A Hybridized DG/Mixed Scheme for Nonlinear Advection-Diffusion Systems, Including the Compressible Navier-Stokes Equations

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We present a novel discretization method for nonlinear convection-diffusion equations and, in particular, for the compressible Navier-Stokes equations. The method is based on a Discontinuous Galerkin (DG) discretization for convection terms, and a mixed method using $H(div)$ spaces for the diffusive terms. Furthermore, hybridization is used to reduce the number of globally coupled degrees of freedom. The method reduces to a DG scheme for pure convection, and to a mixed method for pure diffusion, while for the intermediate case the combined variational formulation requires no additional parameters. We formulate and validate our scheme for nonlinear model problems, as well as compressible flow problems. Furthermore, we compare our scheme to a recently developed Hybridized DG scheme with respect to formulation and convergence behavior.

I. Introduction

The Discontinuous Galerkin (DG) method has been originally formulated as a discretization method for hyperbolic equations,[1] for which it has become more popular than conforming (Petrov-) Galerkin methods. This has arguably been the main motivation behind a renewed interest in discontinuous discretization for elliptic operators,[2] allowing the extension of DG methods to mixed convection-diffusion problems.

However, while discontinuous methods tie in neatly with the wave propagation mechanics of hyperbolic problems via the solution of approximate Riemann problems,[3] there is no similarly intuitive rationale behind such discontinuous discretization for elliptic problems. In fact, for purely elliptic problems, discontinuous discretization has hardly become the method of consensus. The increased number of degrees of freedom associated with DG discretization is certainly harder to justify for such problems. Thus devising discretization methods for advection-diffusion equations is complicated by the fact that the individual subproblems, advection and diffusion, are usually discretized using different methods when they appear by themselves. At least for compressible flow simulation, using discontinuous DG discretization for both advection and diffusion seems most popular, perhaps due to the fact that in compressible flow simulation the nonlinear convective part has traditionally dominated the development of numerical methods.[4]

Very often DG schemes for advection-diffusion are formulated and analyzed for a first order system, formally equivalent to the original PDE. For example, the equation $\nabla \cdot f(w) - \Delta w = 0$ is written as

$$\begin{aligned} \nabla \cdot (f(w) - \sigma) &= 0 \\ \sigma - \nabla w &= 0. \end{aligned}$$

In this nomenclature, DG discretization means simply applying a Galerkin method using discontinuous functional spaces for both the scalar and vector-valued variable, w and σ. In the present paper, essentially a continuation of Ref. 5, we follow a different approach. The diffusive part of the equations is discretized by a dual-mixed method,[6] while the convective part is discretized with a DG method. This is done in such a way that the discretization reduces to a DG method for the purely hyperbolic case, and to a conforming mixed

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method for the purely elliptic case. Our particular approach follows Egger and Schöberl,[7] who proposed a similar method for linear advection-diffusion problems, while the general concept of combining mixed methods and upwinded discretization has other predecessors,[8,9] which actually precede DG methods for advection-diffusion systems.

An immediate consequence of the mixed formulation is the increase of degrees of freedom (DOFs) associated with the vector-valued variable σ. However, a major advantage of many pure DG methods is that σ may be locally eliminated in favor of the scalar variable, possibly using lifting operators.[2] Such a local resolution is not possible in the classical dual-mixed formulation of the elliptic terms. However, it has long been known that *hybridization*, at least for elliptic problems discretized with standard dual-mixed methods, is a way to reduce the number of globally coupled DOFs.[10,11] In fact, hybridization allows the elimination of *both* variables, w and σ in terms of an auxiliary variable with support only on the skeleton of the mesh. Recently, hybridization has been formulated in a unified framework for both DG and standard mixed discretization,[12] and has been extended to advection-diffusion problems for both DG schemes[13,14] and the DG/mixed method.[5,7] The present hybridized formulation is an extension of the method presented in Ref. 7 to both nonlinear scalar equations, and the compressible Navier-Stokes equations. At the same time it presents an alternative to the Hybridized DG (HDG) schemes for the Navier-Stokes equations.

In this paper, we investigate the relationship between a Hybridized DG (HDG) method in the sense of Nguyen et al.[14] and our newly proposed method, for a scalar model problem. It turns out that they differ only in the polynomial approximation order of σ and the hybrid variable, as well as in the definition of a stabilization parameter. However, the convergence behavior can be different in the diffusion-dominated case. We demonstrate the convergence behavior using scalar model equations, and furthermore give numerical results validating the correctness of our Hybrid Mixed method in the context of compressible Navier-Stokes equations.

The paper is structured as follows: In section II, we formulate the underlying equations, and give some preliminary definitions regarding triangulation and discretization spaces. In section III, we present our Hybrid Mixed method. Besides the formulation of the method, we highlight solution procedures for the nonlinear systems of equations, and post processing of the numerical solution to improve the order of convergence. We compare our scheme to an HDG scheme, and show numerical results. Section IV offers conclusions.

II. Underlying Equations and Preliminaries

II.A. Underlying Equations

In this paper, we consider general nonlinear (systems of) convection-diffusion equations. In an abstract way, these equations can be written generically in *mixed* form as

$$\begin{aligned} f_v(w, \nabla w) &= \sigma \quad \forall x \in \Omega \\ \nabla \cdot (f(w) - \sigma) &= g \quad \forall x \in \Omega \end{aligned} \tag{1}$$

equipped with suitable boundary conditions, where w and σ are the unknowns. Without loss of generality, we consider $\Omega \subset \mathbb{R}^2$. The functions f and f_v are called the convective flux, and viscous flux, respectively. The interest in this type of equation is of course due to the fact that the compressible Navier-Stokes equations can be written in this way. However, for convergence results, we also consider the scalar convection-diffusion equation.

II.A.1. Scalar Convection-Diffusion Equation

As a particularly simple nonlinear convection-diffusion system, we consider the viscous Burgers equation defined by using

$$f(w) = \frac{1}{2}(w^2, w^2), \quad f_v(w, \nabla w) = \varepsilon \nabla w. \tag{2}$$

On $\partial\Omega$, we choose zero Dirichlet boundary conditions, i.e., we impose

$$w = 0 \quad \forall x \in \partial\Omega.$$

In the course of this paper, d will denote the dimension of the system. For this scalar equation, $d = 1$. The source function $g \in L^2(\Omega)$ will be modified in such a way that we know the solution in advance.

II.A.2. Navier-Stokes Equations

As a more complicated nonlinear convection-diffusion system, we consider the two-dimensional compressible Navier-Stokes equations with adiabatic, no-slip boundary conditions. This is a system of equations with dimension $d = 4$. In the context of external aerodynamics, $\Omega \subset \mathbb{R}^2$ is an exterior domain, i.e., a domain around an object. The state variable w is given by the vector of conserved variables $w = (\rho, \rho u, \rho v, E)$. Here ρ is the density, (u, v) is the velocity vector, and E is the total specific energy. The functions $f = (f_1, f_2)$ and $f_v(w, \nabla w) = (f_{v,1}, f_{v,2})$ are the convective and diffusive fluxes, respectively, given as

$$\begin{aligned} f_1 &= (\rho u, p + \rho u^2, \rho uv, u(E+p))^T, & f_2 &= (\rho v, \rho uv, p + \rho v^2, v(E+p))^T, \\ f_{v,1} &= (0, \tau_{11}, \tau_{21}, \tau_{11} u + \tau_{12} v + kT_{x_1})^T, & f_{v,2} &= (0, \tau_{12}, \tau_{22}, \tau_{21} u + \tau_{22} v + kT_{x_2})^T, \end{aligned}$$

while the adiabatic, no-slip boundary conditions are formally given as

$$\begin{aligned} u = v &= 0 \quad \forall x \in \partial\Omega, \\ n \cdot \nabla T &= 0 \quad \forall x \in \partial\Omega. \end{aligned}$$

The source function g in (1) is identically zero. Using the ideal gas law, temperature T and pressure p can be related to the conserved variables as

$$T = \frac{\mu\gamma}{k \cdot Pr}\left(\frac{E}{\rho} - \frac{1}{2}(u^2 + v^2)\right) = \frac{1}{(\gamma - 1)c_v}\frac{p}{\rho},$$

where $Pr = \frac{\mu c_p}{k}$ is the Prandtl number, which for air at moderate conditions is constant, with a value of $Pr = 0.72$. The thermal conductivity coefficient is denoted by k, while c_p and c_v are specific heats at constant pressure and constant volume, respectively. These are related via $\gamma = \frac{c_p}{c_v}$, where $\gamma = 1.4$ is again a constant for air at moderate conditions. Given a Newtonian fluid and assuming that the Stokes hypothesis holds, the viscous stress tensor τ can be written as

$$\tau = \mu\left(\nabla\widehat{w} + (\nabla\widehat{w})^T - \frac{2}{3}(\nabla \cdot \widehat{w})Id\right),$$

where we have set $\widehat{w} := (u, v)^T$. The dynamic viscosity μ is taken, using Sutherland's law,[15] as

$$\mu = \frac{C_1 T^{3/2}}{T + C_2}$$

with C_1 and C_2 that can, for air at moderate temperatures, assumed to be constant. Let us note that the adiabatic boundary condition $n \cdot \nabla T = 0$ can, in combination with the no-slip condition, be equivalently written as

$$(\sigma \cdot n)_4 = (f_v(w, \nabla w) \cdot n)_4 = 0 \quad \forall x \in \partial\Omega. \tag{3}$$

The viscous fluxes f_v are linear functions of ∇w, and hence allow a decoupling as

$$f_{v,i}(w) = \sum_{j=1}^{2} B_{ij}(w) w_{x_j} \tag{4}$$

with (nonlinear) matrices $B_{ij}(w)$.

