# A Hybrid Mixed Method for the Compressible Navier-Stokes Equations

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## Abstract

We present a novel discretization method for nonlinear convection-diffusion equations and, in particular, for the compressible Navier-Stokes equations. The method is based on a Discontinuous Galerkin (DG) discretization for convection terms, and a Mixed method using H(div) spaces for the diffusive terms. Furthermore, hybridization is used to reduce the number of globally coupled degrees of freedom. For the scalar case, a local postprocessing procedure is used to enhance the quality of the approximate solution  $w_h$ . The method reduces to a DG scheme for pure convection, and to a Mixed method for pure diffusion, while for the intermediate case the combined variational formulation requires no additional parameters. We formulate and validate our scheme for nonlinear model problems, as well as compressible flow problems.

Keywords: Hybrid Mixed method, Navier-Stokes equations, Euler equations

#### 1. Introduction

The development of high-order methods for the solution of the compressible Navier-Stokes equations on unstructured meshes has received considerable attention during the past few years [1, 2, 3]. Especially methods using discontinuous, piecewise polynomial approximation spaces have become increasingly popular, one very well-known example being the Discontinuous Galer-kin (DG) method [4, 5, 6, 7, 8, 9, 1, 10, 11].

The steady compressible Navier-Stokes equations can be seen as a combination of a convection and a diffusion equation. A commonly used model problem for such types of equations is the scalar convection-diffusion equation, given as

$$\nabla \cdot f(w) - \varepsilon \Delta w = h. \tag{1}$$

If  $\varepsilon = 0$ , the remaining convective part has traditionally been discretized in a DG framework using discontinuous approximate solutions  $w_h$ . This is reasonable from the point of view that the mathematical solution operators are defined in the context of *BV* spaces (i.e., spaces of functions having bounded variation) [12], and as such admit discontinuities. Furthermore, the (discontinuous) Riemann problem, an approximation of which is usually solved at element interfaces,

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ties in neatly with both the physics and the mathematics of the underlying problems. On the other hand, when considering the diffusive part  $\varepsilon \Delta w$  without convection, i.e. an elliptic problem, there is little need for discontinuous approximation spaces, as the problem itself admits solutions that are usually very regular and can, with fewer degrees of freedom, be approximated in continuous function spaces. Thus devising discretization methods for advection-diffusion equations is complicated by the fact that the individual subproblems, advection and diffusion, are usually discretized using different methods when they appear alone. Since in the realm of compressible flow simulation the nonlinear convection terms traditionally dominate the development of numerical schemes, it is not surprising that most high-order schemes for the Navier-Stokes equations discretize both components in a DG framework [1, 2, 3]. Arguably, the use of discontinuous discretization procedures for the combined convection-diffusion problem is primarily a result of the desire to unify the treatment for both the diffusive and the convective part, as well as the higher weight placed on the convection terms.

In this paper, we pursue a different approach, extending an idea by Egger and Schöberl [13] to the nonlinear system case, and combine a DG discretization of the convective operator with a more classical,  $H(\text{Div}, \Omega)$ -based approximation of the viscous part. To illustrate our approach, let us consider equation (1) again. A frequently used concept when deriving a discretization for equations of this type is to rewrite them in mixed form, introducing the auxiliary variable  $\sigma := \varepsilon \nabla w$ , thereby obtaining

$$\sigma = \varepsilon \nabla w \tag{2}$$

$$\nabla \cdot (f(w) - \sigma) = h. \tag{3}$$

The goal of our scheme is to combine both a standard DG scheme for the convective part and a standard Hybrid Mixed method [14] for the diffusive part, the latter yielding a function  $\sigma \in H(\text{Div}, \Omega)$ , i.e., a function possessing a divergence all over the domain.

One potential bottleneck of this method that can immediately be identified is that one needs extra degrees of freedom for the resolution of  $\sigma$ . Due to the continuity requirements of the space  $H(\text{Div}, \Omega)$ , it is not possible to reduce the number of globally coupled degrees of freedom by locally eliminating the variable  $\sigma$  via simple lifting procedures. This is an important difference compared to most DG methods [11]. For purely elliptic equations, this problem has a classic solution by introducing hybridization [15, 16]. Instead of enforcing  $\sigma \in H(\text{Div}, \Omega)$ , one allows  $\sigma$  to be from a completely discontinuous function space, while enforcing the constraint of normal continuity weakly by introducing even more unknowns, the hybrid variables, having support only on the cell boundaries. These hybrid variables can then be used to locally eliminate both w and  $\sigma$  and thus remain the only globally coupled variables. Recently, the concept of hybridization has been extended to DG discretization for purely elliptic problems by Cockburn et al. [17]. Subsequently Nguyen et al. [18, 19] have presented a hybridized DG method for advectiondiffusion problems, which incorporated the hybridization of convective terms, and have since extended their method to the Navier-Stokes equations [3]. In a related paper, Egger and Schöberl [13] have proposed a scheme for linear convection-diffusion problems that uses a DG discretization for the convective terms, and a Hybrid Mixed method for the diffusive terms, recognizing the potential of hybridization to render different discretization techniques compatible. The hybridized schemes for the individual subproblems yield bilinear forms that can be simply added without any compatibility problems, or additional parameters to tune. Our approach follows Ref. [13], extending the concept to nonlinear problems, and, in particular, the compressible Navier-Stokes equations. In particular, as the DG discretization of the nonlinear convective part has

been *hybridized*, it is compatible with the hybrid discretization of the diffusive part. Again the corresponding semi-linear forms may simply be added to obtain a Hybrid Mixed scheme for the nonlinear convection-diffusion equation, which naturally reduces to a pure DG scheme, and a Mixed method for the limiting cases of vanishing and dominating diffusion, respectively.

A very important task in the computation of aerodynamic flows is that of accurately resolving functionals such as lift or drag. Both viscous lift and viscous drag can be written as functions of the unknown solution w and the viscous flux  $\sigma := f_v(w, \nabla w)$ , which in our proposed method is directly discretized as  $\sigma_h$ . If our scheme is indexed with p as a function of the polynomial orders which appear in the approximation of the different variables (for a definition of the underlying spaces, see Section 2), we observe that  $\sigma_h$  converges with order p + 1 in  $L^2$  towards  $\sigma$ , i.e., the  $L^2$ -norm of the error in  $\sigma_h$  converges asymptotically with the same order as the error in  $w_h$ . Furthermore, it is known that for the purely diffusive case, where our method reduces to a standard Mixed method,  $\sigma_h$  even converges with order p + 2 towards  $\sigma$  [14]. If diffusion is dominating, and there is thus no need for stabilization via the numerical flux functions anymore, it is also possible to obtain convergence of order p + 2 towards  $\sigma$  [13] for the advection-diffusion case.

Furthermore, the explicit approximation of  $\sigma$  has the advantage that it is possible to postprocess the approximate solution  $w_h$  in an easy, cell-wise way to increase its order of convergence. In the current paper, we will demonstrate this effect on the scalar convection-diffusion equation. The postprocessing relies on a method proposed by Stenberg [20], and has also been applied by Egger and Schöberl [13] for the linear equation.

By reducing the system to a system in the hybrid variables only, the number of degrees of freedom is asymptotically of the order  $O(p^{d-1})$ , while for 'classical' high-order discretization schemes such as the Discontinuous Galerkin, the Spectral Difference or the Spectral Volume method, it grows as  $O(p^d)$ . In both cases, p is the order of consistency with the underlying differential equation and d the spatial dimension. It has been recognized as a bottleneck for high-order computation of compressible flow problems that the Jacobian matrix, needed for implicit relaxation methods, grows as the square of the locally coupled degrees of freedom  $O(p^{2d})$ . Thus even if one reduces the overall number of degrees of freedom compared to a low order scheme, the memory requirement can still be significantly higher. Hybridization thus reduces this dependence by two orders of magnitude. For high-order discretization, this will surely have an impact.

The paper is organized as follows: In section 2 we introduce the discretization for scalar problems. In particular we present a unified treatment for both linear and non-linear problems. In section 3 we extend our method to compressible flow problems, and present validation both for the Euler equations, and the compressible Navier-Stokes equations. Finally, in section 4, we offer conclusions.

