamma)} &\leq c\|\nabla_{\hat{\Gamma}_h}(u^e - \hat{u}_{h,*})\|_{L^2(\hat{\Gamma}_h)} + \|\nabla_\Gamma(\tilde{u} - u)\|_{L^2(\Gamma)} \\ &\leq ch^m\|u\|_{H^{m+1}(\Gamma)} + \|\nabla_\Gamma(\tilde{u} - u)\|_{L^2(\Gamma)}.\end{aligned} \tag{10.4}$$

We derive a bound for the term $\|\nabla_\Gamma(\tilde{u} - u)\|_{L^2(\Gamma)}$. Let $x \in \hat{\Gamma}_h$ be arbitrary, $y = p_h(x) \in \Gamma_h$, $z = p(y) \in \Gamma$, and $\zeta = p(x) \in \Gamma$. From (7.6) it follows that

$$\|y - \zeta\| \leq ch^{k+1}, \quad \|z - \zeta\| \leq ch^{k+1}, \quad \|x - z\| \leq ch^2. \tag{10.5}$$

Our starting point is the identity $\tilde{u}(z) = u(\zeta)$, which holds a. e. on $\hat{\Gamma}_h$ by definition of $\tilde{u}$. Taking the tangential gradient on $\hat{\Gamma}_h$ yields

$$\hat{\mathbf{P}}(x)Dp_h(x)^T Dp(y)^T \nabla_\Gamma \tilde{u}(z) = \hat{\mathbf{P}}(x)Dp(x)^T \nabla_\Gamma u(\zeta). \tag{10.6}$$

From the smoothness of u and $\|z-\zeta\| \le ch^{k+1}$, we get $\nabla_\Gamma u(\zeta) = \nabla_\Gamma u(z) + r_0$, with $\|r_0\| \le ch^{k+1}\|u\|_{H^2_\infty(\Gamma)}$. We insert this into (10.6) and rearrange the terms to obtain

$$\begin{aligned} &\hat{\mathbf{P}}(x) Dp_h(x)^T Dp(y)^T \nabla_\Gamma\big(\tilde{u}(z) - u(z)\big) \\ &= \hat{\mathbf{P}}(x)\big(Dp(x)^T - Dp_h(x)^T Dp(y)^T\big)\mathbf{P}(z)\nabla_\Gamma u(z) + \hat{\mathbf{P}}(x) Dp_h(x)^T r_0 =: r_1 \end{aligned} \tag{10.7}$$

For the matrix in first term on the right hand-side in (10.7) we have

$$\begin{aligned} &\hat{\mathbf{P}}(x)\big(Dp(x)^T - Dp_h(x)^T Dp(y)^T\big)\mathbf{P}(z) \\ &= \hat{\mathbf{P}}(x)\big(Dp(x)^T - Dp_h(x)^T\big)\mathbf{P}(z) + \hat{\mathbf{P}}(x) Dp_h(x)^T\big(\mathbf{I} - Dp(y)^T\big)\mathbf{P}(z) =: A_0 + A_1. \end{aligned}$$

Using $\|\mathbf{P}(z) - \mathbf{P}(x)\| = \|\mathbf{P}(z) - \mathbf{P}(\zeta)\| \le c\|z-\zeta\| \le ch^{k+1}$ and (8.9) we obtain $\|A_0\| \le ch^{k+1}$. Using (8.1) and $|d(y)| \le ch^{k+1}$ one obtains $Dp(y) = \mathbf{P}(z) + \mathcal{O}(h^{k+1})$. Thus we get $\|A_1\| \le ch^{k+1}$. Using the bounds for A_0, A_1 we get

$$A\nabla_\Gamma\big(\tilde{u}(z) - u(z)\big) = r_1, \quad \|r_1\| \le ch^{k+1}\|u\|_{H^2_\infty(\Gamma)}, \tag{10.8}$$

with $A := \hat{\mathbf{P}}(x) Dp_h(x)^T Dp(y)^T \mathbf{P}(z) = \hat{\mathbf{P}}(x)\big(Dp_h(x)^T + E_1\big)\mathbf{P}(z)$, with $\|E_1\| \le ch^{k+1}$. From (8.3) and $|d_h(x)| \le ch^2$ we get $Dp_h(x) = \mathbf{Q}(y) + \mathcal{O}(h^2)$. Using (8.18), (9.7) and the definitions of the projections we get

$$\|\hat{\mathbf{P}}(x)\mathbf{Q}(y)^T - \mathbf{P}(z)\| \le \|\hat{\mathbf{P}}(x)(\mathbf{Q}(y)^T - \mathbf{I})\| + \|\hat{\mathbf{P}}(x) - \mathbf{P}(z)\| \le ch.$$

Thus we get $A = (\mathbf{I} + E_2)\mathbf{P}(z)$, with $\|E_2\| \le ch$. Using this in (10.8) we get, for h sufficiently small,

$$\|\nabla_\Gamma\big(\tilde{u}(z) - u(z)\big)\| = \|(I + E_2)^{-1} r_1\| \le c\|r_1\| \le ch^{k+1}\|u\|_{H^2_\infty(\Gamma)},$$

which implies $\|\nabla_\Gamma(\tilde{u} - u)\|_{L^2(\Gamma)} \le ch^{k+1}\|u\|_{H^2_\infty(\Gamma)}$. Combining this with the result in (10.4) completes the proof. ∎

11. Geometric error. We study the terms $\|(I - A_\Gamma)\mathbf{P}\|_{L^\infty(\Gamma)}\|\nabla_{\Gamma_h} v_h\|_{L^2(\Gamma_h)}$ and $\tilde{E}_f := \|f - \frac{1}{\mu_h^\ell} f_h^\ell\|_{L^2(\Gamma)}$ that occur in the Strang Lemma, cf. (9.9).