The non-dimensionalized equations depend on the flow conditions only through the Mach number M, the (constant) Prandtl number Pr, the constant γ, and the Reynolds number Re. The latter is defined as

$$Re := \frac{UL\rho_0}{\mu_0},$$

where U is a reference speed, L a reference length (for example the chord length of an airfoil), ρ_0 is a reference density, and μ_0 a reference viscosity.

Letting $Re \to \infty$, and changing the boundary conditions from no-slip to slip, one formally obtains the Euler equations, given as

$$\begin{aligned} \nabla \cdot f(w) &= 0 \quad \forall x \in \Omega \\ (u, v) \cdot n &= 0 \quad \forall x \in \partial\Omega. \end{aligned} \tag{5}$$

We will give numerical results for both Euler and Navier-Stokes equations.

II.B. Preliminaries

To formulate our method in the next chapter, we need some formal definitions here. Let us assume that our domain Ω is triangulated as $\{\Omega_k\}_{k=1}^N$, where

$$\bigcup_{k=1}^{N} \overline{\Omega}_k = \Omega, \qquad \Omega_k \cap \Omega_{k'} = \emptyset \quad \forall k \neq k'.$$

Based on these quantities, we define the set of both interior and boundary edges, called Γ. Following standard nomenclature, we define an interior edge e as an intersection of two neighboring element boundaries $\partial\Omega_k \cap \partial\Omega_{k'}$ having a positive 1-dimensional measure. A boundary edge e is defined as the intersection of an element boundary $\partial\Omega_k$ with the physical boundary $\partial\Omega$. Let us furthermore define $\Gamma_0 \subset \Gamma$ to be the set of all internal edges. Assuming that $\Gamma = \{\Gamma_k\}_{k=1}^{\widehat{N}}$, and the Γ_k are equipped with an orientation given by the direction of the corresponding normal vectors n, we define for a function v, and $x \in \Gamma_k$,

$$v(x)^{\pm} := \lim_{\tau \to 0^+} v(x \pm \tau n).$$

Average and jump operators are defined in a standard way as

$$\{v\} = \frac{v^+ + v^-}{2}, \quad [\![v]\!] = v^- n - v^+ n.$$

Similar definitions hold when considering a function on an element boundary $\partial\Omega_k$.

With this, let us define the Ansatz spaces

$$\begin{aligned}
V_h &:= \{\varphi \in L^2(\Omega) \; |\varphi_{|\Omega_k} \in \Pi^p(\Omega_k) & \forall k = 1, \dots N\}^d \\
H_h &:= \{\tau \in L^2(\Omega)^2 | \tau_{|\Omega_k} \in \Pi^{p+1}(\Omega_k)^2 & \forall k = 1, \dots N\}^d \\
M_h &:= \{\mu \in L^2(\Gamma) \; |\mu_{|e} \; \in \Pi^{p+1}(e) & \forall e \in \Gamma\}^d \\
\widetilde{M_h} &:= \{\mu \in L^2(\Gamma) \; |\mu_{|e} \; \in \Pi^{p}(e) & \forall e \in \Gamma\}^d,
\end{aligned}$$

where Π^p is the space of polynomials up to degree p. These spaces will be needed in the definition of the DG and the newly established Hybrid Mixed method.

Let us make the following remarks about these spaces:

- Standard DG methods use V_h and V_h^2 for the approximation of w and σ, respectively.
- A hybridized DG method in the sense of e.g. Nguyen et al.[13,14] uses the DG spaces V_h and V_h^2 for w and σ, and the additionally introduced quantity $\lambda \approx w_{|\Gamma}$ is discretized in $\widetilde{M_h}$. The convergence order of the L^2-norm for their scheme, which is a hybridized interior penalty scheme, can be proven to be $p + \frac{1}{2}$ for σ and $p + 1$ for w.[16] However, the empirical order of convergence they observe is $p + 1$ for both w and σ. This is, given the underlying polynomial orders, of course optimal convergence. Using postprocessing they are even able to recover a quantity w_h^* that converges with order $p+2$, given that $p > 0$.
- A standard BDM hybrid mixed method[11] uses the spaces V_h for the discretization of w, H_h for the discretization of σ and M_h for the discretization of λ[a]. It is well-known that in the purely elliptic case, the order of convergence is $p + 1$ towards w and $p + 2$ towards σ. A local postprocessing allows the recovery of a quantity w_h^* that converges with order $p + 3$ towards w (again, assuming $p > 0$).[11] However, when introducing numerical stabilization for the discretization of the convective terms, the optimal order of convergence for σ is lost and reduces to $p + 1$. Our scheme, which relies on the BDM spaces, is very similar to the hybridized scheme by Nguyen et al.[13,14] We will demonstrate that in the viscosity-dominated case, when it is possible to work without upwind stabilization via a Riemann flux, we obtain order of convergence $p + 2$ for σ also. This has, in the context of a Raviart-Thomas

[a]As different orders of approximation are used for w and σ, one needs a convention for indexing the order of the method. Here we use the polynomial approximation order of w. In the literature, one often finds the BDM_p method defined as using order $p - 1$ for w, and p for σ. These different conventions can easily be related to each other by means of an obvious index shift.

approximation for σ, been theoretically investigated in Ref. 7. In the convective case, where we cannot switch off the Riemann-flux stabilization, we can very easily change the spaces H_h and M_h to V_h^2 and $\widetilde{M_h}$, respectively, to obtain a standard hybridized scheme. This also gives a new insight on the hybridized DG schemes.

III. A Hybrid Mixed Scheme for the Navier-Stokes Equations

III.A. Method

In this section, we propose a hybrid mixed method for the discretization of the Navier-Stokes equations. This method is an extension to the nonlinear system case of a method proposed for the scalar, linear case by Egger and Schöberl.[7] It can be seen as the combination of a standard mixed method for the diffusive part and a standard DG method for the convective part, made compatible via a hybridization ansatz. This means that in the case of vanishing convection, i.e., $f \equiv 0$ in eq. (1), we approximate the viscous flux σ in $H(\mathrm{div}, \Omega)$, while for vanishing diffusion, i.e., $f_v \equiv 0$, we approximate a DG solution. In-between, the method works without any parameters to tune. To make this clearer, let us subdivide the derivation of the method into two parts, treating the diffusive and convective terms of the equation, respectively.

III.A.1. Diffusive part

Scalar Diffusion Equation   Let us first consider the discretization of the scalar diffusion equation given as

$$\begin{aligned} \varepsilon \nabla w &= \sigma \quad \forall x \in \Omega \\ -\nabla \cdot \sigma &= g \quad \forall x \in \Omega \\ w &= 0 \quad \forall x \in \partial\Omega. \end{aligned}$$

Applying a standard mixed hybrid method to this equation yields the method implicitly defined as

$$N_v(\sigma_h, w_h, \lambda_h; \tau_h, \varphi_h, \mu_h) = \int_\Omega g \varphi_h \, dx \quad \forall (\tau_h, \varphi_h, \mu_h) \in H_h \times V_h \times M_h, \tag{6}$$

where $N_v \equiv N_v(\sigma_h, w_h, \lambda_h; \tau_h, \varphi_h, \mu_h)$ is defined as

$$\begin{aligned} N_v := &\int_\Omega \sigma_h \cdot \tau_h + \varepsilon w_h \nabla \cdot \tau_h - \nabla \cdot \sigma_h \varphi_h \, dx + \int_{\Gamma_0} -\varepsilon \lambda_h \cdot [\![\tau_h]\!] + \mu_h [\![\sigma_h]\!] \, d\sigma \\ &+ \int_{\Gamma \setminus \Gamma_0} -\varepsilon w_{\partial\Omega}(\lambda_h) \tau_h \cdot n + \mu_h (\lambda_h - w_{\partial\Omega}(w_h^-)) \, d\sigma. \end{aligned} \tag{7}$$

We define $w_{\partial\Omega}(w_h^-) = 0$, thereby exactly incorporating given Dirichlet boundary conditions. We note that for arbitrary Dirichlet boundary conditions $w = h$ on $\partial\Omega$, one would set $w_{\partial\Omega}(w_h^-) = h$.