## 2. Method for the Scalar Nonlinear Convection-Diffusion Equation

Before presenting our method for the technically more involved case of the Navier-Stokes equations, we illustrate the ideas in a more convenient setting by considering the (possibly) nonlinear convection-diffusion equation

$$\nabla \cdot f(w) - \varepsilon \Delta w = h \quad \forall x \in \Omega$$

$$w = g \quad \forall x \in \partial \Omega$$
(4)

on a domain  $\Omega \subset \mathbb{R}^d$ . The function *f* is a (possibly nonlinear) smooth function of *w*. Formulated as a mixed equation, eq. (4) can be written as

$$\sigma = \varepsilon \nabla w \quad \forall x \in \Omega$$

$$\nabla \cdot (f(w) - \sigma) = h \qquad \forall x \in \Omega$$

$$w = g \qquad \forall x \in \partial \Omega.$$
(5)

Now let us assume that our domain  $\Omega$  is triangulated in a standard way as  $\Omega = \bigcup_{k=1}^{N} \Omega_k$  with non-overlapping elements  $\Omega_k$ . Also following standard nomenclature, we define an interior edge e as an intersection of two neighboring elements  $\Omega_k$  and  $\Omega_{k'}$  having positive (d-1)-dimensional measure. A boundary edge e is defined as the intersection of an element  $\Omega_k$  with the physical boundary  $\partial\Omega$ .  $\Gamma$  denotes the set of all edges, both interior and boundary. We assume that  $\Gamma$  is enumerated as  $\Gamma =: {\Gamma_k | k = 1, ..., \widehat{N}}$ . The set  $\Gamma_0 \subset \Gamma$  denotes the edges not intersecting the physical boundary  $\partial\Omega$  of the domain. We assume that  $\Gamma_k \in \Gamma$  is equipped with an orientation, expressed by a generic normal  $n_k$ . Considering v on  $\Gamma_k$ , we define  $v^{\pm}(x) := \lim_{\tau \to 0^+} v(x \pm \tau n_k)$ . Considering v on  $\partial\Omega_k$ , we define  $v^{\pm}$  similarly, with the standard convention that a normal to  $\partial\Omega_k$ is pointing out of  $\Omega_k$ .

Based on the sets introduced earlier, we define the necessary function spaces as

$$V_h := \{ f \in L^2(\Omega) | f_{|\Omega_k} \in \Pi^p(\Omega_k) \quad \forall k = 1, \dots N \}^m$$
$$H_h := \{ f \in L^2(\Omega) | f_{|\Omega_k} \in \Pi^{p+1}(\Omega_k) \quad \forall k = 1, \dots N \}^{d \cdot m}$$
$$M_h := \{ f \in L^2(\Gamma) | f_{|\Gamma_k} \in \Pi^{p+1}(\Gamma_k) \quad \forall k = 1, \dots \widehat{N} \}^m.$$

 $\Pi^{p}(\Omega)$  is the space of polynomials up to degree p on  $\Omega$ , and m denotes the dimension of the system. For two-dimensional Navier-Stokes, m = 4, for the scalar equation presented in this section, m = 1. Note that  $V_h$  is a standard space that is also used for Discontinuous Galerkin methods. We will seek numerical approximations  $w_h \in V_h$ , and  $\sigma_h \in H_h$ . Last but not least,  $M_h$  denotes the so-called hybrid space. Unlike other authors [15], we explicitly define  $M_h$  to have support also on the boundary edges. This simplifies the incorporation of boundary conditions in the case of the Navier-Stokes equations, while incurring only a moderate overhead.

Applying a mixed method to (5) would amount to choosing function spaces such that the discrete variable  $w_h \in V_h$ , while  $\sigma_h \in H(\text{Div}, \Omega)$ , with  $H(\text{Div}, \Omega)$  being a function space that allows for a global divergence, more formally,

$$H(\operatorname{div}, \Omega) := \{ f \in L^2(\Omega)^d | \nabla \cdot f \in L^2(\Omega) \}^m.$$

Discrete subspaces of  $H(\text{Div}, \Omega)$  are for example the well-known, classical Raviart-Thomas or the Brezzi-Douglas-Marini [21] spaces. The obvious drawback of this approach is that these function spaces require some degree of continuity, and it is thus not possible to locally eliminate the additional unknown  $\sigma_h$ . This is in contrast to most DG methods, where simple lifting operators are used to produce a global system only in  $w_h$  [11]. However, it is well-known that by weakly enforcing the divergence constraint via hybrid variables  $\lambda_h \in M_h$ , one can locally eliminate *both*  $w_h$  and  $\sigma_h$ , and produce a system that is globally coupled only in the DOFS corresponding to  $\lambda_h$ .

We proceed in two steps: First, we consider the case of pure diffusion  $(f(w) \equiv 0)$  where we derive a (standard) hybrid method that, in the diffusive case, amounts to solving for  $(w_h, \sigma_h) \in V_h \times (H_h \cap H(\text{Div}, \Omega))$ . Subsequently we consider the case of pure convection ( $\varepsilon \equiv 0$ ), where one

solves for  $w_h \in V_h$ , with  $w_h$  being exactly the DG solution to the nonlinear convection equation. The two resulting methods are combined to yield a discretization scheme for (4), where the only globally coupled degrees of freedom are the ones corresponding to  $\lambda_h \in M_h$ .

**Remark.** Let us clarify the convergence rates that can be expected. As  $w_h$  is a function of polynomial order p, it is natural to assume  $||w - w_h||_{L^2} = O(h^{p+1})$  represents the best-case scenario. The same reasoning would lead us to expect that  $||\sigma - \sigma_h||_{L^2} = O(h^{p+2})$ . However, it turns out that this is only obtained in practice if there is no need for stabilization introduced by the Riemann flux, which only holds for diffusion-dominated problems. In general, one can only expect  $||\sigma - \sigma_h||_{L^2} = O(h^{p+1})$ . As we approximate  $\sigma$  explicitly, a simple, element-wise postprocessing procedure is possible for p > 0 (cf. section 2.4.3), to obtain a solution  $w_h^*$ , for which  $||w - w_h^*||_{L^2} = O(h^{\overline{p}+1})$ , where  $\overline{p} = p + 1$  if stabilization is necessary, and  $\overline{p} = p + 2$  if not. As a comparison, for the DG scheme of Nguyen et al. [18], one has  $||w - w_h||_{L^2} = O(h^{p+1})$  and  $||\sigma - \sigma_h||_{L^2} = O(h^{p+1})$ , as they approximate  $\sigma$  in a space of polynomials of order p. Also in their setting, a postprocessing is possible. Results have been summarized in Table 1.

Method	$  w - w_h  _{L^2}$	$\ \sigma - \sigma_h\ _{L^2}$	$\ w - w_h^*\ _{L^2}$
Hybrid Mixed Method	$O(h^{p+1})$	$O(h^{p+1})$	$O(h^{p+2})$
Hybrid Mixed Method (no stabilization)	$O(h^{p+1})$	$O(h^{p+2})$	$O(h^{p+3})$
Hybridized DG Scheme	$O(h^{p+1})$	$O(h^{p+1})$	$O(h^{p+2})$

Table 1: Convergence orders of the quantities for different schemes. In all the cases,  $w_h$  is a cellwise polynomial of order p with p > 0.

#### 2.1. Poisson's Equation

Let us assume for the moment that  $f \equiv 0$  and that we thus approximate Poisson's equation. A standard hybrid method [21] can be applied to this equation, yielding the task of finding the triple  $(\sigma_h, w_h, \lambda_h) \in H_h \times V_h \times M_h$ , such that for all  $(\tau_h, \varphi_h, \mu_h) \in H_h \times V_h \times M_h$ 

$$\sum_{k=1}^{N} \left( \int_{\Omega_{k}} \sigma_{h} \cdot \tau_{h} + \varepsilon w_{h} \nabla \cdot \tau_{h} \, dx - \varepsilon \int_{\partial \Omega_{k} \setminus \partial \Omega} \lambda_{h} \tau_{h}^{-} \cdot n \, d\sigma - \varepsilon \int_{\partial \Omega_{k} \cap \partial \Omega} g \tau_{h}^{-} \cdot n \, d\sigma \right) = 0 \tag{6}$$

$$-\int_{\Omega} \nabla \cdot \sigma_h \varphi_h \, dx = \int_{\Omega} h \varphi_h \, dx \tag{7}$$

$$\int_{\Gamma_0} \mu_h \left( \sigma_h^- \cdot n - \sigma_h^+ \cdot n \right) d\sigma + \int_{\Gamma \setminus \Gamma_0} \mu_h \left( \lambda_h - g \right) d\sigma = 0.$$
(8)

Note that  $\lambda_h$  is set to the  $L^2$ -projection of g onto  $M_h$  on the boundary  $\Gamma \setminus \Gamma_0$ . In principle, it is not necessary to define  $\lambda_h$  on  $\Gamma \setminus \Gamma_0$ , as it is not needed in the presence of Dirichlet boundary data. However, as a preparation for the more complicated case of the Navier-Stokes boundary conditions, we keep this definition for the sake of a unified presentation.

Due to the discontinuous structure of the underlying spaces, it is possible to express the method as

$$\widehat{N}_1(\lambda_h, \mu_h) = \widehat{b}_1(\mu_h) \quad \forall \mu_h \in M_h.$$
(9)

This comes at the expense of higher assembly cost, but yields a much smaller system of equations. Reduction to (9) is possible because one can apply static condensation [14, 22]. Let us therefore define  $\Sigma \in \mathbb{R}^{\dim(H_h)}$ ,  $W \in \mathbb{R}^{\dim(V_h)}$  and  $\Lambda \in \mathbb{R}^{\dim(M_h)}$  to be vectors containing the degrees of freedom corresponding to  $\sigma_h$ ,  $w_h$  and  $\lambda_h$ , respectively. Doing so, (6)-(8) lead to a (linear) system of equations having the form

$$\begin{pmatrix} A & B & C \\ B^T & 0 & 0 \\ C^T & 0 & D \end{pmatrix} \begin{pmatrix} \Sigma \\ W \\ \Lambda \end{pmatrix} = b,$$

where due to the discontinuous structure of the spaces  $H_h$  and  $V_h$ , A and B are block-matrices, and we can cheaply, because locally, invert the contribution

$$\left(\begin{array}{cc}A & B\\B^T & 0\end{array}\right),$$

as it is a block-diagonal matrix. This then leads to a system in  $\Lambda$  only, which is the algebraic equivalent to (9).