THEOREM 11.1. *Let $u_{h,*} \in S_h$ be as in* (10.3). *For h sufficiently small the following estimates hold:*

$$\|(I - A_\Gamma)\mathbf{P}\|_{L^\infty(\Gamma)}\|\nabla_{\Gamma_h} u_{h,*}\|_{L^2(\Gamma_h)} \le ch^{k+1}\|u\|_{H^2(\Gamma)}, \tag{11.1}$$

$$\|f - \frac{1}{\mu_h^\ell} f_h^\ell\|_{L^2(\Gamma)} \le ch^{k+1}\|f\|_{H^1_\infty(U)}. \tag{11.2}$$

Proof. Using $\|d\|_{L^\infty(\Gamma_h)} \le ch^{k+1}$, $\|\frac{1}{\mu_h} - 1\|_{L^\infty(\Gamma_h)} \le ch^{k+1}$, cf. (2.3) and (2.5), in (9.3) yields $A_\Gamma(p(x)) = \mathbf{P}(x)\bar{\mathbf{P}}(x)\mathbf{P}(x) + \mathcal{O}(h^{k+1})$, $x \in \Gamma_h$. From $\mathbf{P}(p(x)) = \mathbf{P}(x)$ and the identity $\mathbf{P}\bar{\mathbf{P}}\mathbf{P} - \mathbf{P} = \mathbf{P}(\mathbf{P} - \bar{\mathbf{P}})(\bar{\mathbf{P}} - \mathbf{P})\mathbf{P}$ we obtain

$$\begin{aligned} \|(I - A_\Gamma)\mathbf{P}\|_{L^\infty(\Gamma)} &\le c\|\mathbf{P}\bar{\mathbf{P}}\mathbf{P} - \mathbf{P}\|_{L^\infty(\Gamma_h)} + ch^{k+1} \\ &\le c\|\mathbf{P} - \bar{\mathbf{P}}\|^2_{L^\infty(\Gamma_h)} + ch^{k+1} \le ch^{k+1}. \end{aligned} \tag{11.3}$$

The last estimate above follows from (2.3). Using (9.5), (5.9) and (10.2) we get

$$\begin{aligned} \|\nabla_{\Gamma_h} u_{h,*}\|_{L^2(\Gamma_h)} &\le c\|\nabla_{\hat{\Gamma}_h} I_h u\|_{L^2(\hat{\Gamma}_h)} \le c\big(\|\nabla_{\hat{\Gamma}_h} u^e\|_{L^2(\hat{\Gamma}_h)} + \|u\|_{H^2(\Gamma)}\big) \\ &\le c\big(\|\nabla_\Gamma u\|_{L^2(\Gamma)} + \|u\|_{H^2(\Gamma)}\big) \le c\|u\|_{H^2(\Gamma)}. \end{aligned} \tag{11.4}$$

Combination of (11.3) and (11.4) yields the proof of (11.1).

We consider the data extension $f_h(x) = f(x) - c_f$, $x \in \Gamma_h$ with c_f as in (5.3). Using the smoothness of f, $\mathrm{dist}(\Gamma, \Gamma_h) \le ch^{k+1}$, (2.5), and $\int_\Gamma f\, ds = 0$ one can derive the bound $|c_f| \le ch^{k+1}\|f\|_{H^1_\infty(U)}$. With this we obtain

$$\|f - \frac{1}{\mu_h^\ell} f_h^\ell\|_{L^2(\Gamma)} \le c\|f - f_h^\ell\|_{L^2(\Gamma)} + ch^{k+1}\|f\|_{L^2(\Gamma)} \le ch^{k+1}\|f\|_{H^1_\infty(U)}$$

which proves the result in (11.2). ∎

12. Quadrature error. In this section we analyze the quadrature error, i.e., we derive bounds for the third and fourth term in the Strang Lemma. Recall from section 5 the affine mapping from the unit reference triangle $\tilde{T}$ to $T \in \mathcal{F}_h$, given by $x = M_T\tilde{x} = B_T\tilde{x} + b_T$, $\tilde{x} \in \tilde{T}$, $x \in T$. Note that $B_T \in \mathbb{R}^{3\times 2}$. Furthermore $\|B_T\| \le ch$ holds. Correspondence of functions on $\tilde{T}$ and T is given by $\tilde{u}(\tilde{x}) = \hat{u}(B_T\tilde{x} + b_T) = \hat{u}(x)$. Note that for $n \in \mathbb{N}$, $\tilde{u} \in C^n(\tilde{T})$ and $\xi_i \in \mathbb{R}^2$, $1 \le i \le n$ we have $D^n\tilde{u}(\tilde{x})(\xi_1, \dots \xi_n) = D^n\hat{u}(x)(B_T\xi_1, \dots, B_T\xi_n) = D^n_T\hat{u}(\hat{x})(B_T\xi_1, \dots, B_T\xi_n)$ (where D_T denotes the tangential derivative along T), and thus as in Theorem 15.1 in [4] we obtain, for $n \in \mathbb{N}$, $p \in [1, \infty]$,

$$|\tilde{u}|_{H^n_p(\tilde{T})} \le c\|B_T\|^n|T|^{-1/p}|\hat{u}|_{H^n_p(T)} \le ch^n|T|^{-1/p}|\hat{u}|_{H^n_p(T)} \quad \text{for } \tilde{u} \in H^n_p(\tilde{T}). \tag{12.1}$$

In the seminorm $|\hat{u}|_{H^n_p(T)}$ only the derivatives of order n are involved and these derivatives are the *tangential* ones along the triangle T. We note that an estimate in the other direction, i.e. bounding derivatives of $\hat{u}$ by those of $\tilde{u}$, causes problems, because the triangle $\hat{T}$ may have arbitrary small angles. Thus the smallest singular value of B_T is not bounded from below by ch with a uniform (w.r.t. T and h) constant $c > 0$.

The quadrature error for the quadrature rule (5.12) is defined by

$$E_{\tilde{T}}(\tilde{u}) = \int_{\tilde{T}} \tilde{u}\, d\tilde{x} - Q_{\tilde{T}}(\tilde{u}), \quad E_T(\hat{u}) = \int_T \hat{u}\, d\hat{x} - Q_T(\hat{u}). \tag{12.2}$$

Note that $E_T(\hat{u}) = |T|E_{\tilde{T}}(\tilde{u})$ holds.

12.1. Smooth approximation of F. In the bilinear form $a_h(u_h, v_h)$ the quadrature rule Q_T is applied to the function $F\nabla_{\hat{\Gamma}_h}\hat{u}_h \cdot \nabla_{\hat{\Gamma}_h}\hat{v}_h$. On each triangle $T \in \hat{\Gamma}_h$ the vector functions $\nabla_{\hat{\Gamma}_h}\hat{u}_h$ and $\nabla_{\hat{\Gamma}_h}\hat{v}_h$ are polynomials and thus have C^∞ smoothness. The matrix $F = F(x) = T_h(x)^{-1}\hat{\mu}_h(x)$, $x \in T$, however, contains derivatives of the function p_h, which is *only Lipschitz*. Hence, F is not even continuous. In this section we show that, on T, this matrix function can be approximated with accuracy $\mathcal{O}(h^{k+1})$ by a smooth matrix function, denoted by F^s. The components $T_h^{-1} = (WW^T)^{-1}$ and $\hat{\mu}_h$ of F are treated in the lemmas 12.1 and 12.2 below.