Testing (7) with $(0, 0, \mu_h)$ yields that $[\![\sigma_h]\!] = 0$, which is equivalent to $\sigma_h \in H(\mathrm{div}, \Omega)$. Consistency follows trivially from the construction principle.

Navier-Stokes Equations   Let us now consider the diffusive part of the Navier-Stokes equations, i.e.,

$$\begin{aligned} f_v(w, \nabla w) &= \sigma \quad \forall x \in \Omega \\ -\nabla \cdot \sigma &= 0 \quad \forall x \in \Omega \\ u = v &= 0 \quad \forall x \in \partial\Omega \\ (\sigma \cdot n)_4 &= 0 \quad \forall x \in \partial\Omega. \end{aligned}$$

This equation resembles of course Poisson's equation and it is thus natural to also be treated in a standard 'Poisson-like' way. Applying a standard hybrid mixed method as above, yields the method implicitly defined as

$$N_v(\sigma_h, w_h, \lambda_h; \tau_h, \varphi_h, \mu_h) = 0 \quad \forall (\tau_h, \varphi_h, \mu_h) \in H_h \times V_h \times M_h, \tag{8}$$

where $N_v \equiv N_v(\sigma_h, w_h, \lambda_h; \tau_h, \varphi_h, \mu_h)$ is defined as

$$N_v := \int_\Omega \sigma_h \cdot \tau_h + w_h \nabla \cdot \left(B(w_h)^T \tau_h\right) - \nabla \cdot \sigma_h \varphi_h \; dx + \int_{\Gamma_0} -\lambda_h \cdot [\![B(w_h)^T \tau_h]\!] + \mu_h [\![\sigma_h]\!] \; d\sigma \\ + \int_{\Gamma \setminus \Gamma_0} -w_{\partial\Omega}(\lambda_h) \cdot (nB(w_h)^T \tau_h) + \mu_h(\lambda_h - w_{\partial\Omega}(w_h^-)) + \mu_h(\sigma_h^- - \sigma_{\partial\Omega}(\sigma_h^-)) \cdot n \; d\sigma. \tag{9}$$

In this definition, $w_{\partial\Omega}(\cdot)$ is a function that maps an input argument to a state that exactly fulfills the boundary conditions the equation poses on w i.e., $w_{\partial\Omega}(w) = (w_1, 0, 0, w_4)^T$. As we have also conditions on σ, we define $\sigma_{\partial\Omega} \equiv \sigma_{\partial\Omega}(\sigma) := (0, \sigma_2 \cdot n, \sigma_3 \cdot n, 0)$ which is an exact state given we use adiabatic boundary conditions, see also (3).

Again, setting τ_h and φ_h to zero directly yields that the jumps of σ_h have to vanish, meaning that σ_h is a function possessing a (weak) divergence, and thus $\sigma_h \in H(\text{div}, \Omega)$.

III.A.2. Convective part

Let us now consider the convective part of equation (1). For the Navier-Stokes equations, the convective part is obviously given by the Euler equations. Discretizing them with a standard DG scheme using a Lax-Friedrichs flux, and weakly enforcing $\lambda_h = \{w_h\}$ on the interior edges, yields the discretization implicitly defined as

$$N_c(w_h, \lambda_h; \varphi_h, \mu_h) = 0 \quad \forall (\varphi_h, \mu_h) \in V_h \times M_h, \tag{10}$$

with $N_c \equiv N_c(w_h, \lambda_h; \varphi_h, \mu_h)$ being defined as

$$N_c := -\int_\Omega f(w_h) \nabla \varphi_h \; dx + \int_{\Gamma_0} [\![\varphi_h (f(\lambda_h) - \alpha(\lambda_h - w_h) n]\!] \; d\sigma + \int_{\Gamma \setminus \Gamma_0} \varphi_h^- f(w_{\partial\Omega}(\lambda_h)) \cdot n \; d\sigma \\ + \int_{\Gamma_0} \mu_h [\![\alpha(\lambda_h - w_h) n]\!] \; d\sigma + \int_{\Gamma \setminus \Gamma_0} \mu_h \alpha(\lambda_h - w_{\partial\Omega}(w_h)) \; d\sigma. \tag{11}$$

α denotes the Lax-Friedrichs coefficient, which can be chosen to be proportional to the largest eigenvalue of $f'(\lambda_h) \cdot n$. For simplicity though, we usually assume it to be constant. Setting φ_h to zero directly yields that $\lambda_h = \{w_h\}$ on Γ_0, while setting μ_h to zero and using the knowledge about λ_h makes it obvious that the method is nothing but a reformulation of a DG method.

III.A.3. Full Convection-Diffusion System

A discretization for the general nonlinear convection-diffusion system is obtained by simply adding N_v from (7) or (9), respectively, and N_c from (11), yielding the task of finding (σ_h, w_h, τ_h), such that

$$N_v(\sigma_h, w_h, \lambda_h; \tau_h, \varphi_h, \mu_h) + N_c(w_h, \lambda_h; \varphi_h, \mu_h) = \int_\Omega g \varphi_h \; dx \quad \forall (\tau_h, \varphi_h, \mu_h) \in H_h \times V_h \times M_h. \tag{12}$$

It is straightforward to see that the method is consistent by substituting the exact solution (σ, w, w). Also, by testing with $(\tau_h, \varphi_h, \mu_h) = (0, \chi_{\Omega_k}, 0)$, it is obvious that the method is locally conservative, meaning that the relation

$$\int_{\partial\Omega_k} f(\lambda_h) \cdot n - \alpha(\lambda_h - w_h^-) - \sigma_h^- \cdot n \; d\sigma = \int_{\Omega_k} g \; dx$$

holds. Due to the fact that the numerical flux is a continuous quantity, which can be gotten by testing (12) with $(0, 0, \mu_h)$, global conservation follows easily.

We will show below numerical results for a Hybrid Mixed scheme where we disable numerical diffusion. This is simply done by considering the scheme as in (12) and setting $\alpha \equiv 0$.

III.A.4. Relation to a hybridized DG method in the scalar case

As already mentioned previously, our method very much resembles the method proposed by Nguyen et al.[14] The only differences are:

- The choice of the spaces for the discretization of σ and λ.
- The coefficient α, which is in our setting only dependent on the convection as it is the Lax-Friedrichs coefficient, and it vanishes as $f \to 0$. In the setting considered by Nguyen et al.,[14] the coefficient α (which they call τ) is dependent on both the convective and the viscous flux, and can be divided into $\tau = \tau_f + \tau_{f_v}$, where the parameter τ_f is basically a Lax-Friedrichs constant, while the parameter τ_{f_v} stems from a local Discontinuous Galerkin discretization[17] of the viscous terms. This parameter does not vanish as $f \to 0$, and yields thus another scheme, at least in the viscous limit. This flexibility is of great interest for us, offering the possibility to switch between a hybridized DG scheme for the convection-dominated case and a hybrid mixed method for the diffusion-dominated case by simply changing the spaces for σ and λ.

Let us make the above considerations a little bit more explicit. In our nomenclature, the method for the scalar case formulated in the paper by Nguyen et al.[14] reads (without loss of generality, we set $\varepsilon \equiv 1$)

$$\int_\Omega \sigma_h \cdot \tau_h + w_h \nabla \cdot \tau \, dx - \int_{\Gamma_0} \lambda_h [\![\tau_h]\!] \, d\sigma + \int_\Omega (\sigma_h - f(w_h)) \cdot \nabla \varphi_h \, dx$$
$$+ \int_{\Gamma_0} [\![\varphi_h \left(-\sigma_h + f(\lambda_h) - \tau(\lambda_h - w_h) n\right)]\!] \, d\sigma + \int_{\Gamma_0} [\![\sigma_h + \tau(\lambda_h - w_h) n]\!] \mu_h \, d\sigma = \int_{\Omega_k} g \varphi_h \, dx. \tag{13}$$

Written in this form, the relation between our method as defined in (12) is obvious. More precisely, the methods only differ in the numerical flux $\widehat{f - \sigma}$, which can be written as

$$\begin{aligned} \left(\widehat{f - \sigma}\right)_{HM} &= f(\lambda_h) - \alpha(\lambda_h - w_h) n - \sigma_h \\ \left(\widehat{f - \sigma}\right)_{HDG} &= f(\lambda_h) - \tau(\lambda_h - w_h) n - \sigma_h. \end{aligned}$$

(We note that the incorporation of boundary conditions in our method is done weakly through $w_{\partial\Omega}$, while Nguyen et al. implement the boundary conditions in the underlying space $\widetilde{M}_h$. This is why there are no boundary terms in their method. For the scalar case, however, results are equivalent.) Let us conclude that the (short) remark above yields new insight on the relation between a hybridized DG and a hybrid mixed discretization. We will numerically demonstrate the benefits of both methods in terms of σ_h and w_h^* convergence, where w_h^* is the post processed variable in the sense of section III.B.