Instead of applying static condensation, we use a conceptually different, yet equivalent, approach: We employ the concept of local solution techniques as introduced by Cockburn and Gopalakrishnan [16], meaning to express both  $w_h$  and  $\sigma_h$  as functions of  $\lambda_h$ . This is similar (although technically more involved) to lifting procedures used with DG schemes. Let us explain this in the context of the simple example of Poisson's equation: We introduce a function

$$M_h \to H_h \times V_h$$
  
 $\mu \mapsto (\sigma_h^k(\mu), w_h^k(\mu))$ 

implicitly given as

$$\int_{\Omega_{k}} \sigma_{h}^{k}(\mu) \cdot \tau_{h} + \varepsilon w_{h}^{k}(\mu) \nabla \cdot \tau_{h} \, dx - \varepsilon \int_{\partial \Omega_{k} \setminus \partial \Omega} \mu \tau_{h}^{-} \cdot n \, d\sigma = 0 \quad \forall \tau_{h} \in H_{h}$$
$$- \int_{\Omega_{k}} \nabla \cdot \sigma_{h}^{k}(\mu) \varphi_{h} \, dx = 0 \quad \forall \varphi_{h} \in V_{h}.$$

Note that this is a local, i.e., cellwise, operator. We furthermore have to introduce the functions  $(\overline{\sigma}_h^k, \overline{w}_h^k)$ , given as

$$\begin{split} \int_{\Omega_k} \overline{\sigma}_h^k \cdot \tau_h + \varepsilon \overline{w}_h^k \nabla \cdot \tau_h \, dx - \varepsilon \int_{\partial \Omega_k \cap \partial \Omega} g \tau_h^- \cdot n \, d\sigma &= 0 \qquad \forall \tau_h \in H_h \\ - \int_{\Omega_k} \nabla \cdot \overline{\sigma}_h^k \varphi_h \, dx &= \int_{\Omega_k} h \varphi_h \, dx \quad \forall \varphi_h \in V_h. \end{split}$$

Now we can make the abstract equation (9) more precise as

$$\widehat{N}_1(\lambda_h,\mu_h) := \int_{\Gamma_0} \mu_h \left( \sigma_h^k(\lambda_h)^- - \sigma_h^{k'}(\lambda_h)^+ \right) \cdot n \, d\sigma + \int_{\Gamma \setminus \Gamma_0} \mu_h \lambda_h \, d\sigma,$$

and

$$\widehat{b}_1(\mu_h) := \int_{\Gamma_0} \mu_h \left( (\overline{\sigma}_h^{k'})^+ - (\overline{\sigma}_h^k)^- \right) \cdot n \, d\sigma + \int_{\Gamma \setminus \Gamma_0} \mu_h g \, d\sigma,$$

where k and k' denote indices to those cells which intersect to form a certain edge e. Such a reduction of degrees of freedom is always possible due to the hybridization procedure.

#### 2.2. Nonlinear Convection Equation

Now we assume that both  $\varepsilon \equiv 0$  and  $h \equiv 0$ . In this case, it is not generally possible to impose Dirichlet boundary conditions on the whole domain, but one rather has to rely on upwinding also in the boundary conditions. For a scalar variable, this results in the (nonlinear) boundary conditions

$$w = g, \quad \forall x \in \partial \Omega^-, \tag{10}$$

where  $\partial \Omega^-$  is the inflow boundary defined as  $\partial \Omega^- := \{x \in \partial \Omega | f'(w) \cdot n < 0\}$ , where *n* is the outward-pointing normal on  $\partial \Omega$ . For the rest of the section, we assume that  $w_{\partial \Omega}(\cdot)$  is an abbreviation for the incorporation of these boundary conditions, i.e.,

$$w_{\partial\Omega}(w) = \begin{cases} g, & x \in \partial\Omega^-\\ w, & x \in \partial\Omega \setminus \partial\Omega^- \end{cases}$$
(11)

A standard DG discretization of the nonlinear convection equation can be written as the task of finding  $w_h \in V_h$ , such that for all  $\varphi_h \in V_h$ 

$$\sum_{k=1}^{N} \left( -\int_{\Omega_{k}} f(w_{h}) \cdot \nabla \varphi_{h} \, dx + \int_{\partial \Omega_{k} \setminus \partial \Omega} f^{num}(w_{h}^{+}, w_{h}^{-}, n) \varphi_{h}^{-} \, d\sigma + \int_{\partial \Omega_{k} \cap \partial \Omega} f(w_{\partial \Omega}(w_{h})) \cdot n \varphi_{h}^{-} \, d\sigma \right) = 0, \tag{12}$$

with  $f^{num}(\cdot, \cdot, \cdot)$  denoting a numerical flux. To present our method, we specifically choose the numerical flux  $f^{num}$  to be a slightly modified Lax-Friedrichs flux, given as

$$f^{num}(u,v,n) := f\left(\frac{u+v}{2}\right) \cdot n - \alpha \frac{u-v}{2}.$$
(13)

Here we assume that  $\alpha$  is constant, although it is also possible to make it a parameter depending on  $\frac{u+v}{2}$ . Introducing an auxiliary variable  $\lambda := \frac{u+v}{2}$ , the numerical flux can be written as

$$f^{num}(\lambda, v, n) := f(\lambda) \cdot n - \alpha (\lambda - v).$$
(14)

Obviously,  $\lambda$  has support on the boundaries of the elements only, and we can thus approximate it by some  $\lambda_h \in M_h$ . An equivalent way of rewriting the DG method as defined in (12) is then the numerical method, formulated as the task of finding  $(w_h, \lambda_h) \in V_h \times M_h$  such that  $\forall (\varphi_h, \mu_h) \in V_h \times M_h$ :

$$\sum_{k=1}^{N} \left( -\int_{\Omega_{k}} f(w_{h}) \cdot \nabla \varphi_{h} \, dx + \int_{\partial \Omega_{k} \setminus \partial \Omega} \varphi_{h}^{-} \left( f(\lambda_{h}) \cdot n - \alpha(\lambda_{h} - w_{h}^{-}) \right) d\sigma + \int_{\partial \Omega_{k} \cap \partial \Omega} \varphi_{h}^{-} f(w_{\partial \Omega}(\lambda_{h})) \cdot n d\sigma \right) = 0.$$
(15)

$$\int_{\Gamma_0} \alpha \mu_h (2\lambda_h - w_h^- - w_h^+) \, d\sigma + \int_{\Gamma \setminus \Gamma_0} \mu_h (\lambda_h - w_{\partial \Omega}(w_h^-)) \, d\sigma = 0.$$
(16)

The method defined above is a *Hybridized Discontinuous Galerkin Method*. At the boundary, because of the nonlinearity of both the flux and the boundary conditions, the resulting method is not fully equivalent to the standard DG method (12). We will, however, numerically demonstrate that the deviation is negligible. In the linear case, the solutions obtained via the DG approach and via the method defined above, are equal, thus justifying the nomenclature.

Again, the method in (16) can be formulated as

$$\widehat{N}_2(\lambda_h, \mu_h) = \widehat{b}_2(\mu_h) \quad \forall \mu_h \in M_h.$$
(17)

This is a consequence of the fact that  $w_h$  on cell  $\Omega_k$  is coupled to  $w_h$  on some neighboring cell  $\Omega_{k'}$  only via  $\lambda_h$ , thus allowing for static condensation via local solvers, in a manner substantially similar to that shown in section 2.1.

#### 2.3. Nonlinear Convection-Diffusion Equation

A discretization for the convection-diffusion equation (4) is obtained in a straightforward manner by adding the bilinear and semilinear forms defined in the previous sections, i.e. (15) and (16) for the convection problem, and (6) – (8) for the Poisson problem. We formulate the scheme as the task of finding  $(\sigma_h, w_h, \lambda_h) \in H_h \times V_h \times M_h$  such that for all  $(\tau_h, \varphi_h, \mu_h) \in H_h \times$  $V_h \times M_h$ :

$$\sum_{k=1}^{N} \left( \int_{\Omega_{k}} \sigma_{h} \cdot \tau_{h} + \varepsilon w_{h} \nabla \cdot \tau_{h} \, dx - \varepsilon \int_{\partial \Omega_{k} \setminus \partial \Omega} \lambda_{h} \tau_{h}^{-} \cdot n \, d\sigma - \varepsilon \int_{\partial \Omega_{k} \cap \partial \Omega} g \tau_{h}^{-} \cdot n \, d\sigma \right) = 0 \tag{18}$$

$$\sum_{k=1}^{N} \left( -\int_{\Omega_{k}} f(w_{h}) \cdot \nabla \varphi_{h} \, dx + \int_{\partial \Omega_{k} \setminus \partial \Omega} \varphi_{h}^{-} \left( f(\lambda_{h}) \cdot n - \alpha(\lambda_{h} - w_{h}^{-}) \right) \, d\sigma \tag{19}$$

$$+ \int_{\partial\Omega_k\cap\partial\Omega} \varphi_h^- f(g) \cdot n \, d\sigma - \int_{\Omega_k} \nabla \cdot \sigma_h \varphi_h \, dx \Big] = \int_{\Omega} h\varphi_h \, dx$$
$$\int_{\Gamma_0} \mu_h \Big( \sigma_h^- \cdot n - \sigma_h^+ \cdot n + \alpha (2\lambda_h - w_h^- - w_h^+) \Big) \, d\sigma + \int_{\Gamma \setminus \Gamma_0} \mu_h (\lambda_h - g) \, d\sigma = 0. \tag{20}$$

Again the method can be formulated as a method in  $\lambda_h$  and  $\mu_h$  only as

$$\widehat{N}(\lambda_h, \mu_h) = \widehat{b}(\mu_h) \quad \forall \mu_h \in M_h.$$
(21)

The discretization is consistent and both locally and globally conservative, thus being in good agreement with the divergence structure of the original PDE. Furthermore, the limiting cases of pure diffusion and pure convection are treated correctly in the sense that the resulting method reduces to well-known, well-working standard methods (Hybrid Mixed and pure DG, respectively).

