An Adjoint Consistency Analysis for a Class of Hybrid Mixed Methods

JOCHEN SCHÜTZ*
Institut für Geometrie und Praktische Mathematik,
Templergraben 55, 52056 Aachen, Germany

AND

GEORG MAY†
Institut for Advanced Study in Computational Engineering Science,
Schinkelstraße 2, 52062 Aachen, Germany

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Abstract

Hybrid methods represent a classic discretization paradigm for elliptic equations. More recently, hybrid methods have been formulated for convection-diffusion problems, in particular compressible fluid flow. In [25], we have introduced a hybrid mixed method for the compressible Navier-Stokes equations as a combination of a hybridized DG scheme for the convective terms, and an $H(\mathrm{div}, \Omega)$-method for the diffusive part. Since hybrid methods are based on Galerkin's principle, the adjoint of a given hybrid discretization may be used for PDE-constraint optimal control problems, or error estimation, provided that the discretization is adjoint consistent. In the present paper, we extend the adjoint consistency analysis, previously reported for many DG schemes to the more complex hybrid methods. We prove adjoint consistency for a class of Hybrid Mixed schemes, which includes the hybridized DG schemes proposed by [19], as well as our recently proposed method ([25]). Hybrid Mixed discretizations, Hybridized Discontinuous Galerkin discretizations, Adjoint Consistency, compressible Navier-Stokes equations

1 Introduction

Recent years have seen tremendous development of solution strategies for high-order consistent discretization of the compressible Navier-Stokes equations, see [2, 13, 21]. Many well-known discretization methods are based on the Discontinuous Galerkin (DG) paradigm, cf. [23, 9, 8, 7, 6, 10, 2, 3, 1]. Such schemes achieve high order of consistency very easily by locally adding degrees of freedom. One drawback of these methods is the high amount of storage that is often

*Corresponding author. Email: schuetz@igpm.rwth-aachen.de
†Email: may@aices.rwth-aachen.de.de

needed when implicit methods are used for the computation of an approximate solution. One approach to reducing the amount of unknowns is to not express the solution in a cell-based fashion, but rather on the edges of the elements. This leads to *hybrid* methods, see, e.g., [5, 25, 11, 19].

Based on the work by [12], we have recently proposed a hybrid-mixed scheme for nonlinear convection-diffusion equations, including the compressible Navier-Stokes equations ([25]). The scheme lies somewhere in-between a hybridized DG and a mixed method. In [26], we have shown similarities between our scheme and a previously developed hybridized DG scheme by [19]. This is interesting from the point of view that we can identify a class of Hybrid Mixed methods, containing a subset of the hybrid methods defined in [11], for which the analysis to be presented applies.

Let a (possibly nonlinear) partial differential equation be given in weak formulation as the task of finding $w \in X$, such that

$$N_c(w; v) = 0 \quad \forall v \in X, \tag{1}$$

where N_c is a semi-linear form, and X is an appropriate function space. Let w_h be a numerical approximation, obtained with a Galerkin method. In engineering applications, one is often interested in the accuracy of a quantity $J(w_h)$ rather than that of w_h itself, where $J : X \to \mathbb{R}$ is a (nonlinear) functional on the solution space X. Naturally, one important task is to estimate the difference $J(w) - J(w_h)$. Applying a first-order Taylor's expansion yields

$$e_h := J(w) - J(w_h) = J'(w_h)(w - w_h) + O(\|w - w_h\|^2). \tag{2}$$

A reasonable approximation for e_h is thus given by the quantity $J'(w_h)(w - w_h)$. However, because of the unavailability of w this expression is usually not directly computable. Nevertheless, it can be approximated by a so-called adjoint procedure, see, e.g., [4, 16].