Another important difference compared to Ref. 18, when extending the scheme to the full Navier-Stokes equations, is given by the definition of the mixed variable σ not to be the gradient of w, but the full viscous flux $f_v(w, \nabla w)$. This, however, is not an inherent difference between the mixed and the DG formulation, but mostly a matter of choice. It may have, however, an impact on the post processing outlined in section III.B.

III.B. Post Processing

One of the very nice features about hybrid methods is that it is possible to post process the numerical solution w_h, yielding an approximation w_h^* to w that converges with a better order of convergence than w_h does. We follow an approach proposed by Stenberg[19] and applied to the proposed scheme by Egger and Schöberl.[7] This post processing relies on the two facts that

- we explicitly compute σ_h, which converges with a better order of accuracy towards $\varepsilon \nabla w$ than the quantity $\varepsilon \nabla w_h$ does, and
- the quantity $(w_h, 1)$ exhibits superconvergence.

For more details and mathematical proofs, we refer to Ref. 7 and 19.

To formulate the post processing algorithm, let us introduce the space of mean-value free polynomials on a given cell Ω_k,

$$\Pi_0^{\overline{p}}(\Omega_k) := \{ f \in \Pi^{\overline{p}}(\Omega_k) | \int_{\Omega_k} f \, dx = 0 \}.$$

As both $\varepsilon\nabla w_h$ and σ_h should approximate the quantity $\sigma = \varepsilon\nabla w$, a very natural algorithm is formulated as the *cell-wise* discretization of a Neumann problem

$$\varepsilon \int_{\Omega_k} \nabla w_h^* \cdot \nabla \varphi_h \, dx = \int_{\Omega_k} \sigma_h \cdot \nabla \varphi_h \, dx \quad \forall \varphi_h \in \Pi_0^{\overline{p}}(\Omega_k) \qquad (14)$$
$$\int_{\Omega_k} w_h \, dx = \int_{\Omega_k} w_h^* \, dx,$$

where $w_h^* \in \Pi^{\overline{p}}(\Omega_k)$. Note that Problem (14) is solved *cell-wise*, i.e. very cheaply.

We have not commented on the choice of $\overline{p}$ yet. If σ_h converges with the *same* order towards σ as w_h does towards w, we choose $\overline{p} = p+1$. (This is the case for the hybridized DG scheme and the Hybrid Mixed scheme with numerical diffusion introduced by the Riemann flux.) If σ_h converges towards σ even better than w_h does towards w, we choose $\overline{p} = p+2$. (This is the case for the Hybrid Mixed scheme with no numerical diffusion.)

It is again emphasized that the post processing routine is defined purely cell-wise, and can thus be performed easily. Numerical results will show that, except in the case $p = 0$, w_h^* will converge towards w_h with one order of accuracy better than σ_h converges towards σ. We have summarized this in Table 1.

Let us mention that it has been proven in Ref. 7 for the linear case, that this post processing algorithm works for the Hybrid Mixed method given that either the mesh size h or the convective flux is small compared to the viscosity. We have also applied this post processing to the Hybridized DG method by Nguyen et al.,[18] with the result that it also gives superconvergence. However, we do not project the flux onto a Raviart-Thomas space as the authors in Ref. 18 do, thus rendering the given post processing scheme somewhat easier to implement.

Method	$e_w := \|w - w_h\|_{L^2}$	$e_\sigma := \|\sigma - \sigma_h\|_{L^2}$	$e_{w^*} := \|w - w_h^*\|_{L^2}$
Hybrid Mixed Method	$p+1$	$p+1$	$p+2$
Hybrid Mixed Method (no stabilization)	$p+1$	$p+2$	$p+3$
Hybridized DG Scheme	$p+1$	$p+1$	$p+2$

Table 1. Convergence of the quantities for different schemes. In all the cases, w_h is a cellwise polynomial of order p with $p > 0$.

III.C. Relaxation Procedure

The nonlinear algebraic system of equations defined in (12) has to be solved. A commonly used approach in the context of Discontinuous Galerkin methods is a *damped Newton procedure.* Given an abstract equation

$$F(x) = 0,$$

the damped Newton method computes a sequence $\{x^n\}_{n\in\mathbb{N}}$, starting from a value x^0, where x^{n+1} is defined as

$$x^{n+1} = x^n + s^n,$$
$$\left(F'(x^n) - \frac{Id}{\Delta t^n}\right) s^n = -F(x^n). \qquad (15)$$

Here Δt^n is a damping parameter; its resemblance to the time step in a linearized backward Euler method is obvious.

The method as proposed in (12) has a Jacobian matrix that is extremely large due to the fact that the method is coupled in σ_h, w_h and λ_h. However, the general idea of using a hybrid method is to reduce the globally coupled degrees of freedom by decoupling, leaving us with a Jacobian in terms of degrees of freedom for λ_h only.

Let us therefore consider an idea going back to Cockburn and Gopalakrishnan[20] and use local solvers to decouple the global degrees of freedom. We make this more precise in the following: The nonlinear, algebraic function F that we want to solve consists of three components

$$F = (T, \Phi, M)^T,$$

corresponding to those parts of (12) that are tested against τ_h, φ_h and μ_h, respectively. The input argument x decouples into $x = (x_\sigma, x_w, x_\lambda)$, where x_σ denotes the vector that represents σ_h in H_h, i.e., $\sigma_h = \sum_{i=1}^{\dim(H_h)} (x_\sigma)_i \tau_i$ for a set of basis vectors τ_i. A similar definition holds true for both x_w and x_λ. Thus, the derivative of F has the shape

$$F' = \begin{pmatrix} T_{x_\sigma}, T_{x_w}, T_{x_\lambda} \\ \Phi_{x_\sigma}, \Phi_{x_w}, \Phi_{x_\lambda} \\ M_{x_\sigma}, M_{x_w}, M_{x_\lambda} \end{pmatrix}.$$

The submatrix

$$\begin{pmatrix} T_{x_\sigma}, T_{x_w} \\ \Phi_{x_\sigma}, \Phi_{x_w} \end{pmatrix}$$

is - due to the construction principle - block-diagonal and can thus be inverted easily, yielding a linear system of equations

$$\widehat{A}' s_\lambda = \widehat{A},$$

that is equivalent to (15) (where s_λ is similarly defined as x_λ). The updates s_w and s_σ have then to be re-derived by local inversion processes. This is the 'traditional' approach to hybrid mixed methods.

It is however also possible to assemble the matrix $\widehat{A}'$ and the right-hand side $\widehat{A}$ on a more PDE-related level. In the linear case (which we in fact have within each Newton iteration), the idea is roughly speaking to have a map $\lambda_h \mapsto (w_h(\lambda_h), \sigma_h(\lambda_h))$, where $w_h(\lambda_h)$ and $\sigma_h(\lambda_h)$ are supposed to fulfill the discretization cell-wise exactly for a given λ_h. For details, we refer to Ref. 20.

The local solves are then characterized as linear, non-homogeneous equations. If one assumes that s_λ is given, the linear system of equations for each cell Ω_k has the form

$$A_k s_{w,\sigma}(s_\lambda) = R_k + b_k(s_\lambda), \tag{16}$$

where b is a linear function in s_λ, and R_k is the (local) residual corresponding to T and Φ. Given that the set $\{\mu_i | i = 1, \ldots, \dim(M_h)\}$ denotes a basis of the space corresponding to x_λ, one can decouple system (16) as

$$\begin{aligned} A_k s_0 &= R_k \\ A_k s(\mu_i) &= b_k(\mu_i) \quad \forall i = 1, \ldots, \dim(M_h), \end{aligned}$$

where s_0 is the homogeneous part. Thus, denoting the non-homogeneous part of the solution by $s_{w,0}$ (and similarly for $s_{\sigma,0}$), x_w^{n+1} allows a decomposition as

$$x_w^{n+1} = x_w^n + s_{w,0} + \sum_{i=1}^{\dim(M_h)} \alpha_i s_w(\mu_i),$$

given that $s_\lambda = (\alpha_0, \ldots, \alpha_{\dim(M_h)})$.