Recall the definition $W(x) = \mathbf{I} - \hat{\mathbf{Q}}(x) + \hat{\mathbf{P}}(x)Dp_h(x)^T$, cf. Lemma 5.1. From (5.8), (8.2), (8.4), and the definition of W, we get for almost all $x \in \hat{\Gamma}_h$:

$$\begin{aligned}\hat{\mathbf{P}}(x)W(x) &= \hat{\mathbf{P}}(x)Dp_h(x)^T = \hat{\mathbf{P}}(x)Dp_h(x)^T\mathbf{Q}(y)^T \\ &= \hat{\mathbf{P}}(x)Dp_h(x)^T\bar{\mathbf{P}}(y) = W(x)\bar{\mathbf{P}}(y), \quad y = p_h(x).\end{aligned} \tag{12.3}$$

This implies the commutator relations (note that W is invertible, cf. (9.6)):

$$\begin{aligned}W(x)W(x)^T\hat{\mathbf{P}}(x) &= \hat{\mathbf{P}}(x)W(x)W(x)^T, \\ \big(W(x)W(x)^T\big)^{-1}\hat{\mathbf{P}}(x) &= \hat{\mathbf{P}}(x)\big(W(x)W(x)^T\big)^{-1}.\end{aligned} \tag{12.4}$$

The oblique projection $\hat{\mathbf{Q}}(x)$, $x \in \hat{\Gamma}_h$, is approximated by:

$$\hat{\mathbf{Q}}^s(x) := \mathbf{I} - \frac{1}{\hat{\alpha}^s(x)} \hat{n}_h(x) n(x)^T, \quad \text{with } \hat{\alpha}^s(x) = \hat{n}_h(x)^T n(x). \tag{12.5}$$

Note that $\hat{n}_h$ is constant on $T \in \hat{\Gamma}_h$ and the normal n is smooth (depending only on the smoothness of Γ). Hence, $\hat{\mathbf{Q}}^s(x)$ is a piecewise smooth matrix function. Similar to (5.8), the following commutation relations hold for $x \in \hat{\Gamma}_h$:

$$\begin{aligned} \hat{\mathbf{Q}}^s(x)\mathbf{P}(x) = \mathbf{P}(x), \quad \mathbf{P}(x)\hat{\mathbf{Q}}^s(x) = \hat{\mathbf{Q}}^s(x), \\ \hat{\mathbf{P}}(x)\hat{\mathbf{Q}}^s(x) = \hat{\mathbf{P}}(x), \quad \hat{\mathbf{Q}}^s(x)\hat{\mathbf{P}}(x) = \hat{\mathbf{Q}}^s(x). \end{aligned} \tag{12.6}$$

We use $\hat{\mathbf{Q}}^s$ to define a piecewise smooth approximation of $W(x)$:

$$W^s(x) := \mathbf{I} - \hat{\mathbf{Q}}^s(x) + \hat{\mathbf{P}}(x) Dp(x)^T, \quad x \in \hat{\Gamma}_h. \tag{12.7}$$

Note that $\hat{\mathbf{P}}(x)$ is constant on $T \in \mathcal{F}_h$ and $Dp(x)$ is a smooth matrix function (depending only on the smoothness of Γ), hence, $W^s(x)$ is smooth for $x \in T$. Using (2.3) and (8.1), we get (for h sufficiently small) $W^s(x) = \mathbf{I} - \hat{\mathbf{Q}}^s(x) + \hat{\mathbf{P}}(x)\mathbf{P}(x) + \mathcal{O}(h^2)$. An elementary computation yields $-\hat{\mathbf{Q}}^s + \hat{\mathbf{P}}\mathbf{P} = \hat{n}_h(\hat{\alpha}^s n - \hat{n}_h)^T - \frac{1}{\hat{\alpha}^s}(\hat{\alpha}^s n - \hat{n}_h)n^T$. Using $|\hat{\alpha}^s - 1| = \frac{1}{2}\|\hat{n}_h - n\|^2 \le ch^2$ we thus get

$$W^s(x) = \mathbf{I} + \mathcal{O}(h), \quad x \in \hat{\Gamma}_h. \tag{12.8}$$

In particular, for h sufficiently small, W^s is invertible. Similar to (12.3) and (12.4), we obtain for almost all $x \in \hat{\Gamma}_h$:

$$\hat{\mathbf{P}}(x) W^s(x) = \hat{\mathbf{P}}(x) Dp(x)^T = W^s(x)\mathbf{P}(x), \tag{12.9}$$

and this yields the commutation relations

$$\begin{aligned} W^s(x)W^s(x)^T\hat{\mathbf{P}}(x) = \hat{\mathbf{P}}(x)W^s(x)W^s(x)^T, \\ \big(W^s(x)W^s(x)^T\big)^{-1}\hat{\mathbf{P}}(x) = \hat{\mathbf{P}}(x)\big(W^s(x)W^s(x)^T\big)^{-1}. \end{aligned} \tag{12.10}$$

Lemma 12.1. *For h sufficiently small the following holds:*

$$\left\| \hat{\mathbf{P}} \left(\left(WW^T\right)^{-1} - \left(W^s W^{s\,T}\right)^{-1} \right) \hat{\mathbf{P}} \right\|_{L^\infty(\hat{\Gamma}_h)} \le ch^{k+1}. \tag{12.11}$$

Proof. We omit the argument x in the notation below. We use the matrix identity $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$ and the commutator relations (12.4), (12.10) to compute

$$\hat{\mathbf{P}}\left((WW^T)^{-1} - (W^sW^{s\,T})^{-1}\right)\hat{\mathbf{P}} = (WW^T)^{-1}\hat{\mathbf{P}}\big(W^sW^{s\,T} - WW^T\big)\hat{\mathbf{P}}(W^sW^{s\,T})^{-1}.$$

From (9.6) and (12.8) we obtain $\|(WW^T)^{-1}\|_{L^\infty(\hat{\Gamma}_h)} \le c$, $\|(W^sW^{s\,T})^{-1}\|_{L^\infty(\hat{\Gamma}_h)} \le c$. The relations (12.3), (12.9) and $Dp(x) = \mathbf{P}(x)Dp(x)$, $Dp_h(x) = \bar{\mathbf{P}}(y)Dp_h(x)$, with $y = p_h(x)$, yield

$$\hat{\mathbf{P}}(W^sW^{s\,T} - WW^T)\hat{\mathbf{P}} = \hat{\mathbf{P}}\Big((Dp - Dp_h)^T\mathbf{P}Dp + Dp_h^T\bar{\mathbf{P}}(y)(Dp - Dp_h)\Big)\hat{\mathbf{P}}.$$

Using (8.9) and (8.10) results in $\|\hat{\mathbf{P}}(W^sW^{s\,T} - WW^T)\hat{\mathbf{P}}\|_{L^\infty(\hat{\Gamma}_h)} \le ch^{k+1}$. ∎

For the estimate in (12.11) to hold the use of the projections $\hat{\mathbf{P}}$ on the left hand-side is essential. Without these projections, only the asymptotically worse upper bound ch^k holds. This difference in the upper bounds is directly related to the different bounds in (8.8) and (8.9).

