YIC GACM 2015
3rd ECCOMAS Young Investigators Conference
6th GACM Colloquium
July 20–23, 2015, Aachen, Germany

Asymptotic Preserving Discontinuous Galerkin Method

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Abstract. *In this work a numerical method for the low Mach isentropic Navier-Stokes equation is devised. Supported by a short analysis, we observe that this algorithm is capable of treating the low Mach equations also in the limit of zero Mach number. The performance of the algorithm is investigated numerically, where one can observe problems with the order of convergence. We conclude with possible reasons and remedies.*

Keywords: discontinuous Galerkin; asymptotic preserving; IMEX

1 INTRODUCTION

Singularly perturbed problems frequently arise in computational physics, in particular, for low Mach number flows, the Navier-Stokes equations constitute such a singularly perturbed problem, as the speed of sound is extremely large in comparison to the fluid's speed. In the limit of zero Mach number, the equations change their type [1].
Devising a numerical method which solves the equations in the low-Mach regime with feasible computational costs has attracted a lot of interest in the past years. For example in [2], [3] and in [4] different numerical methods of low order have been published. Also high-order methods have been introduced, e.g. in [5] for the steady-state case or in [6] a space-time implicit discontinuous Galerkin method has been introduced to solve the equations in entropy variables. See also the references in these publications for a larger overview on this topic.
The goal of this work is to devise a high-order method based on the discontinuous Galerkin method for low Mach flows. The paper is organized as follows. In the first section the governing equation and the corresponding discretization is introduced. In the second section the asymptotic preserving property is shown. The last two sections present numerical results and give a short conclusion and an outlook for further work.

2 IMEX DISCONTINUOUS GALERKIN FOR ISENTROPIC NAVIER-STOKES

2.1 Governing Equations

The nondimensional isentropic Navier-Stokes equations on a domain $\Omega \subset \mathbb{R}^2$ are given in equation ((2)-left), where ρ denotes the density, $p(\rho) = \rho^\gamma$, with $\gamma \geq 1$, is the pressure function and $\vec{u}$ denotes the velocity. Assuming that $w := (\rho, \rho\vec{u})$ can be expanded as

$$(\rho, \rho\vec{u})^T =: w = w_{(0)} + \varepsilon w_{(1)} + \varepsilon^2 w_{(2)} + \mathcal{O}(\varepsilon^3), \tag{1}$$

one can - given suitable initial and boundary conditions - directly compute the limit for $\varepsilon \to 0$ and derive the nondimensional incompressible isentropic Navier-Stokes equations ((2)-right).

$$\begin{aligned} \rho_t + \nabla \cdot (\rho\vec{u}) &= 0 \\ (\rho\vec{u})_t + \nabla \cdot (\rho\vec{u} \otimes \vec{u}) + \tfrac{1}{\varepsilon^2}\nabla p(\rho) &= \tfrac{1}{Re}\Delta\vec{u}. \end{aligned} \qquad \Bigg| \qquad \begin{aligned} \rho_{(0)} &\equiv \text{const} \\ \nabla \cdot \vec{u}_{(0)} &= 0 \\ (\vec{u}_{(0)})_t + \nabla \cdot (\vec{u}_{(0)} \otimes \vec{u}_{(0)}) + \tfrac{\nabla p_{(2)}}{\rho_{(0)}} &= \tfrac{1}{Re}\Delta\vec{u}_{(0)} \end{aligned} \tag{2}$$

2.2 Discretization

Consider a partition of the domain into a finite number of elements $\mathcal{T} = \{T_i\}$, e.g. triangles or rectangles, and introduce the Ansatz space V_h on $\mathcal{T}$,

$$V_h := \{v \in L^2(\Omega) : v|_{T_i} \in \Pi^k(T_i)\ \forall T_i \in \mathcal{T}\}^3, \tag{3}$$

with given polynomial degree k. Then one seeks functions $w_h \in C^1([0,T], V_h)$ and $\sigma_h \in C^1([0,T], V_h^2)$ such that

$$(\sigma_h, \tau_h) - (\nabla w_h, \tau_h) + \langle w_h - \hat{w}, \tau_h \cdot n\rangle = 0 \tag{4}$$

$$\left(\frac{\partial w_h}{\partial t}, v_h\right) - (f_c^\varepsilon(w_h) - f_v(w_h, \sigma_h), \nabla v_h) + \left\langle (\hat{f}_c^\varepsilon - \hat{f}_v) \cdot n, v_h \right\rangle = 0 \tag{5}$$