Roughly speaking, the principle is as follows. Assume that, in addition to the solution of the original (primal) problem (1), there exists an adjoint solution z, such that

$${N_c}'(w)(\mathrm{d}w; z) = J'(w)(\mathrm{d}w) \quad \forall \mathrm{d}w \in X, \tag{3}$$

where ${N_c}'(w)(\mathrm{d}w; z)$ denotes the derivative of $N_c(\cdot;\cdot)$ with respect to the first argument in direction $\mathrm{d}w$. For $\mathrm{d}w := w_h - w$, one can then easily obtain

$$J(w_h) - J(w) + O(\|\mathrm{d}w\|^2) = J'(w)(\mathrm{d}w) = {N_c}'(w)(\mathrm{d}w; z) = N_c(w_h; z) + O(\|\mathrm{d}w\|^2).$$

The essence of this is thus that the adjoint solution relates linearized changes in the functional to linearized changes in the residual. This is at the root of the analytical process we pursue in the following.

There are two obvious ways of solving the adjoint problem:

- Discretizing the adjoint equations (3) independently with a method of choice. This is called *continuous adjoint procedure.*

- Building the adjoint of the discretization that was used to obtain the approximate solution w_h. This is called *discrete adjoint procedure.*

The continuous adjoint approach has advantages in some cases where either the adjoint equation has a certain structure that can be exploited (see, e.g., [27]) or the method is not based on Galerkin's principle as for example in [17]. However, given a Galerkin method, the discrete adjoint procedure offers the advantage that a significant amount of the data structure, which may already be available in a numerical code to solve the primal problem, can often be re-used in the computation of the adjoint solution z_h.

Nevertheless, it is not trivial that such a discrete adjoint approach is viable. Depending on the discretization of the primal problem, the adjoint equations produced in this manner may not be a consistent approximation of the correctly posed adjoint differential equations. One quality measure of the discretization of the primal problem is thus *adjoint consistency*, which means precisely the property that the discrete adjoint is automatically consistent with the adjoint PDE. In the context of DG schemes approximating compressible flow problems, adjoint consistency has been discussed for example by [18], [14], and [20]. Another very important aspect of adjoint consistency is that it allows, under certain conditions, superconvergence of target functionals, as well as optimal L^2-norm convergence. Hence this property is also useful if one is not interested in the adjoint solution.

In this paper, we show adjoint consistency of a class of hybrid mixed methods for nonlinear convection-diffusion problems, including our newly-developed hybrid mixed scheme ([25]), and the hybridized Discontinuous Galerkin scheme introduced by [19].

The paper is outlined as follows. In Sec. 2, we introduce the governing equations, while Sec. 3 treats the associated adjoint equations. In Sec. 4, we briefly introduce a class of hybrid mixed methods and give the underlying Ansatz spaces. Sec. 5, which is the main part of this work, shows that the given class of hybrid mixed method is adjoint consistent. To make the ideas more transparent, these sections are each subdivided into two parts, treating first the simple, scalar convection-diffusion equation, and subsequently the more complex compressible Navier-Stokes equations. Sec. 6 offers conclusions.

2 Underlying Equations

The analysis presented in this paper is done first on the conceptually and technically simple case of the scalar convection-diffusion equation, and is then extended to the compressible Navier-Stokes equations.

2.1 Convection-Diffusion Equation

Consider a scalar (nonlinear) convection-diffusion equation given on a domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\partial\Omega$. In *primal* form, this equation can be written as

$$\begin{aligned} \nabla \cdot f(w) - \varepsilon \Delta w &= h \quad \forall x \in \Omega, \\ w &= g \quad \forall x \in \partial\Omega. \end{aligned} \tag{4}$$

We assume that the functions f, g and h are smooth. The diffusion coefficient $\varepsilon \in \mathbb{R}^+$ is assumed to be constant. Many hybridized discretization methods, including the recently proposed HDG methods by [19], and the DG/mixed method presented in [25], start from the *mixed* form

$$\begin{aligned} \sigma - \varepsilon \nabla w &= 0 \quad \forall x \in \Omega, \\ \nabla \cdot (f(w) - \sigma) &= h \quad \forall x \in \Omega, \\ w &= g \quad \forall x \in \partial\Omega. \end{aligned} \tag{5}$$

This will be used as a prototype for the more complicated compressible Navier-Stokes equations.