Both $w := \sum_{i=1}^{\dim(V_h)} (x_w)_i \varphi_i$ and $\sigma := \sum_{i=1}^{\dim(H_h)} (x_\sigma)_i \tau_i$ can be substituted into (12), yielding a system in terms of degrees of freedom for M_h only. The Jacobian of this system is thus only of size $\dim(M_h)$, instead of size $\dim(M_h) + \dim(V_h) + \dim(H_h)$. However, the local solves that need to be performed, yield an overhead which is not present in a non-hybridized DG method.

The local solves are thus needed *twice*, once for the assembly of the Jacobian matrix and once for the update of both w_h and σ_h at the end of a damped Newton iteration step. In principle, one could store the local solves. However, there is then an obvious trade off between memory requirement and runtime. The reduction of memory requirements is a key issue of hybrid methods. In our code, we thus retain an option of switching between storing (small problem sizes, faster) and computing the local solves twice (bigger problems, less memory).

III.D. Numerical Results

In this section, we present numerical results for our proposed scheme. In the scalar case, we furthermore compare it to a hybridized DG scheme and a modified version of our scheme with the diffusion coefficient α turned off.

III.D.1. Scalar Equation

In this subsection, we compare our Hybrid Mixed scheme to a modified Hybrid Mixed scheme (without upwind stabilization) and a standard Hybridized DG scheme. Let us remark that, following Nguyen et al.,[14] we make α a discontinuous value over the edges, however with a rather crude choice (we choose $\alpha^- = 0.8$, $\alpha^+ = 1.2$). It seems however that this crude choice has no influence on the convergence rates, and actually, a similar 'crude' choice has been proposed in Ref. 13. For the hybrid DG method, this is needed to obtain optimal order convergence, because only then is the method an interior penalty method with guaranteed optimal order convergence.[16] The Hybrid Mixed method does not need α to be discontinuous. However, for comparison, we choose α in the same way.

Let us consider a boundary layer test case that was proposed in Ref. 7. The solution is assumed to be given as

$$w(x_1, x_2) = \left(x_1 + \frac{e^{x_1/\varepsilon} - 1}{1 - e^{1/\varepsilon}} \right) \cdot \left(x_2 + \frac{e^{x_2/\varepsilon} - 1}{1 - e^{1/\varepsilon}} \right).$$

In contrast to Ref. 7, we consider a nonlinear problem as given in (1) with Burgers flux (2). The right hand side $g \equiv g(x, \varepsilon)$ was chosen in such a way that w solves (1), for an arbitrary choice of ε. The smaller ε is, the more distinct is the boundary layer. A contour plot corresponding to $\varepsilon = 0.1$ can be seen in Fig. 1.

Numerical results corresponding to $\varepsilon = 1$ and $\varepsilon = 0.1$ can be seen for the Hybrid Mixed method in Tables 2 and 5, for the Hybridized DG method in Tables 4 and 7, and for the Hybrid Mixed method without numerical diffusion in Tables 3 and 6. We recall the notation

$$e_w := \|w - w_h\|_{L^2}, \quad e_\sigma := \|\sigma - \sigma_h\|_{L^2}, \quad e_{w^*} := \|w - w_h^*\|_{L^2}.$$

For $\varepsilon = 0.1$, we need of course more degrees of freedom to accurately resolve the boundary layer on our uniform grid (or use a better grid, such as a Shishkin mesh[21]), and thus the order is slightly deteriorated for coarse grids. We can, however, see that the convergence rates from Table 1 are always approached in grid refinement.

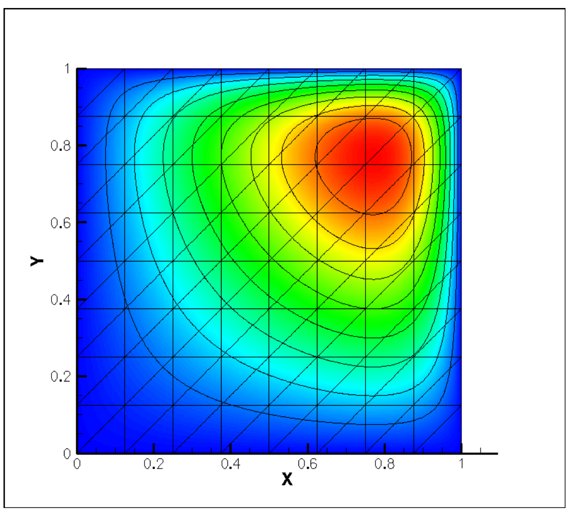


Figure 1. Boundary layer problem corresponding to $\varepsilon = 0.1$ - Contour Plot. Solution has been computed with a Hybrid Mixed Method without Numerical Diffusion and $p = 5$.

N	p	e_w	order	e_σ	order	e_{w^*}	order	p	e_w	order	e_σ	order	e_{w^*}	order
2	0	4.55E-003		1.60E-002		4.04E-003		1	4.18E-003		1.04E-002		2.09E-003	
8		4.08E-003	0.16	1.19E-002	0.43	2.46E-003	0.71		1.30E-003	1.68	3.16E-003	1.72	4.98E-004	2.07
32		2.42E-003	0.75	7.00E-003	0.77	1.37E-003	0.84		3.45E-004	1.92	8.48E-004	1.90	7.36E-005	2.76
128		1.28E-003	0.92	3.73E-003	0.91	7.29E-004	0.91		8.72E-005	1.98	2.18E-004	1.96	9.63E-006	2.93
2	2	1.04E-003		2.10E-003		8.21E-004		3	5.06E-004		9.07E-004		1.51E-004	
8		2.33E-004	2.16	4.61E-004	2.19	6.71E-005	3.61		3.80E-005	3.73	7.77E-005	3.54	5.73E-006	4.71
32		3.24E-005	2.85	6.73E-005	2.77	4.39E-006	3.93		2.48E-006	3.94	5.46E-006	3.83	1.89E-007	4.92
128		4.15E-006	2.96	8.90E-006	2.92	2.76E-007	3.99		1.57E-007	3.98	3.59E-007	3.93	6.01E-009	4.98
2	4	7.75E-005		1.48E-004		1.66E-005		5	7.49E-006		1.49E-005		1.31E-006	
8		2.74E-006	4.82	5.81E-006	4.67	3.13E-007	5.73		1.30E-007	5.84	2.82E-007	5.72	1.21E-008	6.76
32		8.84E-008	4.95	1.98E-007	4.88	5.21E-009	5.91		2.10E-009	5.96	4.73E-009	5.89	1.01E-010	6.92
128		2.78E-009	4.99	6.41E-009	4.95	8.34E-011	5.96		3.30E-011	5.99	7.63E-011	5.96	8.03E-013	6.97

Table 2. Hybrid Mixed Method: Convergence for $\varepsilon = 1$

N	p	e_w	order	e_σ	order	e_{w^*}	order	p	e_w	order	e_σ	order	e_{w^*}	order
2	0	4.47E-003		1.54E-002		1.47E-003		1	3.92E-003		7.87E-003		1.02E-003	
8		3.73E-003	0.26	7.07E-003	1.13	8.46E-004	0.79		1.25E-003	1.65	1.32E-003	2.58	9.53E-005	3.42
32		2.08E-003	0.85	2.15E-003	1.72	2.40E-004	1.82		3.40E-004	1.88	1.83E-004	2.85	6.77E-006	3.82
128		1.06E-003	0.97	5.74E-004	1.91	6.19E-005	1.95		8.69E-005	1.97	2.39E-005	2.94	4.44E-007	3.93
2	2	1.03E-003		1.51E-003		1.80E-004		3	5.01E-004		1.77E-004		1.99E-005	
8		2.32E-004	2.15	1.10E-004	3.78	6.77E-006	4.73		3.79E-005	3.72	6.30E-006	4.81	3.52E-007	5.82
32		3.23E-005	2.84	7.17E-006	3.94	2.20E-007	4.94		2.48E-006	3.93	2.05E-007	4.94	5.68E-009	5.95
128		4.15E-006	2.96	4.55E-007	3.98	6.98E-009	4.98		1.57E-007	3.98	6.48E-009	4.98	8.97E-011	5.99
2	4	7.69E-005		1.56E-005		1.51E-006		5	7.44E-006		1.12E-006		9.43E-008	
8		2.73E-006	4.81	2.74E-007	5.83	1.34E-008	6.81		1.30E-007	5.84	9.78E-009	6.84	4.16E-010	7.82
32		8.83E-008	4.95	4.42E-009	5.95	1.09E-010	6.95		2.10E-009	5.96	7.88E-011	6.96	1.68E-012	7.95
128		2.78E-009	4.99	6.97E-011	5.99	8.58E-013	6.98		3.30E-011	5.99	6.21E-013	6.99	6.62E-015	7.99