For given $T \in \mathcal{F}_h$ let $U = U_T$ be the orthogonal matrix from Lemma 4.1. We let

$$\hat{\mu}^s(x) := \sqrt{\det(U^T Dp(x)^T Dp(x) U)}, \quad x \in \hat{\Gamma}_h. \tag{12.12}$$

Since U is constant on each T and $Dp(x)$ is a smooth matrix function (depending only on the smoothness of Γ) we have that $\hat{\mu}^s$ is a smooth matrix function on each $T \in \mathcal{F}_h$.

LEMMA 12.2. *For h sufficiently small the following holds:*

$$\|\hat{\mu}_h - \hat{\mu}^s\|_{L^\infty(\hat{\Gamma}_h)} \leq ch^{k+1}. \tag{12.13}$$

Proof. Let $x \in \hat{\Gamma}_h$, $y = p_h(x)$, $\zeta = p(x)$, and $U \in \mathbb{R}^{3\times 2}$ be the orthogonal matrix from Lemma 4.1. Note that $\hat{\mathbf{P}}(x)U = U$ holds. We define $A = Dp(x)U \in \mathbb{R}^{3\times 2}$ and $B = Dp_h(x)U \in \mathbb{R}^{3\times 2}$, hence $\hat{\mu}^s(x)^2 - \hat{\mu}_h(x)^2 = \det(A^TA) - \det(B^TB)$ holds. Using $Dp(x) = \mathbf{P}(x) + \mathcal{O}(h^2) = \hat{\mathbf{P}}(x) + \mathcal{O}(h)$, cf. (9.7), and $Dp_h(x) = \hat{\mathbf{P}}(x) + \mathcal{O}(h)$, we get with $\hat{\mathbf{P}}(x)U = U$ that $A^TA = \mathbf{I} + \mathcal{O}(h)$ and $B^TB = \mathbf{I} + \mathcal{O}(h)$. hold. Define $M(t) := A^TA + t(B^TB - A^TA)$ and $f(t) := \det M(t)$, $t \in [0,1]$. For h sufficiently small the inequality $|\det M(t) - 1| \leq \frac{1}{2}$, $t \in [0,1]$, holds. Hence, $M(t)$ is invertible and its condition number is uniformly bounded. There exists $s \in (0,1)$ with

$$\det B^TB - \det A^TA = f(1) - f(0) = f'(s) = f(s)\,\mathrm{tr}\big(M(s)^{-1}(B^TB - A^TA)\big).$$

With this we obtain

$$\left|\det B^TB - \det A^TA\right| \leq 3|\det M(s)|\,\|M(s)^{-1}\|\|B^TB - A^TA\| \leq c\left\|B^TB - A^TA\right\|.$$

Using $\hat{\mathbf{P}}U = U$, $Dp_h(x) = \bar{\mathbf{P}}(y)Dp_h(x)$, $Dp(x) = \mathbf{P}(x)Dp(x)$ and the estimates in Lemma 8.2 we obtain

$$\|A^TA - B^TB\| \leq \|(B-A)^TB\| + \|A^T(B-A)\| \leq ch^{k+1}.$$

Both $\hat{\mu}^s(x)$ and $\hat{\mu}_h(x)$ are strictly positive and uniformly bounded away from 0 for h sufficiently small. Hence, we obtain

$$|\hat{\mu}^s(x) - \hat{\mu}_h(x)| = \frac{|\hat{\mu}^s(x)^2 - \hat{\mu}_h(x)^2|}{|\hat{\mu}^s(x) + \hat{\mu}_h(x)|} \leq c\left|\det B^TB - \det A^TA\right| \leq ch^{k+1}.$$

□

The results in the Lemmas 12.1, 12.2, induce a piecewise (on $T \in \mathcal{F}_h$) smooth approximation of the matrix $F(x) = \hat{\mu}(x)\big(W(x)W(x)^T\big)^{-1}$.

COROLLARY 12.3. *Define*

$$F^s(x) = \hat{\mu}^s(x)\big(W^s(x)W^s(x)^T\big)^{-1}, \quad x \in T \in \mathcal{F}_h.$$

Take $m \in \mathbb{N}$, $m \geq 1$. Provided Γ is sufficiently smooth, the entries of the matrix F^s have the smoothness property $F^s_{ij} \in H^m_\infty(T)$, $1 \leq i,j \leq 3$, for all $T \in \mathcal{F}_h$. Furthermore *the estimates*

$$\max_{T\in\mathcal{F}_h} \|F^s\|_{H^m_\infty(T)} \le c, \tag{12.14}$$

$$\max_{T\in\mathcal{F}_h} \left\|\hat{\mathbf{P}}(F-F^s)\hat{\mathbf{P}}\right\|_{L^\infty(T)} \le ch^{k+1}, \tag{12.15}$$

hold, with constants c independent of h.

Proof. From (2.1) it is clear that the smoothness of p and n, in a small neighborhood of Γ, depends only on the smoothness of Γ. On $T \in \mathcal{F}_h$ the normal $\hat{n}_h$ is constant, hence the smoothness of (the entries of) the matrix $W^s(x)^{-1} = \big(\mathbf{I} - \hat{\mathbf{Q}}^s(x) + \hat{\mathbf{P}}(x)Dp(x)^T\big)^{-1}$ depends only on the smoothness of the matrix Dp and of the vector field n. Similarly, on $T \in \mathcal{F}_h$ the orthogonal matrix $U = U_T$ is constant and thus the smoothness of $\hat{\mu}^s$ depends only on the smoothness of the matrix Dp. On $T \in \mathcal{F}_h$, (higher) derivatives of F^s_{ij} can be estimated by bounds that depend only on bounds for (higher) derivatives of p and n. If Γ is sufficiently smooth, these bounds are uniform w.r.t. $T \in \mathcal{F}_h$. From these observations it follows that for all entries of the matrix F^s, we have $F^s_{ij} \in H^m_\infty(T)$ for all $T \in \mathcal{F}_h$ and that the result (12.14) holds. The result (12.15) directly follows from (12.13) and Lemma 12.1. ∎

12.2. Bound on the quadrature error in the bilinear form. We derive a bound for the term $\sup_{w_h\in S_h/\mathbb{R}} \frac{a(v_h,w_h)-a_h(v_h,w_h)}{\|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)}}$ in the Strang Lemma, where we take $v_h = u_{h,*}$ as in Theorem 10.1. The technique used in the error analysis below is very similar to the one used in the analysis of the quadrature error in [4].