2.2 Navier-Stokes Equations

The compressible Navier-Stokes equations describe viscous, compressible flow in a domain $\Omega \subset \mathbb{R}^2$. We write these equations as

$$\begin{aligned} \sigma - f_v(w, \nabla w) &= 0 \quad \forall x \in \Omega, \\ \nabla \cdot (f(w) - \sigma) &= 0 \quad \forall x \in \Omega, \end{aligned} \tag{6}$$

subject to suitable boundary conditions, stated below. The state variable w is given by the vector of conserved variables $w = (\rho, \rho u, \rho v, E)$. Here ρ is the density, (u, v) is the velocity vector, and E is the total specific energy. The functions $f = (f_1, f_2)$ and $f_v(w, \nabla w) = (f_{v,1}, f_{v,2})$ are the convective and diffusive fluxes, respectively, given as

$$\begin{aligned} f_1 &= (\rho u, p + \rho u^2, \rho uv, u(E+p))^T, & f_2 &= (\rho v, \rho uv, p + \rho v^2, v(E+p))^T, \\ f_{v,1} &= (0, \tau_{11}, \tau_{21}, \tau_{11} u + \tau_{12} v + k T_{x_1})^T, & f_{v,2} &= (0, \tau_{12}, \tau_{22}, \tau_{21} u + \tau_{22} v + k T_{x_2})^T. \end{aligned}$$

Using the ideal gas law, temperature T and pressure p can be related to the conserved variables as

$$T = \frac{\mu\gamma}{k \cdot Pr} \left(\frac{E}{\rho} - \frac{1}{2}(u^2 + v^2) \right) = \frac{1}{(\gamma - 1)c_v} \frac{p}{\rho},$$

where $Pr = \frac{\mu c_p}{k}$ is the Prandtl number, which for air at moderate conditions is constant, with a value of $Pr = 0.72$. The thermal conductivity coefficient is denoted by k, while c_p and c_v are specific heats at constant pressure and constant volume, respectively. These are related via $\gamma = \frac{c_p}{c_v}$, where $\gamma = 1.4$ is again a constant for air at moderate conditions. Given a Newtonian fluid and assuming that the Stokes hypothesis holds, the viscous stress tensor τ can be written as

$$\tau = \mu \left(\nabla \widehat{w} + (\nabla \widehat{w})^T - \frac{2}{3} (\nabla \cdot \widehat{w}) Id \right),$$

where we have set $\widehat{w} := (u, v)^T$. The dynamic viscosity μ is taken, using Sutherland's law (cf. [28]), as

$$\mu = \frac{C_1 T^{3/2}}{T + C_2}$$

with C_1 and C_2 that can, for air at moderate temperatures, assumed to be constant. The viscous fluxes f_v are linear functions of ∇w, and hence allow a decoupling as

$$f_{v,i}(w) = \sum_{j=1}^{2} B_{ij}(w) w_{x_j} =: B(w)\nabla w \tag{7}$$

with matrices $B_{ij}(w)$ that depend nonlinearly on w.

Boundary conditions are imposed by setting $U_\Gamma(w) = 0 \ \forall x \in \partial\Omega$, where

$$U_\Gamma(w) := \begin{cases} (u, v, n \cdot \nabla T)^T & \text{boundary is adiabatic} \\ (u, v, T - T_{wall})^T & \text{boundary is isothermal.} \end{cases} \tag{8}$$

Let us note that the adiabatic boundary condition implies

$$(\sigma \cdot n)_4 = (f_v(w, \nabla w) \cdot n)_4 = 0 \quad \forall x \in \partial\Omega. \tag{9}$$

3 The Adjoint Equations

In this section, we derive the adjoint equations for both the convection-diffusion and the Navier-Stokes equations. This means essentially deriving a concrete expression of (3) for the weak formulation of both these equations. This will then be used as the correctly posed adjoint problem with which we require consistency when analyzing the discrete adjoint approach.