holds for every $\tau_h \in V_h^2$ and $v_h \in V_h$. $(\cdot,\cdot)$ denotes the sum over integrals over the cells Ω_i and $\langle\cdot,\cdot\rangle$ the sum over all integrals over the cell boundaries $\partial\Omega_i$. Furthermore, f_c^ε denotes the convective part of ((2)-left) and f_v the viscous part of ((2)-left). This discontinuous Galerkin formulation is closed by the numerical flux functions $\hat{f}_c^\varepsilon, \hat{f}_v$ and $\hat{w}$ and a suitable time integration method. For example one can choose the local Lax-Friedrichs numerical flux for the convective part and the Bassi-Rebay [7] numerical flux for the viscous part, which is done in the following.
The equations are extremely stiff for small values of ε. Therefore an explicit method would have to deal with a large amount of time steps. On the other hand an implicit method would produce unnecessary damping for larger time steps and one has to solve a nonlinear system of equations in every step. A possible workaround is splitting the convective part of the equation, which is responsible for the stiffness, into a part which should be solved explicitly and into a part which should be solved implicitly. Then, one can use an IMEX scheme, which is a combination of an explicit and an implicit time integration method. There are several IMEX schemes available in literature, e.g. IMEX Runge-Kutta methods [8, 9, 10] or IMEX linear multistep methods [11].
Next, one has to find a suitable splitting for using an IMEX time integration method. The resulting scheme should be stable for a CFL restriction independently of ε, both parts of the splitting should induce a hyperbolic system and the implicit part should be linear to reduce the computational costs. Finding a stable splitting is, even for the linear one dimensional case, nontrivial, see also [12]. There is a possible splitting available for the isentropic Navier-Stokes equations by Haack et al. in [4], which is given by

$$\begin{pmatrix}\rho_t\\(\rho\vec{u})_t\end{pmatrix} + \underbrace{\begin{pmatrix}\tau\nabla\cdot(\rho\vec{u})\\ \nabla\cdot(\rho\vec{u}\otimes\vec{u}) + \frac{1}{\varepsilon^2}\nabla\left(p(\rho) - a(t)\rho\right)\end{pmatrix} - \begin{pmatrix}0\\ \frac{1}{Re}\Delta\vec{u}\end{pmatrix}}_{\text{explicit}} + \underbrace{\begin{pmatrix}(1-\tau)\nabla\cdot(\rho\vec{u})\\ \frac{1}{\varepsilon^2}\nabla\left(a(t)\rho\right)\end{pmatrix}}_{\text{implicit}} = \begin{pmatrix}0\\0\end{pmatrix}, \tag{6}$$

where $\tau = c\varepsilon^2$, with a constant c, and $a = \min p'(\rho)$. With this choice one can take the local Lax-Friedrichs numerical flux function with a viscous stabilization coefficient in $\mathcal{O}(1)$ for the explicit and $\mathcal{O}(\varepsilon^{-1})$ for the implicit part. Note that the implicit part of the splitting is only linear if one treats the splitting coefficient $a(t)$ in an explicit way, i.e., evaluated at time t^n. Furthermore, there could be problems with the convergence order in the setting of singular perturbation problems if one uses an IMEX Runge-Kutta method [13]. In [14], Boscarino derived a third order IMEX Runge-Kutta which shows no order degeneration. For IMEX linear multistep methods no such effects occur [15].

3 ASYMPTOTIC PRESERVING

For small values of ε the analytical solution is very close to the corresponding solution of the limiting equations. Therefore it is useful to check if the limiting numerical method, viz. for $\varepsilon \to 0$, is a discretization of the limiting equation ((2)-right). This is the so-called asymptotic preserving (AP) property [16] and it can be verified with the help of an asymptotic expansion.

Lemma 1. *The given numerical method with an arbitrary polynomial degree and IMEX-Euler time discretization is asymptotic preserving.*

The proof of this lemma is very technical and too long for this short overview. Therefore only the main ideas are given.

- Expand all quantities with the help of an asymptotic expansion (see eq. (1)) and collect the terms in power of ε.

- The stabilization term of the numerical flux function of the continuity equation gives continuity of the limit density.
- From the ε^{-2} terms of the momentum equation and the continuity equation one can conclude that the limit density is constant in space and time.
- Rearranging the terms and inserting expansions of different quantities delivers a discretization of the limit equation with additional stabilization and splitting terms.

This lemma shows the AP-property for the IMEX Euler time discretization only. However, it can directly be extended to IMEX multistep methods. Note that in this case, high order might be lost for $\varepsilon \to 0$. Recovering a uniform (in ε) high order is a difficult task, we refer to [13]. In the context of DG methods, this is left for future work. Note that the AP property depends on the numerical flux through the viscosity coefficient, e.g. choosing $\mathcal{O}(\varepsilon^{-2})$ for the continuity equation and $\mathcal{O}(1)$ for the momentum equation, one can also show that the density $\rho_{(1)}$ is constant in space. Currently, this proof is only done for the local Lax Friedrichs flux, therefore choosing a complete different flux function, e.g. the Roe flux, could also lead to a scheme which is AP, but this might be part of prospective investigations.