Due to the generic structure of both the equation and the presentation of the method, the adaptation to the full Navier-Stokes equation is a rather easy task as will be demonstrated in Section 3.

## 2.4. Properties of the Discretization

In the following, let us explain some of the most prominent properties of the Hybrid Mixed discretization of the nonlinear convection-diffusion equation.

#### 2.4.1. Characterization of $\lambda$

 $\lambda_h$  is a Lagrange parameter designed to reduce the globally coupled degrees of freedom. Thus, the idea in the implementation is to express  $\sigma_h$  and  $w_h$  as functions of  $\lambda_h$ . Nevertheless, for a better insight into the method, it is also useful to have a characterization of  $\lambda_h$  in terms of the other two variables. Let us derive one here, under the assumption that there are no hanging nodes in the triangulation: Due to (20), we have that, on interior faces  $\Gamma_k \in \Gamma_0$ , the equality

$$\int_{\Gamma_k} \mu_h \left( 2\alpha \lambda_h - \alpha w_h^- - \alpha w_h^+ - \sigma_h^+ \cdot n + \sigma_h^- \cdot n \right) d\sigma = 0$$

has to hold. Given that  $\alpha$  is, on each edge, a constant with value other than zero, we can conclude that  $\lambda_h$  is given as

$$\lambda_{h} = \frac{w_{h}^{-} + w_{h}^{+}}{2} - \frac{[\sigma_{h}]}{2\alpha},$$
(22)

while on the boundary edges, we have  $\lambda_h = \prod_{M_h} g$ , with  $\prod_{M_h}$  being the  $L^2$ -projection onto  $M_h$ .  $[\sigma_h]$  denotes the jump in  $\sigma_h$ , i.e., the quantity  $\sigma_h^- \cdot n - \sigma_h^+ \cdot n$ . Eq. (22) is a standard characterization which can be found in one way or the other in all hybrid methods, showing that  $\lambda_h \approx w$  on the element boundaries.

## 2.4.2. Consistency and Conservativity

It is a straightforward operation to show that the method proposed here is consistent with the weak formulation of the nonlinear convection-diffusion equation by substituting  $w_h = w$ ,  $\sigma_h = \varepsilon \nabla w$  and  $\lambda_h = w$  in (18)-(20). We omit the details. The method proposed is both locally and globally conservative and thus in good agreement with the physical meaning of conservation laws. Similar to the approach taken in [13], local conservativity can be shown by testing the method with  $\varphi_h = \chi_{\Omega_k}$ , where  $\chi_{\Omega_k}$  is the characteristic function of mesh element  $\Omega_k$ . If  $\Omega_k$  does not intersect the boundary, this yields

$$\int_{\partial\Omega_k} \left( f(\lambda_h) \cdot n - \alpha(\lambda_h - w_h^-) - \sigma_h^- \cdot n \right) \, d\sigma = \int_{\Omega_k} h \, dx. \tag{23}$$

This is the definition of local conservativity. Global conservation can then be shown by using (23) to write the flux across the intersection of two cell boundaries, i.e. the flux across an interior edge  $\Gamma_k$ , as

$$\int_{\Gamma_k} -\alpha(2\lambda_h - w_h^- - w_h^+) - \sigma_h^- \cdot n + \sigma_h^+ \cdot n \, d\sigma.$$

Due to (20), this expression vanishes, which shows discrete conservation in our hybrid scheme.

#### 2.4.3. Postprocessing

As already mentioned in the introduction, a desirable side-effect of the explicit computation of  $\sigma_h$  (instead of, as in the DG case, elimination via lifting operations), is that it is possible to post process the numerical solution  $w_h$ , yielding an approximation  $w_h^*$  to w that converges with a better order of convergence than  $w_h$  does. We follow an approach proposed by Stenberg [20] and applied to the proposed scheme by Egger and Schöberl [13]. This post processing relies on the two facts that

- we explicitly compute  $\sigma_h$ , which converges with a better order of accuracy towards  $\varepsilon \nabla w$  than the quantity  $\varepsilon \nabla w_h$  does, and
- the quantity  $\int_{\Omega_{\ell}} w_h dx$  exhibits, for p > 0, superconvergence.

For more details and mathematical proofs, we refer to Refs. [13] and [20].

To formulate the post processing algorithm, let us introduce the space of mean-value free polynomials on a given cell  $\Omega_k$ ,

$$\Pi_0^{\overline{p}}(\Omega_k) := \{ f \in \Pi^{\overline{p}}(\Omega_k) | \int_{\Omega_k} f \, dx = 0 \}.$$

As both  $\varepsilon \nabla w_h$  and  $\sigma_h$  should approximate the quantity  $\sigma = \varepsilon \nabla w$ , a very natural algorithm is formulated as the *cell-wise* discretization of a Neumann problem

$$\varepsilon \int_{\Omega_{k}} \nabla w_{h}^{*} \cdot \nabla \varphi_{h} \, dx = \int_{\Omega_{k}} \sigma_{h} \cdot \nabla \varphi_{h} \, dx \quad \forall \varphi_{h} \in \Pi_{0}^{\overline{p}}(\Omega_{k})$$

$$\int_{\Omega_{k}} w_{h} \, dx = \int_{\Omega_{k}} w_{h}^{*} \, dx,$$
(24)

where  $w_h^* \in \Pi^{\overline{p}}(\Omega_k)$ . Note that Problem (24) is solved *cell-wise*, i.e., very cheaply.

We have not commented on the choice of  $\overline{p}$  yet. If  $\sigma_h$  converges with the *same* order towards  $\sigma$  as  $w_h$  does towards w, we choose  $\overline{p} = p + 1$ . (This is the case for the Hybrid Mixed scheme with numerical diffusion introduced by the Riemann flux, i.e.,  $\alpha \neq 0$ .) If  $\sigma_h$  converges towards  $\sigma$  even better than  $w_h$  does towards w, we choose  $\overline{p} = p + 2$ . (This is the case for the Hybrid Mixed scheme with no numerical diffusion, i.e.,  $\alpha = 0$ .)

It is again emphasized that the post processing routine is defined purely cell-wise, and can thus be performed easily. Numerical results will show that, except in the case p = 0,  $w_h^*$  will converge towards  $w_h$  with one order of accuracy better than  $\sigma_h$  converges towards  $\sigma$ . The results have been summarized in Table 1.

Let us mention that it has been proven in Ref. [13] for the linear case, that this post processing algorithm works for the Hybrid Mixed method given that either the mesh size h or the convective flux is small compared to the viscosity.

## 2.5. Numerical Results

In this subsection, we validate the scheme (18)-(20) by numerical convergence studies. In particular we consider the viscous two-dimensional Burgers' equation on the unit square  $\Omega = [0; 1]^2$  as a special case of (4):

$$\frac{1}{2}\nabla \cdot (w^2, w^2) - \Delta w = h \quad \forall x \in \Omega,$$
$$w = 0 \quad \forall x \in \partial \Omega$$

where  $h \equiv h(x_1, x_2)$  is such that the solution is given by

$$w(x_1, x_2) := \sin(2\pi x_1) \sin(2\pi x_2).$$

We have performed convergence studies for different orders of polynomials and mesh sizes. This example is highly diffusion-dominated, and it is possible to neglect the stabilization, which amounts to setting  $\alpha \equiv 0$  in the definition of the hybrid method. We have computed the problem once with  $\alpha = 0$ , and once with  $\alpha = 1$ . Results can be seen in Tables 2 and 3, respectively. As expected, the quantity  $||w - w_h||_{L^2}$  converges in both cases with the correct order p + 1. For the test with stabilization, i.e., with  $\alpha \neq 0$ , the quantity  $||\sigma - \sigma_h||_{L^2}$  converges suboptimally with order

p+1, while it converges with the correct order p+2 for  $\alpha = 0$ , i.e. no stabilization. Consequently, the postprocessed quantity converges with order p+2 for the stabilized computation, and with order p+3 for the non-stabilized one (with the expectation of p=0, where one only achieves p+2). These results are to be expected from Table 1.