THEOREM 12.4. *Assume that the quadrature rule $Q_{\tilde{T}}$ is exact for all polynomials of degree $2m-2$ and that* (10.1) *and assumption 10.1 hold. For $u_{h,*}$ from* (10.3)*, the following holds for h sufficiently small:*

$$\sup_{w_h\in S_h/\mathbb{R}} \frac{a(u_{h,*},w_h)-a_h(u_{h,*},w_h)}{\|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)}} \le ch^m\|u\|_{H^{m+1}(\Gamma)} + ch^{k+1}\|u\|_{H^2(\Gamma)}.$$

Proof. We write v_h, $\hat{v}_h$ for $u_{h,*}$ and $\hat{u}_{h,*}$. Note that

$$\int_{\hat{\Gamma}_h} F\nabla_{\hat{\Gamma}_h}\hat{v}_h\cdot\nabla_{\hat{\Gamma}_h}\hat{w}_h\,d\hat{s}_h = \int_{\hat{\Gamma}_h} \hat{\mathbf{P}}F\hat{\mathbf{P}}\nabla_{\hat{\Gamma}_h}\hat{v}_h\cdot\nabla_{\hat{\Gamma}_h}\hat{w}_h\,d\hat{s}_h$$

holds, and similarly with F replaced by F^s. We use the splitting

$$\begin{aligned} &a(v_h,w_h)-a_h(v_h,w_h) = \int_{\hat{\Gamma}_h} \hat{\mathbf{P}}(F-F^s)\hat{\mathbf{P}}\nabla_{\hat{\Gamma}_h}\hat{v}_h\cdot\nabla_{\hat{\Gamma}_h}\hat{w}_h\,d\hat{s}_h \\ &+\sum_{T\in\mathcal{F}_h} E_T(F^s\nabla_{\hat{\Gamma}_h}\hat{v}_h\cdot\nabla_{\hat{\Gamma}_h}\hat{w}_h) + \sum_{T\in\mathcal{F}_h} Q_T\big(\hat{\mathbf{P}}(F^s-F)\hat{\mathbf{P}}\nabla_{\hat{\Gamma}_h}\hat{v}_h\cdot\nabla_{\hat{\Gamma}_h}\hat{w}_h\big) \\ &=: A+B+C, \end{aligned} \tag{12.16}$$

cf. (12.2), (5.13), (5.14). For the first term in (12.16) we get, using (12.15) and $\|\hat{v}_h\|_{H^1(\hat{\Gamma}_h)} = \|I_hu\|_{H^1(\hat{\Gamma}_h)} \le \|u^e\|_{H^1(\hat{\Gamma}_h)} + ch\|u\|_{H^2(\Gamma)} \le c\|u\|_{H^2(\Gamma)}$, cf. (10.1) and (9.5),

$$|A| \le ch^{k+1}\|\nabla_{\hat{\Gamma}_h}\hat{v}_h\|_{L^2(\hat{\Gamma}_h)}\|\nabla_{\hat{\Gamma}_h}\hat{w}_h\|_{L^2(\hat{\Gamma}_h)} \le ch^{k+1}\|u\|_{H^2(\Gamma)}\|\nabla_{\Gamma_h}w_h\|_{L^2(\Gamma_h)}. \tag{12.17}$$

For the third term, we use the positivity of the quadrature weights to obtain

$$|C| \le \sum_{T\in\mathcal{F}_h} \|\hat{\mathbf{P}}(F^s-F)\hat{\mathbf{P}}\|_{L^\infty(T)}\|\nabla_{\hat{\Gamma}_h}\hat{v}_h\|_{L^\infty(T)}\|\nabla_{\hat{\Gamma}_h}\hat{w}_h\|_{L^\infty(T)}\,Q_T(1).$$

Clearly, $Q_T(1) = |T|$. We use the local estimate $\sqrt{|T|}\|f\|_{L^\infty(T)} \leq c\|f\|_{L^2(T)}$ which is valid for FE functions f on arbitrarily shaped triangles. We again apply (12.15) and combine this with a Cauchy-Schwarz inequality to obtain

$$\begin{aligned} |C| &\leq ch^{k+1} \sum_{T\in\mathcal{F}_h} \|\nabla_{\hat{\Gamma}_h} \hat{v}_h\|_{L^2(T)} \|\nabla_{\hat{\Gamma}_h} \hat{w}_h\|_{L^2(T)} \\ &\leq ch^{k+1} \|\nabla_{\hat{\Gamma}_h} \hat{v}_h\|_{L^2(\hat{\Gamma}_h)} \|\nabla_{\hat{\Gamma}_h} \hat{w}_h\|_{L^2(\hat{\Gamma}_h)} \leq ch^{k+1} \|u\|_{H^2(\Gamma)} \|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)}. \end{aligned} \tag{12.18}$$

In the second term in (12.16) we have smooth integrands F_{ij}^s, $\partial_i^\Gamma \hat{v}_h$, $\partial_j^\Gamma \hat{w}_h$, on each $T \in \mathcal{F}_h$. The latter two are polynomials of degree $m-1$. For the derivation of a bound we can apply an analysis as in [4]. The result (28.16) in [4] states:

$$|E_{\tilde{T}}(\tilde{a}\tilde{v}\tilde{w})| \leq c\Big(\sum_{j=0}^{m-1} |\tilde{a}|_{H_\infty^{m-j}(\tilde{T})} |\tilde{v}|_{H^j(\tilde{T})}\Big) \|\tilde{w}\|_{L^2(\tilde{T})}$$

for all $\tilde{a} \in H_\infty^m(\tilde{T})$, $\tilde{v} \in P_{m-1}(\tilde{T})$, $\tilde{w} \in P_{m-1}(\tilde{T})$. With the result in (12.1) (note that $H^j(T) := H_2^j(T)$) and using $E_T(\hat{\phi}) = |T| E_{\tilde{T}}(\tilde{\phi})$ we get

$$\begin{aligned} |E_T(avw)| &\leq ch_T^m \Big(\sum_{j=0}^{m-1} |a|_{H_\infty^{m-j}(T)} |v|_{H^j(T)}\Big) \|w\|_{L^2(T)} \\ &\leq ch_T^m \|a\|_{H_\infty^m(T)} \|v\|_{H^{m-1}(T)} \|w\|_{L^2(T)} \end{aligned}$$

for all $a \in H_\infty^m(T)$, $v \in P_{m-1}(T)$, $w \in P_{m-1}(T)$. For the second term in (12.16), we take $a = F_{ij}^s$, $v = \partial_i^\Gamma \hat{v}_h$, $w = \partial_j^\Gamma \hat{w}_h$, and using (12.14) we get

$$\begin{aligned} |B| &= \Big| \sum_{T\in\mathcal{F}_h} \sum_{i,j=1}^{3} E_T(F_{ij}^s \partial_i^\Gamma \hat{v}_h \partial_j^\Gamma \hat{w}_h) \Big| \leq ch^m \sum_{T\in\mathcal{F}_h} \|\hat{v}_h\|_{H^m(T)} \|\nabla_{\hat{\Gamma}_h} \hat{w}_h\|_{L^2(T)} \\ &\leq ch^m \Big(\sum_{T\in\mathcal{F}_h} \|\hat{v}_h\|_{H^m(T)}^2\Big)^{\frac{1}{2}} \|\nabla_{\hat{\Gamma}_h} \hat{w}_h\|_{L^2(\hat{\Gamma}_h)} \\ &\leq ch^m \|u\|_{H^{m+1}(\Gamma)} \|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)}. \end{aligned} \tag{12.19}$$