4 NUMERICAL RESULTS

Two examples are taken from literature, see [4] and Table 1, to investigate the performance of the method. The DG method is coupled to an IMEX Runge-Kutta time discretization to compute different high-order solutions. In this case for the 2nd order method the SSP-332 [9], for the 3rd order method the BHR-553 [14] and for the 4th order method the ARK-436lsa [10] IMEX Runge-Kutta methods are chosen. To investigate the convergence orders, the numerical solution is compared with the numerical solution on a finer grid, namely on the grid which results from one uniform refinement.

Table 1: The settings of the numerical examples 'Vortex in a Box' and 'Periodic Flow'.

	Values		**Vortex in a Box**	**Periodic Flow**
Ω	$[0,1]^2$	$\rho(0,x,y)$	$1-\frac{\varepsilon^2}{2}\tanh\left(y-\frac{1}{2}\right)$	$1+\varepsilon^2\sin^2(2\pi(x+y))$
T	0.125 or 0.01	$u(0,x,y)$	$2\sin^2(\pi x)\sin(\pi y)\cos(\pi y)$	$\sin(2\pi(x-y))$
Re	100	$v(0,x,y)$	$-2\sin(\pi x)\cos(\pi x)\sin^2(\pi y)$	$\sin(2\pi(x-y))$
ε	10^{-4}	Boundary	$(u,v)\|_{\partial\Omega}=0$	periodic
		γ	1.4	2

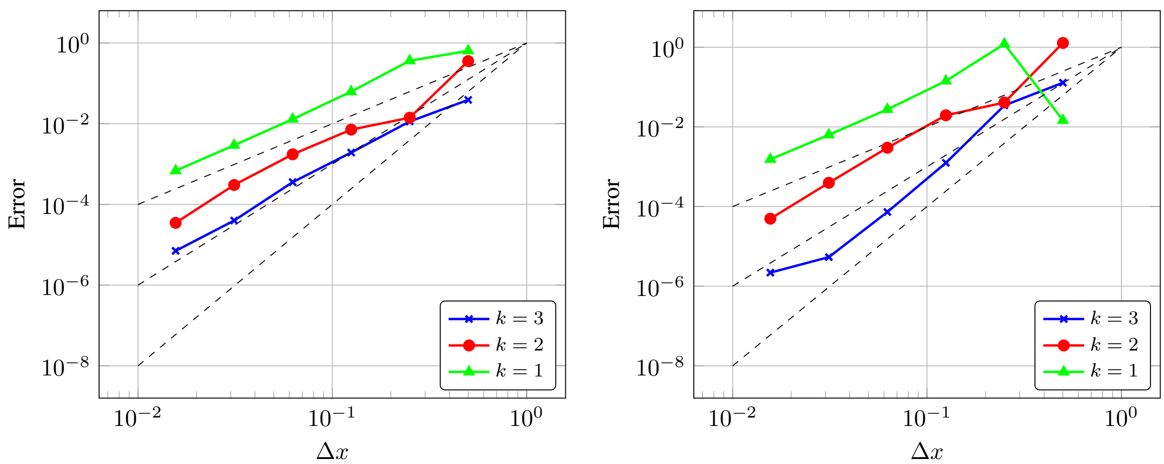

Figure 1: Convergence study for the 'Vortex in a Box' (left) and 'Periodic Flow' (right) examples with different high-order methods.

In Figure 1-left one can see the convergence for different polynomial degrees k. The low order method seems to deliver the desired results, but increasing the polynomial degree does not result in an increase of the convergence

order. Especially polynomials of degree $k = 3$, viz. a 4^{th} order method, only show third order convergence. Again one can take a look at the reference method of Haack et al. in [4] who observe a similar behavior in the case of their 2^{nd} order method.
The second example, namely the 'Periodic Flow' example, shows a slightly better behavior. One can see in Figure 1-right, that for all polynomial degrees the desired convergence order is given. Only the 4^{th} order method starts with the desired convergence order but then slows down.

5 CONCLUSION AND OUTLOOK

Coupling the splitting of Haack, Jin and Liu [4] with a high-order IMEX discontinuous Galerkin method results in an asymptotic preserving scheme, but it does not deliver the desired convergence order for all test-cases. Of course this could have several explanations. One possible explanation could be, that the 'Vortex in a Box' test case might not be designed for high-order methods. Other explanations could be the order degeneration mentioned before or an effect due to the splitting combined with a discontinuous Galerkin discretization. Similar observations have been made in [4].
Therefore the next step would be to figure out what exactly constitutes the reason for these problems. One possible solution to overcome the convergence problems and devise a stable high-order numerical method is considering a different splitting. The so-called RS-IMEX splitting, applied to an ODE problem in [15], delivers very promising results for simple test equations and could be an improvement. This splitting also has a direct extension to the non-isentropic Navier-Stokes equations, which is not given for the splitting of Haack, Jin and Liu.

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