N	р	$  w - w_h  _2$	order	$\ \sigma - \sigma_h\ _2$	order	$  w - w^*  _2$	order	р	$  w - w_h  _2$	order	$\ \sigma - \sigma_h\ _2$	order	$  w - w^*  _2$	order
2		4.83E-01		4.86E+00		4.81E-01			4.64E-01		3.27E+00		4.73E-01	
8		2.96E-01	0.71	2.16E+00	1.17	1.28E-01	1.90		2.82E-01	0.72	1.57E+00	1.06	1.71E-01	1.47
32	0	2.61E-01	0.18	1.31E+00	0.72	1.25E-01	0.04	1	7.41E-02	1.93	2.19E-01	2.84	1.41E-02	3.60
128		1.33E-01	0.97	3.78E-01	1.79	3.53E-02	1.82		1.95E-02	1.92	2.95E-02	2.89	1.03E-03	3.77
512		6.58E-02	1.01	9.82E-02	1.94	9.18E-03	1.95		4.95E-03	1.98	3.78E-03	2.97	6.74E-05	3.94
2		3.49E-01		2.45E+00		2.50E-01			2.71E-01		1.85E+00		2.10E-01	
8		6.70E-02	2.38	2.29E-01	3.42	1.31E-02	4.26		4.78E-02	2.50	1.74E-01	3.40	1.05E-02	4.32
32	2	1.63E-02	2.04	3.66E-02	2.64	1.33E-03	3.31	3	2.86E-03	4.06	5.14E-03	5.09	1.50E-04	6.13
128		2.16E-03	2.91	2.40E-03	3.93	4.12E-05	5.01		1.89E-04	3.92	1.70E-04	4.91	2.43E-06	5.95
512		2.75E-04	2.98	1.52E-04	3.99	1.27E-06	5.02		1.20E-05	3.98	5.42E-06	4.97	3.83E-08	5.99
2		1.30E-01		6.90E-01		5.91E-02			9.65E-02		5.11E-01		4.40E-02	
8		5.01E-03	4.70	1.30E-02	5.73	5.39E-04	6.78		3.63E-03	4.73	9.51E-03	5.75	4.05E-04	6.76
32	4	4.13E-04	3.60	6.21E-04	4.38	1.52E-05	5.15	5	5.05E-05	6.17	6.54E-05	7.18	1.39E-06	8.19
128		1.36E-05	4.93	1.02E-05	5.93	1.26E-07	6.92		8.26E-07	5.93	5.34E-07	6.94	5.69E-09	7.94
512		4.30E-07	4.98	1.62E-07	5.98	9.99E-10	6.98		1.31E-08	5.98	4.22E-09	6.98	2.25E-11	7.98

Table 2: Burgers equation, smooth testcase *without* stabilization: Convergence table for the approximation of the solution, its postprocessed version, and the gradient.

N	р	$  w - w_h  _2$	order	$\ \sigma - \sigma_h\ _2$	order	$  w - w^*  _2$	order	p	$  w - w_h  _2$	order	$\ \sigma - \sigma_h\ _2$	order	$  w - w^*  _2$	order
2		4.83E-001		4.85E+000		4.71E-001			4.47E-01		3.23E+00		4.92E-01	
8		3.03E-001	0.67	2.26E+000	1.10	2.67E-001	0.82		2.97E-01	0.59	1.77E+00	0.87	2.02E-01	1.29
32	0	2.75E-001	0.14	1.68E+000	0.43	1.80E-001	0.57	1	7.64E-02	1.96	3.39E-01	2.39	3.40E-02	2.57
128		1.46E-001	0.91	7.22E-001	1.22	7.60E-002	1.24		1.97E-02	1.96	6.92E-02	2.29	4.16E-03	3.03
512		7.25E-002	1.01	3.12E-001	1.21	3.26E-002	1.22		4.96E-03	1.99	1.50E-02	2.21	4.92E-04	3.08
2		3.57E-001		2.55E+000		3.58E-001			2.77E-01		1.93E+00		2.36E-01	
8		6.79E-002	2.40	2.79E-001	3.19	7.33E-002	2.29		4.84E-02	2.51	2.38E-01	3.02	1.51E-02	3.97
32	2	1.64E-002	2.05	6.51E-002	2.10	4.95E-003	3.89	3	2.87E-03	4.08	1.03E-02	4.53	6.94E-04	4.44
128		2.17E-003	2.92	6.77E-003	3.27	3.12E-004	3.99		1.89E-04	3.92	5.67E-04	4.18	2.20E-05	4.98
512		2.75E-004	2.98	7.46E-004	3.18	1.93E-005	4.01		1.20E-05	3.98	3.26E-05	4.12	6.91E-07	4.99
2		1.33E-001		7.75E-001		1.48E-001			9.81E-02		5.91E-01		5.54E-02	
8		5.03E-003	4.72	1.88E-002	5.37	5.77E-003	4.68		3.65E-03	4.75	1.51E-02	5.29	6.54E-04	6.41
32	4	4.14E-004	3.60	1.40E-003	3.75	8.55E-005	6.08	5	5.05E-05	6.17	1.62E-04	6.55	9.07E-06	6.17
128		1.36E-005	4.93	3.91E-005	5.16	1.38E-006	5.96		8.27E-07	5.93	2.30E-06	6.14	7.26E-08	6.96
512		4.30E-007	4.98	1.13E-006	5.11	2.16E-008	6.00		1.31E-08	5.98	3.37E-08	6.09	5.68E-10	7.00

Table 3: Burgers equation, smooth testcase *with* stabilization: Convergence table for the approximation of the solution, its postprocessed version, and the gradient.

A more challenging test case that includes a boundary layer was proposed in [13]. The solution is assumed to be given as

$$w(x_1, x_2) = \left(x_1 + \frac{e^{x_1/\varepsilon} - 1}{1 - e^{1/\varepsilon}}\right) \cdot \left(x_2 + \frac{e^{x_2/\varepsilon} - 1}{1 - e^{1/\varepsilon}}\right).$$

In contrast to [13], we chose to make the problem nonlinear by considering again Burgers' flux  $f(w) = 0.5(w^2, w^2)^T$ . The right hand side  $h \equiv h(x, \varepsilon)$  was chosen in such a way that w solves (4), for an arbitrary choice of  $\varepsilon$ , setting  $g \equiv 0$ . The smaller  $\varepsilon$  is, the more distinct is the boundary layer. A contour plot corresponding to  $\varepsilon = 0.1$  can be seen in Fig. 1. Also in this case, we obtained spectral convergence (see Tables 4 - 5). Naturally, for small  $\varepsilon$ , we need more degrees of freedom to accurately resolve the boundary layer (or, which would be a better thing to do, but is not part of this work, use a Shishkin mesh [23], a mesh specifically designed to better resolve the boundary layer). One can observe that the optimal order is attained.



Figure 1: Boundary layer problem corresponding to  $\varepsilon$  = 0.1 - Contour Plot.

N	р	$  w - w_h  _2$	order	$\ \sigma - \sigma_h\ _2$	order	$  w - w^*  _2$	order	p	$  w - w_h  _2$	order	$\ \sigma - \sigma_h\ _2$	order	$  w - w^*  _2$	order
2		4.55E-003		1.60E-002		4.04E-003			4.18E-003		1.04E-002		2.09E-003	
8	0	4.08E-003	0.16	1.19E-002	0.43	2.46E-003	0.71	1	1.30E-003	1.68	3.16E-003	1.72	4.98E-004	2.07
32	0	2.42E-003	0.75	7.00E-003	0.77	1.37E-003	0.84	1	3.45E-004	1.92	8.48E-004	1.90	7.36E-005	2.76
128		1.28E-003	0.92	3.73E-003	0.91	7.29E-004	0.91		8.72E-005	1.98	2.18E-004	1.96	9.63E-006	2.93
2		1.04E-003		2.10E-003		8.21E-004			5.06E-004		9.07E-004		1.51E-004	
8	2	2.33E-004	2.16	4.61E-004	2.19	6.71E-005	3.61	2	3.80E-005	3.73	7.77E-005	3.54	5.73E-006	4.71
32	2	3.24E-005	2.85	6.73E-005	2.77	4.39E-006	3.93	3	2.48E-006	3.94	5.46E-006	3.83	1.89E-007	4.92
128		4.15E-006	2.96	8.90E-006	2.92	2.76E-007	3.99		1.57E-007	3.98	3.59E-007	3.93	6.01E-009	4.98
2		7.75E-005		1.48E-004		1.66E-005			7.49E-006		1.49E-005		1.31E-006	
8	4	2.74E-006	4.82	5.81E-006	4.67	3.13E-007	5.73	6	1.30E-007	5.84	2.82E-007	5.72	1.21E-008	6.76
32	4	8.84E-008	4.95	1.98E-007	4.88	5.21E-009	5.91	1 3	2.10E-009	5.96	4.73E-009	5.89	1.01E-010	6.92
128		2.78E-009	4.99	6.41E-009	4.95	8.34E-011	5.96		3.30E-011	5.99	7.63E-011	5.96	8.03E-013	6.97

Table 4: Boundary layer problem for  $\varepsilon = 1$ : Convergence table for the approximation of the solution, its postprocessed version, and the gradient.