Table 3. Hybrid Mixed Method without Numerical Diffusion: Convergence for $\varepsilon = 1$

N	p	e_w	order	e_σ	order	e_{w^*}	order	p	e_w	order	e_σ	order	e_{w^*}	order
2	0	2.72E-002		4.27E-002		2.66E-002		1	5.04E-003		1.46E-002		1.43E-003	
8		1.28E-002	1.08	2.09E-002	1.03	1.22E-002	1.12		2.05E-003	1.29	6.35E-003	1.21	4.19E-004	1.77
32		6.52E-003	0.98	1.16E-002	0.85	6.17E-003	0.98		5.68E-004	1.85	1.79E-003	1.83	6.25E-005	2.74
128		3.25E-003	1.00	6.07E-003	0.93	3.07E-003	1.01		1.46E-004	1.96	4.65E-004	1.94	8.35E-006	2.90
2	2	2.29E-003		8.05E-003		8.39E-004		3	6.67E-004		1.46E-003		1.35E-004	
8		4.01E-004	2.52	1.24E-003	2.70	6.58E-005	3.67		5.01E-005	3.74	1.07E-004	3.77	4.94E-006	4.77
32		5.43E-005	2.88	1.65E-004	2.91	4.30E-006	3.93		3.27E-006	3.93	7.08E-006	3.92	1.61E-007	4.94
128		6.95E-006	2.97	2.11E-005	2.97	2.71E-007	3.99		2.07E-007	3.98	4.52E-007	3.97	5.08E-009	4.99
2	4	9.52E-005		1.70E-004		1.41E-005		5	9.05E-006		1.50E-005		1.09E-006	
8		3.39E-006	4.81	6.16E-006	4.79	2.57E-007	5.78		1.59E-007	5.83	2.69E-007	5.79	9.80E-009	6.79
32		1.10E-007	4.95	2.03E-007	4.92	4.22E-009	5.92		2.55E-009	5.96	4.44E-009	5.92	8.05E-011	6.93
128		3.45E-009	4.99	6.49E-009	4.97	6.73E-011	5.97		4.01E-011	5.99	7.08E-011	5.97	6.41E-013	6.97

Table 4. Hybridized DG Method: Convergence for $\varepsilon = 1$

N	p	e_w	order	e_σ	order	e_{w^*}	order	p	e_w	order	e_σ	order	e_{w^*}	order
2	0	1.24E-001		9.69E-002		1.12E-001		1	8.87E-002		5.79E-002		9.75E-002	
8		9.83E-002	0.34	6.58E-002	0.56	8.18E-002	0.46		6.14E-002	0.53	4.37E-002	0.40	4.69E-002	1.06
32		9.28E-002	0.08	5.73E-002	0.20	7.84E-002	0.06		2.46E-002	1.32	2.61E-002	0.74	1.57E-002	1.58
128		7.45E-002	0.32	5.23E-002	0.13	6.75E-002	0.22		7.17E-003	1.78	1.09E-002	1.26	3.51E-003	2.16
512		5.06E-002	0.56	3.96E-002	0.40	4.75E-002	0.51		1.82E-003	1.98	3.54E-003	1.63	5.80E-004	2.60
2	2	7.00E-002		4.35E-002		5.62E-002		3	3.07E-002		2.29E-002		2.48E-002	
8		2.09E-002	1.74	1.96E-002	1.15	1.39E-002	2.01		6.09E-003	2.33	7.03E-003	1.70	4.41E-003	2.49
32		4.87E-003	2.10	6.34E-003	1.63	2.57E-003	2.44		1.12E-003	2.44	1.59E-003	2.14	5.47E-004	3.01
128		8.43E-004	2.53	1.39E-003	2.19	2.80E-004	3.20		1.13E-004	3.31	1.97E-004	3.02	3.19E-005	4.10
512		1.17E-004	2.85	2.24E-004	2.64	2.12E-005	3.72		8.11E-006	3.80	1.62E-005	3.60	1.24E-006	4.69
2	4	1.11E-002		1.13E-002		1.02E-002		5	4.88E-003		6.20E-003		5.26E-003	
8		2.34E-003	2.25	2.97E-003	1.93	1.72E-003	2.56		9.36E-004	2.38	1.24E-003	2.32	6.12E-004	3.10
32		2.70E-004	3.11	4.15E-004	2.84	1.16E-004	3.89		5.56E-005	4.07	9.05E-005	3.78	2.08E-005	4.88
128		1.40E-005	4.27	2.59E-005	4.00	3.41E-006	5.09		1.44E-006	5.27	2.77E-006	5.03	3.01E-007	6.11
512		5.06E-007	4.79	1.05E-006	4.62	6.68E-008	5.67		2.61E-008	5.78	5.55E-008	5.64	2.93E-009	6.68

Table 5. Hybrid Mixed Method: Convergence for $\varepsilon = 0.1$

N	p	e_w	order	e_σ	order	e_{w^*}	order	p	e_w	order	e_σ	order	e_{w^*}	order
2		2.09E-001		1.26E-001		2.47E-001			1.81E-001		1.15E-001		2.13E-001	
8		1.49E-001	0.49	1.00E-001	0.34	1.59E-001	0.63		5.57E-002	1.70	3.50E-002	1.72	3.08E-002	2.79
32	0	7.04E-002	1.09	3.92E-002	1.35	4.56E-002	1.81	1	2.09E-002	1.41	9.06E-003	1.95	4.26E-003	2.86
128		3.66E-002	0.94	1.26E-002	1.63	1.24E-002	1.88		6.46E-003	1.69	1.68E-003	2.43	4.17E-004	3.35
512		1.87E-002	0.97	3.54E-003	1.84	3.19E-003	1.95		1.74E-003	1.89	2.43E-004	2.79	3.07E-005	3.77
2		6.84E-002		4.85E-002		5.00E-002			2.78E-002		2.17E-002		1.33E-002	
8		1.93E-002	1.83	1.22E-002	1.98	6.60E-003	2.92		5.84E-003	2.25	4.97E-003	2.12	2.44E-003	2.45
32	2	4.58E-003	2.07	2.29E-003	2.42	7.32E-004	3.17	3	1.08E-003	2.44	5.46E-004	3.19	1.52E-004	4.01
128		8.21E-004	2.48	2.32E-004	3.30	3.79E-005	4.27		1.11E-004	3.28	2.85E-005	4.26	4.04E-006	5.23
512		1.16E-004	2.82	1.69E-005	3.78	1.34E-006	4.82		8.07E-006	3.78	1.04E-006	4.78	7.31E-008	5.79
2		1.11E-002		1.14E-002		7.71E-003			5.09E-003		6.27E-003		4.36E-003	
8		2.25E-003	2.31	1.88E-003	2.60	8.49E-004	3.18		8.96E-004	2.51	6.22E-004	3.33	2.51E-004	4.12
32	4	2.63E-004	3.10	1.10E-004	4.10	2.66E-005	5.00	5	5.44E-005	4.04	1.89E-005	5.04	3.99E-006	5.98
128		1.38E-005	4.25	2.89E-006	5.25	3.55E-007	6.23		1.43E-006	5.25	2.49E-007	6.25	2.66E-008	7.23
512		5.04E-007	4.78	5.25E-008	5.78	3.23E-009	6.78		2.61E-008	5.78	2.25E-009	6.78	1.20E-010	7.79

Table 6. Hybrid Mixed Method without Numerical Diffusion: Convergence for $\varepsilon = 0.1$