In the last inequality we used $\sum_{T\in\mathcal{F}_h} \|\hat{v}_h\|_{H^m(T)}^2 = \sum_{T\in\mathcal{F}_h} \|I_h u\|_{H^m(T)}^2 \leq c\|u\|_{H^{m+1}(\Gamma)}^2$, which follows from (10.1). Combining the bounds (12.17), (12.18), (12.19) with the splitting (12.16) completes the proof. ◻

12.3. Bound on the quadrature error in the right hand-side functional. We finally analyze the term $\sup_{w_h\in S_h/\mathbb{R}} \frac{l(w_h)-l_h(w_h)}{\|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)}}$ in the Strang Lemma. The idea of the analysis is the same as used above: We replace the Lipschitz functions $f_h \circ p_h\, \hat{\mu}_h$ and $f_h^q \circ p_h\, \hat{\mu}_h$ by piecewise smooth ones and split the error into three terms. As f_h and f_h^q only differ by a constant, the estimates are very similar. For the approximation of $\hat{\mu}_h$ we use $\hat{\mu}^s$ defined in (12.12), with error bound as in (12.13). For the approximation of $f \circ p_h$ we use $f^s := f \circ p \,(= f^e)$.

Theorem 12.5. *Assume that the quadrature rule $Q_{\tilde{T}}$ is exact for all polynomials of degree $2m-2$. Assume that $f \in H_\infty^1(U) \cap H_\infty^m(\Gamma)$. The following holds:*

$$\sup_{w_h\in S_h/\mathbb{R}} \frac{l(w_h) - l_h(w_h)}{\|\nabla_{\Gamma_h} w_h\|_{L^2(\Gamma_h)}} \leq ch^m \|f\|_{H_\infty^m(\Gamma)} + ch^{k+1} \|f\|_{H_\infty^1(U)}. \tag{12.20}$$

Proof. The proof uses very similar arguments as in the previous section. It is based on the error splitting

$$
\begin{aligned}
l(w_h) - l_h(w_h) = & \int_{\hat{\Gamma}_h} \big((f_h \circ p_h)\hat{\mu}_h - f^s \hat{\mu}^s\big)\hat{w}_h \, d\hat{s}_h \\
& + \sum_{T \in \mathcal{F}_h} E_T(f^s \hat{\mu}^s \hat{w}_h) \\
& + \sum_{T \in \mathcal{F}_h} Q_T\big((f^s \hat{\mu}^s - f_h^q \hat{\mu}_h)\hat{w}_h\big).
\end{aligned} \tag{12.21}
$$

Bounds for the three terms on the right-hand side are given in theorem 12.5 in [13]. ∎

13. Main theorem. Combining the results derived in the previous sections with the Strang Lemma we obtain a discretization error bound. This main result and the key assumptions are summarized in the following theorem. We assume that Γ is sufficiently smooth, but do not specify the required smoothness.

THEOREM 13.1. *Assume that the FE level set function ϕ_h^k satisfies* (2.2). *To construct a quasi-normal field we apply a gradient recovery method to ϕ_h^k that satisfies Assumption 3.1. On $\hat{\Gamma}_h$ (zero level of ϕ_h^1) we use a FE space $\hat{S}_h$ that has the approximation property* (10.1)*, with $m \geq 1$. Assume that $f \in H_\infty^m(\Gamma) \cap H_\infty^1(U)$ and that the solution u of* (1.1) *has regularity $u \in H^{m+1}(\Gamma) \cap H_\infty^2(\Gamma)$. We consider the discrete problem* (5.16) *with data extension f_h as in* (5.3) *and with a quadrature rule Q_T that is exact for all polynomials of degree $2m-2$. Then there exist constants $h_0 > 0$ and c such that for all $h \leq h_0$ the error in the solution u_h^q of* (5.16) *is bounded by*

$$
\begin{aligned}
& \big\| \nabla_\Gamma \big(u - (u_h^q)^\ell\big) \big\|_{L^2(\Gamma)} \\
& \leq c h^m \big(\|u\|_{H^{m+1}(\Gamma)} + \|f\|_{H_\infty^m(\Gamma)}\big) + c h^{k+1} \big(\|u\|_{H_\infty^2(\Gamma)} + \|f\|_{H_\infty^1(U)}\big).
\end{aligned}
$$

14. Numerical experiments. As surface we take the unit sphere $\Gamma = \{x \in \mathbb{R}^3 \mid \phi(x) = 0\}$, $\phi(x) := \|x\| - 1$, and $\Omega = (-2, 2)^3$. A family $\{\mathcal{T}_l\}_{l \geq 0}$ of tetrahedral triangulations of Ω is used. We triangulate Ω starting with a uniform subdivision into 48 tetrahedra with mesh size $h_0 = \sqrt{3}$. Adaptive red-green refinement (implemented in the software package DROPS [8]) is applied; in each refinement step the tetrahedra that contain the (approximate) surface are refined such that on level $l = 1, 2, \ldots$ there holds $h_T \leq \sqrt{3}\ 2^{-l}$ in a small neighborhood of Γ. The family $\{\mathcal{T}_l\}_{l \geq 0}$ is consistent and quasi-uniform (in a neighborhood of Γ). As piecewise linear approximation of ϕ we take $\hat{\phi}_h := \phi_h^1 := I^1(\phi)$ where I^1 is the nodal interpolation operator on $\mathcal{T}_l$ for piecewise linear FEs. The piecewise linear interface is given by $\hat{\Gamma}_h := \{\, x \in \Omega \mid \hat{\phi}_h(x) = 0 \,\}$. For the approximation of ϕ two choices are considered. A piecewise quadratic approximation of ϕ is given by $\phi_h := \phi_h^2 := I^2(\phi)$ where I^2 is the nodal interpolation operator on $\mathcal{T}_l$ for piecewise quadratic FEs. This choice satisfies (2.2) for $k = 2$. The higher order interface is $\Gamma_h := \{\, x \in \Omega \mid \phi_h(x) = 0 \,\}$. We also consider the choice $\phi_h := \hat{\phi}_h$, hence $\Gamma_h = \hat{\Gamma}_h$, which satisfies (2.2) with $k = 1$. These choices will show the dependence on both m and k of the bound in Theorem 13.1.