N		II	andan		andan		and an		II	and an		and an	II	
11	p	$  w - w_h  _2$	order	$\ v - v_h\ _2$	order	$  w - w  _2$	order	p	$  w - w_h  _2$	order	$\ v - v_h\ _2$	order	$  w - w  _2$	order
2		1.24E-001		9.69E-002		1.12E-001			8.87E-002		5.79E-002		9.75E-002	
8		9.83E-002	0.34	6.58E-002	0.56	8.18E-002	0.46		6.14E-002	0.53	4.37E-002	0.40	4.69E-002	1.06
32	0	9.28E-002	0.08	5.73E-002	0.20	7.84E-002	0.06	1	2.46E-002	1.32	2.61E-002	0.74	1.57E-002	1.58
128		7.45E-002	0.32	5.23E-002	0.13	6.75E-002	0.22		7.17E-003	1.78	1.09E-002	1.26	3.51E-003	2.16
512		5.06E-002	0.56	3.96E-002	0.40	4.75E-002	0.51		1.82E-003	1.98	3.54E-003	1.63	5.80E-004	2.60
2048		3.03E-002	0.74	2.54E-002	0.64	2.88E-002	0.72		4.50E-004	2.01	1.01E-003	1.81	8.44E-005	2.78
2		7.00E-002		4.35E-002		5.62E-002			3.07E-002		2.29E-002		2.48E-002	
8		2.09E-002	1.74	1.96E-002	1.15	1.39E-002	2.01		6.09E-003	2.33	7.03E-003	1.70	4.41E-003	2.49
32	2	4.87E-003	2.10	6.34E-003	1.63	2.57E-003	2.44	3	1.12E-003	2.44	1.59E-003	2.14	5.47E-004	3.01
128		8.43E-004	2.53	1.39E-003	2.19	2.80E-004	3.20		1.13E-004	3.31	1.97E-004	3.02	3.19E-005	4.10
512		1.17E-004	2.85	2.24E-004	2.64	2.12E-005	3.72		8.11E-006	3.80	1.62E-005	3.60	1.24E-006	4.69
2048		1.50E-005	2.96	3.11E-005	2.85	1.41E-006	3.91		5.26E-007	3.95	1.14E-006	3.84	4.15E-008	4.90
2		1.11E-002		1.13E-002		1.02E-002			4.88E-003		6.20E-003		5.26E-003	
8		2.34E-003	2.25	2.97E-003	1.93	1.72E-003	2.56		9.36E-004	2.38	1.24E-003	2.32	6.12E-004	3.10
32	4	2.70E-004	3.11	4.15E-004	2.84	1.16E-004	3.89	5	5.56E-005	4.07	9.05E-005	3.78	2.08E-005	4.88
128		1.40E-005	4.27	2.59E-005	4.00	3.41E-006	5.09		1.44E-006	5.27	2.77E-006	5.03	3.01E-007	6.11
512		5.06E-007	4.79	1.05E-006	4.62	6.68E-008	5.67		2.61E-008	5.78	5.55E-008	5.64	2.93E-009	6.68
2048		1.64E-008	4.95	3.64E-008	4.85	1.13E-009	5.89		4.25E-010	5.94	9.52E-010	5.87	3.76E-011	6.28

Table 5: Boundary layer problem for  $\varepsilon = 0.1$ : Convergence table for the approximation of the solution, its postprocessed version, and the gradient.

## 3. Method for the Navier-Stokes Equations

In this section, we extend the method previously formulated for a scalar problem to the full Navier-Stokes equations. These equations can be written as

$$\nabla \cdot (f(w) - f_v(w, \nabla w)) = 0 \quad \forall x \in \Omega,$$
(25)

where w is the vector of conserved variables  $w = (\rho, \rho u, \rho v, E)^T$  i.e., density, momentum and total energy. The convective flux  $f = (f_1, f_2)$  is defined as

$$f_{1} = (\rho u, p + \rho u^{2}, \rho uv, u(E + p))^{T}$$

$$f_{2} = (\rho v, \rho uv, p + \rho v^{2}, v(E + p))^{T}.$$
(26)

Here p is the pressure, related to the other variables by an equation of state. In the present paper we formulate the Navier-Stokes equations for external aerodynamics applications, and hence use the ideal gas law,

$$p = (\gamma - 1) \left( E - \frac{1}{2} \rho (u^2 + v^2) \right), \tag{27}$$

where  $\gamma = 1.4$  is the ratio of specific heats for air.

The diffusive flux  $f_v = (f_{v,1}, f_{v,2})$  is defined as

$$f_{\nu,1} = (0, \tau_{11}, \tau_{21}, \tau_{11}u + \tau_{12}v + kT_{x_1})^T$$

$$f_{\nu,2} = (0, \tau_{12}, \tau_{22}, \tau_{21}u + \tau_{22}v + kT_{x_2})^T.$$
(28)

Here  $\tau$  denotes the stress tensor, T is the temperature, and k is the thermal conductivity coefficient. For Newtonian fluids, the viscous flux  $f_v$  can be written as  $f_v(w, \nabla w) = B(w)\nabla w$  with a non-linear function B only depending on w and not on  $\nabla w$ . We consider a domain  $\Omega$  that is usually a domain around an airfoil, i.e., an external domain. At the airfoil boundary we impose the no-slip condition (u, v) = 0, and the adiabatic wall condition  $n \cdot \nabla T = 0$ .

The extension of the previously defined method to the full Navier-Stokes equations is - considering the underlying paradigms - a straightforward operation. We thus formulate the method first as the task of finding the triple  $(\sigma_h, w_h, \lambda_h) \in H_h \times V_h \times M_h$  such that we have for all  $(\tau_h, \varphi_h, \mu_h) \in H_h \times V_h \times M_h$ :

$$\sum_{k=1}^{N} \left( \int_{\Omega_{k}} \sigma_{h} \cdot \tau_{h} \, dx - \int_{\Omega_{k}} \nabla w_{h} \cdot (B(w_{h})^{T} \tau_{h}) \, dx - \int_{\partial \Omega_{k} \setminus \partial \Omega} (\lambda_{h} - w_{h}) \cdot (B(w_{h}^{-})^{T} \tau_{h}^{-} n) \, d\sigma - \int_{\partial \Omega_{k} \cap \partial \Omega} (w_{\partial \Omega}(\lambda_{h}) - w_{h}) \cdot (B(w_{h}^{-})^{T} \tau_{h}^{-} n) \, d\sigma \right) = 0$$
(29)

$$\sum_{k=1}^{N} \left( -\int_{\Omega_{k}} f(w_{h}) \nabla \varphi_{h} \, dx - \int_{\Omega_{k}} \nabla \cdot \sigma_{h} \varphi_{h} \, dx + \int_{\partial \Omega_{k} \setminus \partial \Omega} \varphi_{h}^{-} \left( f(\lambda_{h}) \cdot n - \alpha(\lambda_{h} - w_{h}^{-}) \right) \, d\sigma + \int_{\partial \Omega_{k} \cap \partial \Omega} \varphi_{h}^{-} \left( f(w_{\partial \Omega}(\lambda_{h})) \cdot n - \alpha(\lambda_{h} - w_{\partial \Omega}(w_{h}^{-})) \right) \, d\sigma \right) = 0 \tag{30}$$

$$\int_{\Gamma_0} \mu_h \alpha (2\lambda_h - w_h - w_h) + \mu_h (\sigma_h \cdot n - \sigma_h \cdot n) \, d\sigma$$
$$+ \int_{\Gamma \setminus \Gamma_0} \mu_h \alpha (\lambda_h - w_{\partial\Omega}(w_h)) + \mu_h (\sigma_h^- \cdot n - \sigma_{\partial\Omega} \cdot n) \, d\sigma = 0.$$
(31)

Here  $\alpha$  denotes again the Lax Friedrichs constant. In the above definition, the adiabatic boundary conditions are implemented by setting  $\sigma_{\partial\Omega} \equiv \sigma_{\partial\Omega}(\sigma)$  at the wall, where

$$\sigma_{\partial\Omega}(\sigma) \cdot n := (0, \sigma_2 \cdot n, \sigma_3 \cdot n, 0), \tag{32}$$

while  $w_{\partial\Omega}$ , defined as

$$w_{\partial\Omega}(\rho,\rho u,\rho v,E) := (\rho,0,0,E), \tag{33}$$

imposes the no-slip boundary conditions. For some test cases, we need slightly adapted boundary conditions. In that case, one has to adapt both  $w_{\partial\Omega}$  and  $\sigma_{\partial\Omega}$  accordingly. For Dirichlet boundary conditions for example, one takes  $\sigma_{\partial\Omega}(\sigma) = \sigma$  and  $w_{\partial\Omega}(w) = g$ , if g is the given Dirichlet boundary data. Slip boundary conditions for the Euler equations can be implemented similarly. Considering the implementation of the boundary conditions in the Navier-Stokes setting, it is obvious that defining  $\lambda_h$  also on the physical boundary is advantageous.

Let us mention that also in the scenario of the Navier-Stokes equations, the two limiting cases of pure diffusion ( $f(w) \equiv 0$ ) and pure convection ( $B(w) \equiv 0$ ) are treated correctly in the sense that for pure diffusion, the method reduces to a standard H(div)-method, yielding a diffusive flux  $\sigma_h \in H(\text{div})$ , while for pure convection, the method reduces to a standard discontinuous Galerkin method.

Thus, as already mentioned in [13], the method is both in the diffusive and in the convective case a well-established method, and in-between a method without any parameters to tune.