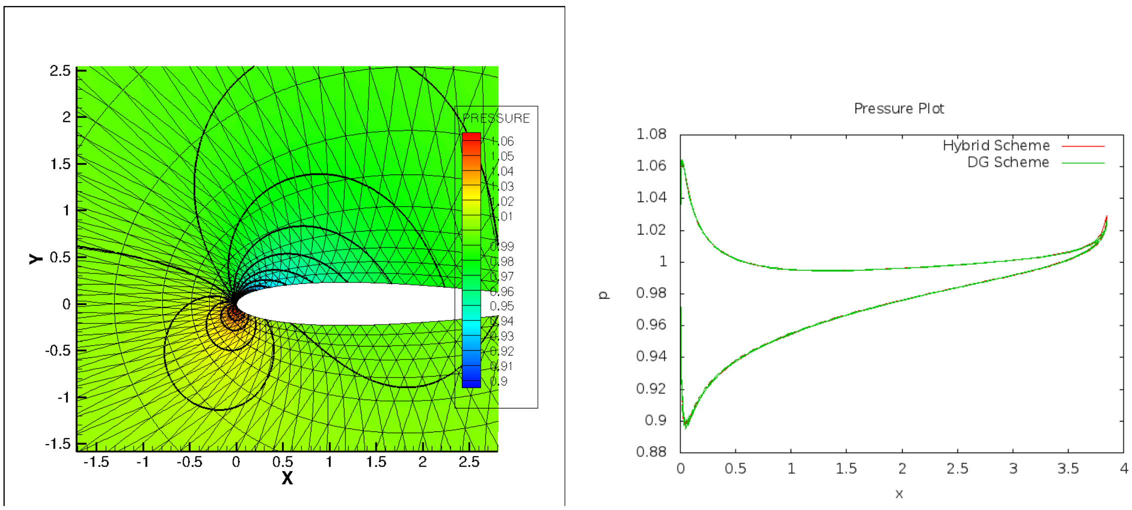
N	p	e_w	order	e_σ	order	e_{w^*}	order	p	e_w	order	e_σ	order	e_{w^*}	order
2		3.63E-001		2.36E-001		3.45E-001			7.04E-002		8.29E-002		8.78E-002	
8		1.71E-001	1.09	1.59E-001	0.57	1.29E-001	1.42		5.74E-002	0.29	5.53E-002	0.58	3.57E-002	1.30
32	0	8.46E-002	1.01	8.93E-002	0.83	5.18E-002	1.32	1	2.49E-002	1.21	2.83E-002	0.96	1.06E-002	1.75
128		4.97E-002	0.77	5.40E-002	0.73	3.47E-002	0.58		7.81E-003	1.67	1.07E-002	1.41	2.30E-003	2.21
512		2.83E-002	0.81	3.25E-002	0.73	2.15E-002	0.69		2.08E-003	1.91	3.20E-003	1.74	3.75E-004	2.62
2		6.55E-002		5.52E-002		5.16E-002			3.02E-002		2.98E-002		2.18E-002	
8		2.18E-002	1.59	2.49E-002	1.15	1.24E-002	2.06		6.65E-003	2.19	1.05E-002	1.51	4.10E-003	2.41
32	2	5.39E-003	2.02	8.10E-003	1.62	2.25E-003	2.46	3	1.24E-003	2.42	2.13E-003	2.29	4.94E-004	3.05
128		9.74E-004	2.47	1.63E-003	2.31	2.38E-004	3.24		1.31E-004	3.24	2.26E-004	3.24	2.71E-005	4.19
512		1.39E-004	2.81	2.45E-004	2.74	1.77E-005	3.75		9.72E-006	3.75	1.71E-005	3.72	1.02E-006	4.73
2		1.19E-002		1.65E-002		9.30E-003			5.10E-003		9.26E-003		5.05E-003	
8		2.47E-003	2.26	4.39E-003	1.91	1.66E-003	2.49		9.69E-004	2.40	1.65E-003	2.49	5.85E-004	3.11
32	4	2.97E-004	3.06	4.99E-004	3.14	1.05E-004	3.99	5	6.13E-005	3.98	9.93E-005	4.06	1.85E-005	4.98
128		1.62E-005	4.19	2.69E-005	4.21	2.85E-006	5.20		1.68E-006	5.19	2.70E-006	5.20	2.50E-007	6.21
512		6.05E-007	4.75	1.02E-006	4.72	5.38E-008	5.73		3.12E-008	5.75	5.11E-008	5.72	2.35E-009	6.73

Table 7. Hybridized DG Method: Convergence for $\varepsilon = 0.1$

III.D.2. Navier-Stokes Equations

In this section, we demonstrate the performance of our Hybrid Mixed method for inviscid and viscous compressible flow problems.

Inviscid Flow As a first test case, we consider inviscid flow, i.e. the underlying flow equations are the Euler equations (5), with Ω an external domain around a NACA0012 airfoil. Flow conditions are characterized by a free stream Mach number of $M = 0.3$, with an angle of attack $\alpha = 4°$. We compare our Hybrid Mixed scheme against a standard (non-hybridized) DG method, with very satisfactory results. A plot of the pressure contours can be seen in Fig. 2(a). A plot of the pressure distribution along the airfoil can be seen in Fig. 2(b), both for the above-mentioned standard DG scheme, and for our newly developed scheme. The plots lie nearly on top of each other. The slight deviation that can be observed at the trailing edge is due to the fact that the pressure at the airfoil boundary for the hybrid scheme is of course evaluated with the hybrid variable λ_h.



(a) Pressure distribution

(b) Pressure contours

Figure 2. An inviscid NACA0012 test case. Free-stream conditions are $M = 0.3$, $\alpha = 4.0°$.

Viscous Flow Let us first consider initial validation of our scheme, similar to the validation carried out in Ref. 5. We begin with viscous flow around a cylinder at different Reynolds numbers. The viscous drag coefficient c_d can be split into a pressure-induced component, and a friction drag, i.e., $c_d = c_{Dp} + c_{Df}$. In Fig. 3, we compare results for both c_{Dp} and c_{Df} to data from Henderson.[22] Results have been computed with third order polynomials in w_h. The figure shows very good agreement of the drag coefficient.

A second test case based on compressible Couette flow has been performed in Ref. 18. The computational domain is given by $\Omega = [0; 1]^2$, while the solution is given by smooth flow, with flow field variables defined as

$$w = (\rho, \rho u, \rho v, E) = (1/T, \frac{y \log(1+y)}{T}, 0, \frac{p}{\gamma - 1} + \frac{u^2 + v^2}{2T}), \tag{17}$$

where $p := \frac{1}{\gamma M^2}$ is a constant and T is defined as (see Ref. 18)

$$T = T_0 + y(T_1 - T_0) + \frac{M^2(\gamma - 1)}{2} Pr\ y(1 - y). \tag{18}$$

T_0, T_1 and M are constants, which we choose as in Ref. 18 to be 0.8, 0.85 and 0.15, respectively. A source term is incorporated in such a way that w is indeed a solution to the Navier-Stokes equations. In Tables (8) and (9), we computed the L^2- norms of both the error in w_h approximating w and in σ_h approximating $\sigma := f_v(w, \nabla w)$ for our Hybrid Mixed method and the Hybridized DG method. The stability constant α was in both cases chosen to be constant with value 2. As in Ref. 18, we observe that all the quantities converge with order $p + 1$, even for the case $p = 0$.

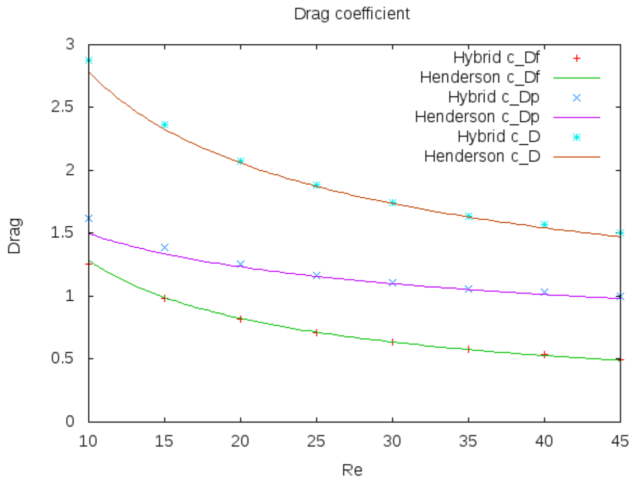


Figure 3. Laminar flow around a cylinder; Total, pressure-induced and viscosity-induced drag in comparison with results from the literature[22]