For the case $\phi_h = \phi_h^2$ ($k = 2$) the quasi normal field n_h is a vector-valued, continuous, piecewise quadratic FE function. It is computed as described in Remark 1. The skew projection $y = p_h(x) \in \Gamma_h$, $x \in U$, is computed as in [19, Sec. 5]. Given

l	$\|Dp_h - Dp\|_{L^\infty(\hat{\Gamma}_h)}$	factor	EE	factor	$\|\hat{\mu}_h - \hat{\mu}^s\|_{L^\infty(\hat{\Gamma}_h)}$	factor
1	0.09229	–	0.02021	–	0.02880	–
2	0.02704	3.4	3.584e-3	5.6	4.851e-3	5.9
3	7.004e-3	3.9	4.645e-4	7.7	6.441e-4	7.5
4	1.722e-3	4.1	6.837e-5	6.8	9.422e-05	6.8
5	4.579e-4	3.8	8.919e-6	7.7	1.239e-05	7.6
6	1.148e-4	4.0	1.141e-6	7.8	1.585e-06	7.8

TABLE 14.1

Error of the (projected) Jacobian ($EE := \|\mathbf{P}(Dp_h - Dp)\hat{\mathbf{P}}\|_{L^\infty(\hat{\Gamma}_h)}$) and of the functional determinant.

x and y, one can compute $n_h(y)$, $Dn_h(y)$, and $d_h(x) = \langle x - y, n_h(y)\rangle$. The exact normal on Γ_h can be determined from $\bar{n}_h(y) = \|\nabla\phi_h(y)\|^{-1}\nabla\phi_h(y)$. Hence, $Dp_h(x)$ can be computed using (8.3), and $\hat{\mu}_h$ can be computed as in Lemma 4.1.

Experiment 1. We perform an experiment to show that the estimates in Lemma 8.2 and Lemma 12.2 are sharp. These estimates are crucial in the error analysis for bounding the errors resulting from variational crimes by $\mathcal{O}(h^{k+1})$ instead of $\mathcal{O}(h^k)$ terms. For the unit sphere, one computes $Dp(x) = \|x\|^{-1}(\mathbf{I} - xx^T/\|x\|^2)$. For given $x \in \hat{\Gamma}_h$, the Jacobian $Dp_h(x)$ can be determined as explained above. The corresponding error $\|Dp_h - Dp\|_{L^\infty(\hat{\Gamma}_h)}$ is approximated by taking the maximum of $\|Dp_h(x) - Dp(x)\|$ over the vertices x of all triangles $T \in \mathcal{F}_h$. In Table 14.1 this error is given. The results show the $\mathcal{O}(h^2)$ behavior as proven in (8.8).

The projected error $EE := \|\mathbf{P}(Dp_h - Dp)\hat{\mathbf{P}}\|_{L^\infty(\hat{\Gamma}_h)}$ (approximated in the same way as explained above) is also given in Table 14.1 and shows a $\mathcal{O}(h^3)$ behavior as proven in (8.9). Finally in Table 14.1, we give the error quantity $\|\hat{\mu}_h - \hat{\mu}^s\|_{L^\infty(\hat{\Gamma}_h)}$ which has an $\mathcal{O}(h^3)$ behavior, as proven in Lemma 12.2.

Experiment 2. We apply the discretization method to the Laplace-Beltrami equation (1.1) on two different surfaces Γ, cf. [17]. As a first example we take Γ and Ω as above. The right-hand side f is such that the solution is given by $u(x) = \frac{12}{\|x\|^3}(3x_1^2x_2 - x_2^3)$, $x = (x_1, x_2, x_3) \in \Omega$. The function u is an eigenfunction of the Laplace-Beltrami operator. The right-hand side f satisfies $\int_\Gamma f\,ds = 0$, likewise does u. Note that u and f are constant along normals of Γ, i.e. $u \equiv u^e$, $f \equiv f^e$.

The triangulations $\mathcal{T}_l$ and $\hat{\phi}_h$ are the same as explained above. For the FE space $\hat{S}_h$, cf. (5.1), we use the *trace of the outer piecewise quadratic FE space*, as explained in Remark 2. Thus, we have $m = 2$ in (10.1). For $\phi_h = \phi_h^k$ we consider the choices with $k \in \{1, 2\}$ as explained above.

We outline the approach for the evaluation of a_h in (5.14) and l_h in (5.15). For the quadrature rule Q_T in (5.12), we use a fifth order accurate formula with positive weights on the reference triangle. For $x \in \hat{\Gamma}_h$, $\hat{\mu}_h(x)$ can be evaluated as described above. With these data, l_h can be computed as in (5.15). For a_h, Lemma 5.1 and (5.10) yield an expression for F in (5.14). The numerical solution u_h^q is normalized such that $\sum_{T\in\mathcal{F}_h} Q_T(\hat{u}_h^q \circ p_h\,\hat{\mu}_h) = 0$. We solve the discrete problem with a CG method with symmetric Gauss-Seidel preconditioner to a relative tolerance of 10^{-7}.

We start with the case $k = 1$ ($\Gamma_h = \hat{\Gamma}_h$), $m = 2$. Theorem 13.1 implies the $H^1(\Gamma_h)$-error-bound $ch^2 + ch^2 = \mathcal{O}(h^2)$. This can be observed in Table 14.2. Since the geometric errors are of the order $\mathcal{O}(h^2)$ we expect that $\|u^e - u_h^q\|_{L^2(\Gamma_h)}$ is dominated

level l	$\|u^e - u_h^q\|_{L^2(\Gamma_h)}$	factor	$\|\nabla_{\Gamma_h}(u^e - u_h^q)\|_{L^2(\Gamma_h)}$	factor
1	0.1431	–	0.6911	–
2	0.03239	4.4	0.1636	4.2
3	7.986e-3	4.1	0.04219	3.9
4	1.968e-3	4.1	0.01054	4.0
5	4.935e-4	4.0	2.689e-3	3.9
6	1.230e-4	4.0	6.685e-4	4.0

TABLE 14.2
Sphere, $m = 2$, $k = 1$: Discretization errors and error reduction.

level l	$\|u^e - u_h^q\|_{L^2(\Gamma_h)}$	factor	$\|\nabla_{\Gamma_h}(u^e - u_h^q)\|_{L^2(\Gamma_h)}$	factor
1	0.03910	–	0.5615	–
2	4.541e-3	8.6	0.1319	4.3
3	6.197e-4	7.3	0.03452	3.8
4	7.772e-5	8.0	8.647e-3	4.0
5	1.006e-5	7.7	2.224e-3	3.9
6	1.243e-6	8.1	5.444e-4	4.1

TABLE 14.3
Sphere, $m = 2$, $k = 2$: Discretization errors and error reduction.

by this term. The $L^2(\Gamma_h)$-error is given in Table 14.2 and scales like $\mathcal{O}(h^2)$. The number of iterations needed in nthe PCG solver on level $l = 1, 2, \ldots, 6$, is 21, 40, 68, 147, 272, 588.