### 3.1. Implementation

While the method as proposed in (29)-(31) appears to have even more degrees of freedom than a non-hybridized method, the degrees of freedom can be decoupled, as we have already pointed out in the context of the convection-diffusion equation. This is precisely the idea behind using hybrid methods. Let us again consider the idea introduced by Cockburn and Gopalakrishnan [16] and define local solvers  $(w_h^k(\lambda_h), \sigma_h^k(\lambda_h))$  fulfilling *for a given*  $\lambda_h$  (where for ease of notation, we omitted the argument  $(\lambda_h)$ )

$$\begin{split} \int_{\Omega_{k}} \sigma_{h}^{k} \cdot \tau_{h} \, dx - \int_{\Omega_{k}} \nabla w_{h}^{k} \cdot (B(w_{h}^{k})^{T} \tau_{h}) \, dx - \int_{\partial\Omega_{k} \setminus \partial\Omega} \left(\lambda_{h} - w_{h}^{k}\right) \cdot (B(w_{h}^{k-})^{T} \tau_{h}^{-} n) \, d\sigma \\ &+ \int_{\partial\Omega_{k} \cap \partial\Omega} \left(w_{\partial\Omega}(\lambda_{h}) - w_{h}^{k}\right) \cdot (B(w_{h}^{k-})^{T} \tau_{h}^{-} n) \, d\sigma = 0 \quad (34) \\ &- \int_{\Omega_{k}} f(w_{h}^{k}) \nabla \varphi_{h} \, dx - \int_{\Omega_{k}} \nabla \cdot \sigma_{h}^{k} \varphi_{h} \, dx + \int_{\partial\Omega_{k} \setminus \partial\Omega} \varphi_{h}^{-} \left(f(\lambda_{h}) \cdot n - \alpha(\lambda_{h} - w_{h}^{k-})\right) \, d\sigma \\ &+ \int_{\partial\Omega_{k} \cap \partial\Omega} \varphi_{h}^{-} \left(f(w_{\partial\Omega}(\lambda_{h})) \cdot n - \alpha(\lambda_{h} - w_{\partial\Omega}(w_{h}^{k-}))\right) \, d\sigma = 0 \quad (35) \end{split}$$

for all  $(\tau_h, \varphi_h) \in H_h \times V_h$ .

Now defining  $w_h(\lambda_h) := \sum_{k=1}^N w_h^k(\lambda_h)$  and similarly for  $\sigma_h(\lambda_h)$ , the original method can be re-formulated as the task of finding  $\lambda_h \in M_h$  such that  $\forall \mu_h \in M_h$ 

$$\widehat{N}(\lambda_{h},\mu_{h}) := \int_{\Gamma_{0}} \mu_{h} \left( \alpha (2\lambda_{h} - w_{h}(\lambda_{h})^{-} - w_{h}(\lambda_{h})^{+}) + \sigma_{h}(\lambda_{h})^{-} \cdot n - \sigma_{h}(\lambda_{h})^{+} \cdot n \right) d\sigma$$

$$+ \int_{\Gamma \setminus \Gamma_{0}} \mu_{h} \alpha (\lambda_{h} - w_{\partial \Omega}(w_{h}(\lambda_{h}))) + \mu_{h}(\sigma_{h}(\lambda_{h})^{-} \cdot n - \sigma_{\partial \Omega}(\sigma_{h}(\lambda_{h})) \cdot n) d\sigma = 0.$$
(36)

Thus, we obtain a nonlinear system whose dimension is  $\dim(M_h)$  and not  $\dim(M_h)+\dim(V_h)+\dim(V_h)+\dim(H_h)$  any more. Naturally, this is a significant reduction of memory and workload for iterative solvers used as part of a Newton procedure. However, the assembly step for a Jacobian matrix of the method is much more involved compared to a standard DG procedure, as one has to solve many ('small', because local) linear systems of equations during the assembly.

## 3.2. Numerical Results for the Navier-Stokes Equations

In this section, we present numerical results to validate our HM methods for both inviscid and viscous flow, governed by the Euler equations, and the Navier-Stokes, respectively.

#### 3.2.1. Smooth Inviscid Flow

We start by considering inviscid, subsonic flow around a NACA0012 airfoil. We set the freestream conditions to Mach number M = 0.4 and angle of attack  $\alpha = 5^{\circ}$ . In Fig. 2 both the Mach number distribution and the underlying grid are plotted. The grid consists of 2,560 elements and 3,920 faces, including 160 faces at the boundary. Discretizing  $w_h$  using third order polynomials, this results in 102,400 degrees of freedom for the unknown  $w_h$ , and 78,400 degrees of freedom for the unknown  $\lambda_h$ . The latter are of course the only coupled unknowns. Note that for purely inviscid flow, one does not need to explicitly compute  $\sigma_h$ , as it is identically 0. In Fig. 2(c), we have plotted the logarithm (to the base 10) of the mean-value of the entropy error along the airfoil, thereby yielding quantitative information on the behavior of the algorithm. More precisely, we plotted the mean-value of the quantity

$$\log_{10}\left|\frac{p}{\rho^{\gamma}}-1\right|.$$

For exact solutions, the entropy error should of course be zero. Compared to the resolution of the mesh, the values achieved are quite satisfactory.

In Fig. 2(d), we have plotted the pressure along the airfoil for both a DG method, and our Hybrid method. These solutions are nearly identical.

## 3.2.2. Nonsmooth Inviscid Flow

This test case was chosen to demonstrate that it is also possible to incorporate the approximation of discontinuities. We have used shock-capturing based on artificial viscosity. To this end, one adds to equation (30) the expression  $\int_{\Omega_k} \varepsilon(w_h) \nabla w_h \nabla \varphi_h dx$ . Following Hartmann [2], we choose  $\varepsilon(w)$  to depend on the (continuous) residual. More precisely, we choose

$$\varepsilon(w)_{|\Omega_k|} = \frac{Ch_k^{2-\beta}}{|\Omega_k|} \int_{\Omega_k} \|\nabla \cdot f(w)\|_{l^1} dx$$
(37)

for some parameters  $\beta$  and *C*. In our experiments, we chose  $\beta = 0$  and C = 0.05. In the definition of  $\varepsilon(w)$ ,  $h_k$  denotes the diameter of cell  $\Omega_k$ .

Note that this choice is reasonable, as  $\varepsilon(w)$  vanishes for smooth solutions up to the level of consistency with the original PDE, while for non-smooth solutions, it is *active*. Furthermore, it only depends on w in the interior of a cell  $\Omega_k$  and not on a jump term on  $\partial \Omega_k$ . This, in combination with the fact that we only use the term  $\int_{\Omega_k} \varepsilon(w_h) \nabla w_h \nabla \varphi_h$  as a shock-capturing term, makes this choice of  $\varepsilon$  especially attractive in our method, as the shock capturing terms only affect the definition of the local solver.

For demonstration, we have chosen a standard test case characterized by a free-stream Mach number of M = 0.8 and an angle of attack  $\alpha = 1.25^{\circ}$ . Results are shown in Fig. 3, where we have plotted both the Mach number contours and the logarithm of the mean-value of the entropy error. Again we have used third order polynomials to represent  $w_h$ , while the underlying mesh is the same as in the previous test case. Thus the number of degrees of freedom remains unchanged.

#### 3.2.3. Viscous Laminar Flow Around a Cylinder

To validate the viscous terms, we have computed laminar flow around a cylinder. This is a well-known test case for which both experimental and computational results are available in the



Figure 2: An inviscid NACA0012 test case. Free-stream conditions are M = 0.4 and  $\alpha = 5^{\circ}$ .



Figure 3: An inviscid NACA0012 test case. Free-stream conditions are M = 0.8 and  $\alpha = 1.25^{\circ}$ .

literature [24]. The viscous drag can be defined as

$$J \equiv c_D = c_{Dp} + c_{Df} := \int_{\Gamma_w} p\beta \cdot n - \tau(w, \nabla w)\beta \cdot n \, d\sigma, \tag{38}$$

where  $c_{Dp}$  is the pressure contribution of the drag and  $c_{Df}$  is the viscous contribution of the drag. The vector  $\beta$  is defined as

$$\beta = \frac{1}{C_{\infty}} (\cos(\alpha), \sin(\alpha))^T.$$

As usual,  $\alpha$  denotes the angle of attack while  $C_{\infty}$  is a normalized reference value defined as

$$C_{\infty} = \frac{1}{2} \left( \gamma p_{\infty} M_{\infty}^2 l \right).$$

*l* is the chord length of the airfoil, while  $p_{\infty}$  and  $M_{\infty}$  are the values of pressure and Mach number at free-stream conditions, i.e., values in the far field.

Henderson [24] has given a functional dependency for both  $c_{Dp}$  and  $c_{Df}$  of the form  $c(Re) = \frac{a_0}{Re^{a_1}}$  with parameters  $a_i$  he determined empirically. In Fig. 4(d), we have plotted the results computed with our hybrid solver and compared them to the functional dependency of Henderson. Very good agreement of the drag coefficient can be observed. The results have been computed using third order polynomials for  $w_h$  on a rather coarse grid that can be seen in Fig. 4(c). The grid consists of 1,596 elements, with 2,460 faces, resulting in 63,840 degrees of freedom for  $w_h$ , 191,520 degrees of freedom for  $\sigma_h$ , and 49,200 degrees of freedom for  $\lambda_h$ . Here again, the only globally coupled degrees are those corresponding to  $\lambda_h$ .