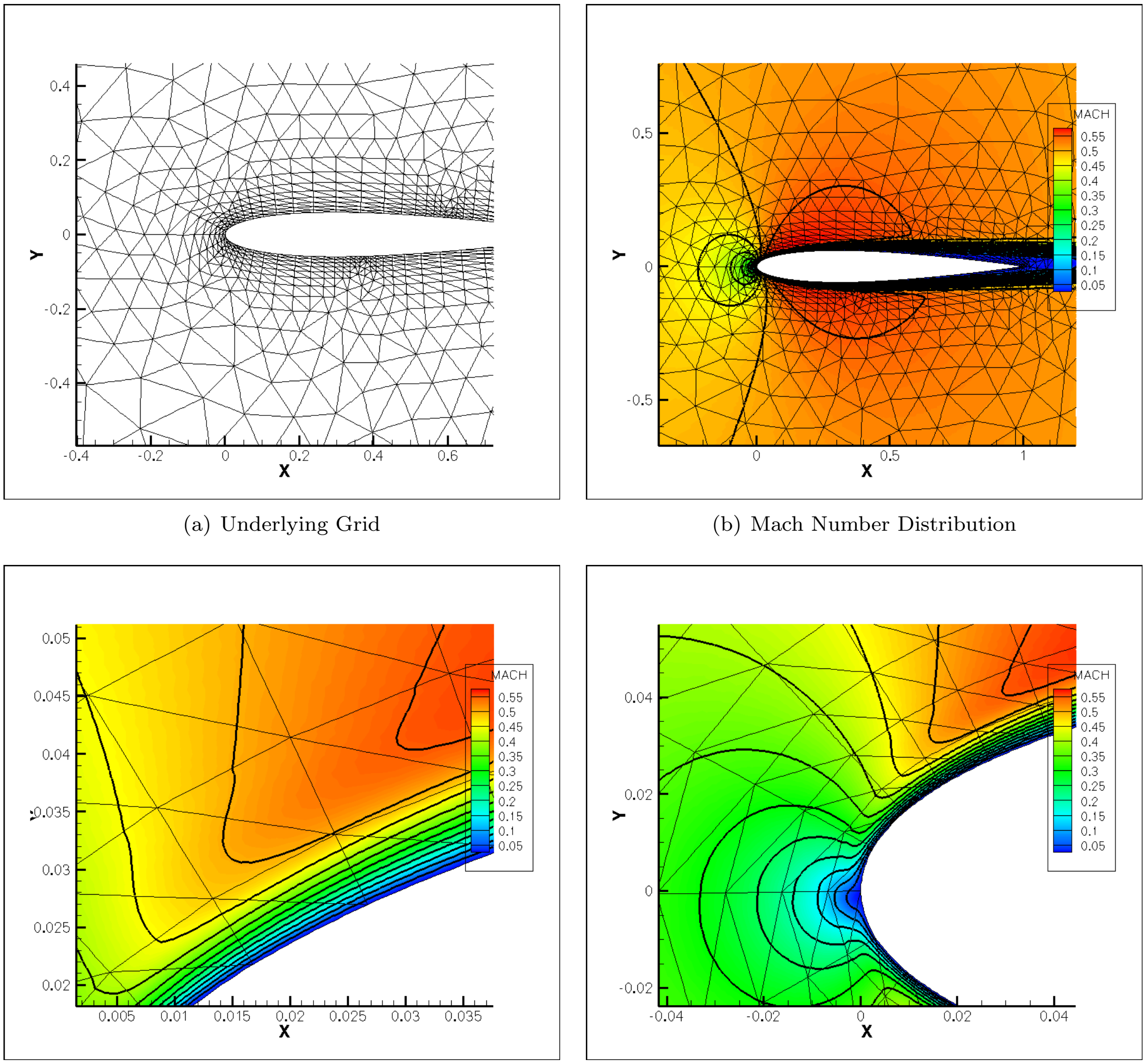
N	p	e_w	order	e_σ	order	p	e_w	order	e_σ	order
2		2.35E-001		1.67E+000			2.10E-001		1.72E-001	
8		6.31E-001	-1.43	1.52E+000	0.13		3.93E-002	2.42	5.61E-002	1.62
32	0	3.28E-001	0.94	9.64E-001	0.66	1	1.03E-002	1.92	1.74E-002	1.69
128		1.65E-001	0.99	5.46E-001	0.82		2.39E-003	2.11	4.99E-003	1.81
512		7.59E-002	1.12	2.91E-001	0.90		4.85E-004	2.30	1.34E-003	1.90
2		1.55E-002		2.50E-002			1.24E-003		1.60E-003	
8		2.78E-003	2.48	4.28E-003	2.54		1.39E-004	3.16	2.00E-004	3.01
32	2	3.75E-004	2.89	6.37E-004	2.75	3	1.01E-005	3.77	1.72E-005	3.54
128		4.41E-005	3.09	8.75E-005	2.86		6.83E-007	3.89	1.24E-006	3.80
512		4.83E-006	3.19	1.15E-005	2.93		4.40E-008	3.96	8.22E-008	3.91
2		1.81E-004		2.55E-004			2.40E-005		4.13E-005	
8		1.07E-005	4.08	1.88E-005	3.76		8.39E-007	4.84	1.67E-006	4.63
32	4	4.09E-007	4.71	8.56E-007	4.46	5	1.82E-008	5.52	4.02E-008	5.38
128		1.33E-008	4.94	3.13E-008	4.77		3.29E-010	5.79	7.49E-010	5.75
512		4.09E-010	5.02	1.04E-009	4.90		3.52E-012	6.55	1.25E-011	5.90

Table 8. Hybrid Mixed Method, Couette Flow Convergence

N	p	e_w	order	e_σ	order	p	e_w	order	e_σ	order
2		5.49E-001		4.53E-001			1.97E-001		1.32E-001	
8		6.26E-001	-0.19	5.69E-001	-0.33		3.48E-002	2.51	4.11E-002	1.68
32	0	2.86E-001	1.13	3.64E-001	0.64	1	8.36E-003	2.06	1.18E-002	1.80
128		1.12E-001	1.35	1.99E-001	0.87		1.96E-003	2.09	3.20E-003	1.89
512		4.27E-002	1.39	1.03E-001	0.95		4.28E-004	2.20	8.31E-004	1.94
2		1.61E-002		2.13E-002			1.66E-003		1.97E-003	
8		1.93E-003	3.06	3.15E-003	2.75		1.28E-004	3.69	2.15E-004	3.19
32	2	2.69E-004	2.84	4.47E-004	2.82	3	9.89E-006	3.70	1.84E-005	3.55
128		3.23E-005	3.05	5.96E-005	2.91		6.55E-007	3.92	1.31E-006	3.81
512		3.85E-006	3.07	7.69E-006	2.95		4.29E-008	3.93	8.71E-008	3.91
2		2.23E-004		3.40E-004			3.77E-005		6.01E-005	
8		1.10E-005	4.34	2.19E-005	3.96		1.01E-006	5.22	2.23E-006	4.75
32	4	4.53E-007	4.61	9.90E-007	4.47	5	2.25E-008	5.49	5.45E-008	5.36
128		1.54E-008	4.88	3.59E-008	4.79		4.05E-010	5.79	1.02E-009	5.74
512		5.05E-010	4.93	1.18E-009	4.92		7.02E-012	5.85	1.72E-011	5.89

Table 9. Hybridized DG Method, Couette Flow Convergence

As an example for viscous flow with a higher Reynolds number of $Re = 5{,}000$, we consider a test case, again using a NACA0012 airfoil, with free stream conditions of $M = 0.5$ and $\alpha = 1°$. The grid using stretched boundary layer elements is shown in Fig. 4(a). The computation was performed using polynomials of order $p = 3$ in w_h, and $p + 1 = 4$ in both σ_h and λ_h. The mesh consists of $N = 2{,}120$ elements with $\widehat{N} = 3{,}228$ edges. For this test case, this thus results in 84,800 degrees of freedom for w_h, 254,400 for σ_h and 64,560 for λ_h. The globally coupled degrees are only those of λ_h.



(a) Underlying Grid

(b) Mach Number Distribution

(c) Close up view of the boundary layer

(d) Close up view of the leading edge

Figure 4. Underlying mesh and Mach number distribution of a compressible, viscous flow around a NACA0012 airfoil. Free stream conditions are Mach 0.5, $\alpha = 1^\circ$, $Re = 5000$. The underlying polynomial order is $p = 3$. Method: Hybrid Mixed

IV. Conclusion and Outlook

We have developed a hybridized DG/mixed method for compressible flow. The general approach has been validated for nonlinear scalar model problems and both Euler equations and Navier-Stokes equations. We have furthermore given a connection between well-known Hybridized Discontinuous Galerkin schemes and our method. In the diffusive limit, our method performs, in terms of convergence rates, better than an HDG method. However, in the convective limit, the HDG method shows the same order of accuracy as our method, while having fewer degrees of freedom, as both σ_h and λ_h are computed with polynomial order of one degree lower. An upcoming paper will treat the possibility of mixing both methods in the appropriate regimes, thus optimizing the overall quality of the approximated solution while at the same time reducing the number of degrees of freedom.

Furthermore, the correct formulation of the Hybrid Mixed method for the compressible Navier-Stokes equations is not entirely clear at this point. From the point of view of post processing the solution, it seems more natural to define $\sigma := \nabla w$ rather than $\sigma := f_v(w, \nabla w)$. This might also be a better choice if turbulence equations are included. Further research will be conducted in both the correct formulation and the possibility of post processing in our framework.

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