We finally consider the case $k = m = 2$, i.e. a *higher order* approximation. From Theorem 13.1, we know that $H^1(\Gamma_h)$-error is bounded by $ch^2 + ch^3 = \mathcal{O}(h^2)$. This order can be observed in Table 14.3. Since the geometric errors are of the order $\mathcal{O}(h^3)$ we expect, cf. the analysis in [6], that $\|u^e - u_h^q\|_{L^2(\Gamma_h)}$ is of the order $\mathcal{O}(h^3)$. The error $\|u^e - u_h^q\|_{L^2(\Gamma_h)}$ is given in Table 14.3. These results show that the error indeed scales like $\mathcal{O}(h^3)$. Hence our method, based on piecewise quadratics both for the surface approximation and for the Galerkin discretization, has third order convergence. The number of PCG iterations needed on level $l = 1, 2, \ldots, 6$, is 21, 39, 68, 147, 272, 588.

As a second example we take a torus instead of the unit sphere. Let $\phi(x) = d(x) \cdot q(x)$, $\Gamma = \{x \in \Omega \mid \phi(x) = 0\}$, where $d(x)$ is the signed distance function of a torus with radii $R = 1$, $r = 0.6$ and $q(x) = 9 + 4\cos(10x_1x_2/(10^{-10} + |x|))$ is a perturbation which makes $\phi(x)$ steep and oscillatory [19]. As before, the numerical method uses only $I^2\phi$, but neither d nor ϕ. In the coordinate system (ρ, φ, θ) with $x = R\big(\cos\varphi, \sin\varphi, 0\big)^T + \rho\big(\cos\varphi\cos\theta, \sin\varphi\cos\theta, \sin\theta\big)^T$, the ρ-direction $\frac{\partial x}{\partial \rho}$ is normal to Γ. Thus, the solution $u(x) = \sin(3\varphi)\cos(3\theta + \varphi)$ (and the corresponding right-hand side f) are constant in normal direction. Both u and f have zero mean on Γ. In the discretization all components are the same as in the previous example. We only present the results for $k = m = 2$. The error $\|u^e - u_h^q\|_{L^2(\Gamma_h)}$ is shown in Table 14.4. We observe the expected $\mathcal{O}(h^3)$ behavior.

REFERENCES

level l	2	3	4	5	6	7
$\|u^e - u_h^q\|_{L^2(\Gamma_h)}$	0.08688	0.007165	0.0009576	1.179e-4	1.433e-05	1.861e-06
factor	–	12	7.5	8.1	8.2	7.7

TABLE 14.4
Torus: Discretization errors and error reduction.

[1] M. Ainsworth and J. Oden, *A Posteriori Error Estimation in Finite Element Analysis*, Wiley, New York, 2000.

[2] M. Bertalmio, G. Sapiro, L.-T. Cheng, and S. Osher, *Variational problems and partial differential equations on implicit surfaces*, J. Comp. Phys., 174 (2001), pp. 759–780.

[3] Y. Chen and C. B. Macdonald, *The Closest Point Method and multigrid solvers for elliptic equations on surfaces*, SIAM J. Sci. Comput., 37 (2015).

[4] P. Ciarlet, *Basic error estimates for elliptic problems*, in Handbook of Numerical Analysis, P. Ciarlet and J.-L. Lions, eds., North-Holland, Amsterdam, 1991, pp. 17–351.

[5] K. Deckelnick, G. Dziuk, C. M. Elliott, and C.-J. Heine, *An h-narrow band finite element method for elliptic equations on implicit surfaces*, IMA Journal of Numerical Analysis, 30 (2010), pp. 351–376.

[6] A. Demlow, *Higher-order finite element methods and pointwise error estimates for elliptic problems on surfaces*, SIAM J. Numer. Anal., 47 (2009), pp. 805–827.

[7] A. Demlow and M. Olshanskii, *An adaptive surface finite element method based on volume meshes*, SIAM J. Numer. Anal., 50 (2012), pp. 1624–1647.

[8] *DROPS package*. http://www.igpm.rwth-aachen.de/DROPS/.

[9] G. Dziuk, *Finite elements for the Beltrami operator on arbitrary surfaces*, in Partial differential equations and calculus of variations, S. Hildebrandt and R. Leis, eds., vol. 1357 of Lecture Notes in Mathematics, Springer, 1988, pp. 142–155.

[10] G. Dziuk and C. Elliott, *A fully discrete evolving surface finite element method*, SIAM J. Numer. Anal., 50 (2012), pp. 2677–2694.

[11] ———, *Finite element methods for surface PDEs*, Acta Numerica, (2013), pp. 289–396.

[12] J. Grande, *Analysis of highly accurate finite element based algorithms for computing distances to level sets*, IGPM Report 419, IGPM, RWTH Aachen University, 2015.

[13] J. Grande and A. Reusken, *A higher order finite element method for partial differential equations on surfaces*, Preprint 403, IGPM, RWTH Aachen, 2014.

[14] C.-J. Heine, *Isoparametric finite element approximation of curvature on hypersurfaces*, Tech. Report 26, Fakulät für Mathematik und Physik, Universität Freiburg, 2004.

[15] J. Nedelec, *Curved finite element methods for the solution of singular integral equations on surfaces in R^3*, Comput. Methods in Appl. Mech. Eng., 8 (1976), pp. 61 – 80.

[16] M. Olshanskii and A. Reusken, *A finite element method for surface PDEs: matrix properties*, Numer. Math., 114 (2009), pp. 491–520.

[17] M. Olshanskii, A. Reusken, and J. Grande, *A finite element method for elliptic equations on surfaces*, SIAM Journal on Numerical Analysis, 47 (2009), pp. 3339–3358.

[18] M. A. Olshanskii, A. Reusken, and X. Xu, *On surface meshes induced by level set functions*, Comp. Vis. Sci., 15 (2012), pp. 53–60.

[19] A. Reusken, *A finite element level set redistancing method based on gradient recovery*, SIAM J. Numer. Anal., 51 (2013), pp. 2723–2745.

[20] A. Reusken, *Analysis of trace finite element methods for surface partial differential equations*, IMA Journal of Numerical Analysis, (2014).

[21] I. von Glehn, T. März, and C. B. Macdonald, *An embedded method-of-lines approach to solving partial differential equations on surfaces*, tech. report, Mathematical Institute, University of Oxford, 2014.

[22] Z. Zhang and A. Naga, *A new finite element gradient recovery method: superconvergence property*, SIAM J. Sci. Comput., 26 (2005), pp. 1192–1213.