#### 3.2.4. Compressible Couette Flow

A test case based on compressible Couette flow has been proposed in [3]. The computational domain is given by  $\Omega = [0; 1]^2$ , while the solution is given by smooth flow, with flow field variables defined as

$$w = (\rho, \rho u, \rho v, E) = (1/T, \frac{y \log(1+y)}{T}, 0, \frac{p}{\gamma - 1} + \frac{u^2 + v^2}{2T}),$$
(39)

where *p* is a constant and *T* is defined as (see [3])

$$T = T_0 + y(T_1 - T_0) + \frac{M^2(\gamma - 1)}{2} Pr y(1 - y).$$
(40)

 $T_0$ ,  $T_1$  and M are constants, which we choose, following [3], to be 0.8, 0.85 and 0.15, respectively. A source term is incorporated in such a way that w is indeed a solution to the Navier-Stokes equations. In Table (6), we computed the  $L^2$ - norms of both the error in  $w_h$  approximating w and in  $\sigma_h$  approximating  $\sigma := f_v(w, \nabla w)$ . As in [3], we observe that all the quantities converge with order p + 1, even for the case p = 0.

#### 3.2.5. Viscous Flow Around the NACA0012 profile

In this section, we present numerical results for viscous flow on standard NACA0012 airfoils. Some of the test cases presented in this subsection have been used previously, for example in [25] and [1].



(a) Mach Number distribution and Streamline Plot: Re = 10.





(b) Mach Number distribution and Streamline Plot: Re = 35.



(d) Total, pressure-induced and viscosity-induced drag in comparison with results from the literature [24].

Figure 4: Laminar flow around a cylinder.

N	р	$  w - w_h  _{L^2}$	order	$\ \sigma - \sigma_h\ _{L^2}$	order	p	$  w - w_h  _{L^2}$	order	$\ \sigma - \sigma_h\ _{L^2}$	order
2		3.46E-01		2.16E+00			2.54E-01		2.83E-01	
8		7.51E-01	-1.12	1.94E+00	0.15		4.97E-02	2.35	9.68E-02	1.55
32	0	3.88E-01	0.95	1.24E+00	0.65	1	1.29E-02	1.94	2.97E-02	1.70
128		1.94E-01	1.00	7.13E-01	0.80		3.01E-03	2.10	8.45E-03	1.82
512		9.01E-02	1.11	3.88E-01	0.88		6.30E-04	2.26	2.26E-03	1.90
2		1.94E-02		3.91E-02			1.65E-03		2.72E-03	
8		3.39E-03	2.52	7.12E-03	2.46		1.75E-04	3.24	3.45E-04	2.98
32	2	4.52E-04	2.91	1.08E-03	2.72	3	1.26E-05	3.80	2.98E-05	3.53
128		5.34E-05	3.08	1.50E-04	2.85		8.37E-07	3.91	2.14E-06	3.80
512		5.95E-06	3.17	1.98E-05	2.92		5.36E-08	3.96	1.42E-07	3.91
2		2.33E-04		4.37E-04			3.07E-05		6.81E-05	
8		1.32E-05	4.13	3.18E-05	3.78		1.03E-06	4.89	2.73E-06	4.64
32	4	5.02E-07	4.72	1.44E-06	4.47	5	2.22E-08	5.54	6.53E-08	5.38
128		1.63E-08	4.94	5.27E-08	4.77		3.96E-10	5.81	1.21E-09	5.75
512		5.04E-10	5.02	1.76E-09	4.90		5.79E-12	6.10	2.05E-11	5.88

Table 6: Solution and viscous-flux convergence of the compressible Couette flow.

We begin by considering flow characterized by a Reynolds number Re = 73, a free-stream Mach number M = 0.8 and an angle of attack  $\alpha = 10^{\circ}$ . The results have been computed on a rather coarse mesh consisting of 668 elements and 1,022 faces, which can be seen in Fig. 6. Since w is discretized with polynomials of order p = 3, this results in 26,720 unknowns for  $w_h$ , 80,160 unknowns for  $\sigma_h$  and 20,440 unknowns for  $\lambda_h$ . Again, the coupled degrees of freedom are those corresponding to  $\lambda_h$ . These are less than those corresponding to  $w_h$ . In Fig. 5, we have plotted both a contour plot of the pressure distribution over the complete domain, and the pressure distribution over the airfoil. The obtained results compare very well to the literature [1]. We have furthermore plotted the mean skin friction coefficient of this testcase, which we computed as

$$C_f = \frac{2\mu}{\rho U_{\infty}^2} \left( t_1 \nabla u \cdot n + t_2 \nabla v \cdot n \right).$$

 $t = (t_1, t_2)$  denotes the vector tangential to the surface of the airfoil, while  $\mu$  denotes the viscosity. Results have been plotted in Fig. 5(c). The bad behavior of both the pressure and the skin friction coefficient at the trailing edge is due to the poor resolution of the mesh.

A similar test case for the same geometry is defined by Reynolds number Re = 500, Mach number M = 0.8, and an angle of attack  $\alpha = 10^{\circ}$ . The underlying polynomial order was p = 3for  $w_h$  (and consequently, p = 4 for both the discretization of  $\sigma_h$  and  $\lambda_h$ ). The grid can be seen in Fig. 2(a). It is the same grid that has been used to compute the inviscid test case in sections 3.2.1 and 3.2.2. This then yields 102,400 degrees of freedom for  $w_h$ , 307,200 degrees of freedom for  $\sigma_h$  and 78,400 degrees of freedom for  $\lambda_h$ . In Fig. 7, we have plotted both the Mach number distribution and the pressure distribution around the airfoil. The Mach number distribution was computed using the primal variables  $w_h$ , while the pressure distribution along the airfoil was computed using the hybrid variables  $\lambda_h$ . The close up view of the Mach number distribution reveals the smoothness of the underlying approximate solution, which indicates adequate resolution. As we did in the previous testcase, we also computed the skin friction coefficient, see Fig. 7(d).

System (36) is driven to steady-state using a damped Newton method, using diagonal damping terms that renders the relaxation method equivalent to a linearized backward Euler method using local (pseudo-) time step. The damping parameter is adjusted via the CFL number, so that



Figure 5: A viscous NACA0012 test case. Free-stream conditions are M = 0.8,  $\alpha = 10^{\circ}$ , Re = 73.



Figure 6: Coarse Grid. Number of elements: 668, number of faces: 1,022.



Figure 7: A viscous NACA0012 test case. Free-stream conditions are M = 0.8,  $\alpha = 10^{\circ}$ , Re = 500.

as  $CFL \rightarrow \infty$ , a Newton's iteration is recovered. In Fig. 8(a), we have documented the convergence towards steady state including the CFL number for a direct initialization based on freestream values. It is also possible to use an increasing-order initialization approach, exploiting the inherent hierarchy in the polynomial approximation. In this approach we start approximating with p = 0, then use this as initialization for a p = 1 computation, and so on. Convergence toward steady state for this case is shown in Fig. 8(b).

As a final example, we consider again flow around a NACA0012 airfoil, but this time at a higher Reynolds number of Re = 1500. Free stream mach number and angle of attack are set as M = 0.8 and  $\alpha = 1.25^{\circ}$ , respectively. This test case exhibits a boundary layer, which can, for low order schemes, only be resolved by adding enough stretched cells to the boundary. To demonstrate the effectiveness of our proposed method, we took the same mesh as in the above test cases, which means that we did not use stretched cells along the boundary layer. The results are shown in Fig. 9. Again we observe adequate resolution, which demonstrates the inherent good approximation properties of a high order scheme.



Figure 8: Convergence history belonging to Re = 500 test case.



(a) Mach number distribution

(b) Mach number distribution - close up view



(c) Skin Friction coefficient

Figure 9: A viscous NACA0012 test case. Free-stream conditions are M = 0.8,  $\alpha = 1.25^{\circ}$ , Re = 1500.

## 4. Conclusions and Future Work

We have presented a method for the Navier-Stokes equations that combines well-known discretization methods for the convective (DG) and the diffusive (Mixed) part via *Hybridization*. The method performs well on both scalar model problems and the full Navier-Stokes equations. We have demonstrated convergence properties for the approximated function  $w_h$ , the viscous flux  $\sigma_h$  and (in the scalar case) the postprocessed quantity  $w_h^*$  numerically. Extending the method to other kinds of convection-diffusion equations is straightforward.

It remains to extend the strategy of postprocessing to the Navier-Stokes equations. This may make it necessary to re-think the formulation and choose  $\sigma := \nabla w$  rather than  $\sigma := f_v(w, \nabla w)$ . This is subject to ongoing work. Furthermore, it seems as if the Hybrid Method is more diffusive than a standard DG method. Also this aspect will be treated in more detail in an upcoming publication.

As the method is based on Galerkin's principle, it is straightforward to incorporate adaptivity via an adjoint based adaptation criterion. In an upcoming paper [26], we will show that the method proposed is *adjoint consistent* [27], and thus the discrete adjoint approach can be used.

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