3.1 Adjoint Convection-Diffusion Equation

Definition 1 (Functional of interest for the convection-diffusion equation)**.** *Let* $\zeta \in L^2(\Omega)$ *and* $\xi \in L^2(\partial\Omega)$*. We define the functional of interest for the convection-diffusion equation, or* target functional*, for short, as*

$$J(w) := \int_\Omega \zeta w \, dx + \int_{\partial\Omega} \xi(\varepsilon \nabla w \cdot n) \, d\sigma. \tag{10}$$

The goal in an adjoint computation is to relate the linearized residual to the linearized functional error. This, however, is meaningful only for certain variations that respect the boundary conditions. To place this in a more precise framework, let us make the following definition:

Definition 2 (Suitable variations for the convection-diffusion equation.)**.** *Let* w *solve* (4)*. We call a function* $\mathrm{d}w : \Omega \to \mathbb{R}$ *a suitable variation to* w*, if* $w + \mathrm{d}w$ *fulfills the same boundary conditions as* w *does. More precisely,*

$$w(x) + \mathrm{d}w(x) = g(x) \quad \forall x \in \partial\Omega,$$

which implies $\mathrm{d}w(x) = 0 \ \forall x \in \partial\Omega$.

Remark 1. *Let us clarify the notion of suitable variations. In a variational formulation of* (4) *we typically seek a solution* $w \in H^1_g(\Omega) := \{f \in H^1(\Omega) | f = g \text{ on } \partial\Omega\}$*. If we consider variations* $\mathrm{d}w$ *to* w*, we must ensure that* $w + \mathrm{d}w$ *is still in* $H^1_g(\Omega)$*, which is essentially the condition of* suitable variations *as defined in Def. 2. The reason why we do* not *use the Sobolev spaces is the fact that for the compressible Navier-Stokes equations, suitable spaces are not known any more. The generalization of* suitable variations *is, however, straightforward.*

With these preliminaries in mind, we can state the adjoint equations for the convection-diffusion equation.

Lemma 1. *For J as given in* (10)*, the adjoint equation to the convection-diffusion equation* (4) *is given as*

$$\begin{aligned} -f'(w)^T \nabla z - \varepsilon \Delta z = \zeta, \qquad \forall x \in \Omega \\ z = -\xi, \quad \forall x \in \partial\Omega. \end{aligned} \tag{11}$$

Proof. The proof is well-known. We merely repeat the most important facts: The derivative of the convection-diffusion operator, i.e., the left-hand side of (4), with respect to w is given as $\mathcal{R}'(w)(\mathrm{d}w) := \nabla \cdot (f'(w)\mathrm{d}w) - \varepsilon \Delta \mathrm{d}w$. It is easy to check that the function z as defined in (11) fulfills

$$\int_\Omega \mathcal{R}'(w)\mathrm{d}wz \, dx = J'(w)(\mathrm{d}w)$$

for all suitable variations $\mathrm{d}w$ in the sense of Def. 2. □

Remark 2. *The proof of La. 1 reveals that, for this type of equation, the only suitable functionals are precisely given as in* (10)*.*

3.2 Adjoint Navier-Stokes Equations

In this section, we give a short overview of the adjoint Navier-Stokes equations, only repeating those details that are necessary for the adjoint consistency analysis. For a more thorough investigation, we refer to the work of [14], [15] and [24].

Definition 3 (Functional of interest for the Navier-Stokes equations)**.** *Let $\beta \in \mathbb{R}^2$ be a given vector, and $\partial\Omega$ be the wall-boundary with normal n pointing into the wall. We define the functional of interest for the Navier-Stokes equations (*target functional*) as*

$$J(w) := \int_{\partial\Omega} p(w)\beta \cdot n - (\tau\beta) \cdot n \, d\sigma. \tag{12}$$

Remark 3. *Note that upon choosing β as either*

$$\begin{aligned} \beta_d &= \frac{1}{C_\infty}(\cos(\alpha), \sin(\alpha))^T, \\ \beta_l &= \frac{1}{C_\infty}(-\sin(\alpha), \cos(\alpha))^T, \end{aligned}$$

the functional $J(w)$ represents the drag and lift coefficient, respectively, of a body submerged in a flow field. As usual, α denotes the angle of attack while C_∞ is a normalized reference value defined as $C_\infty = \frac{1}{2}\left(\gamma p_\infty M_\infty^2 l\right)$. Here l is a reference length, while p_∞ and M_∞ are reference values of pressure and Mach number, respectively.

Before stating the adjoint Navier-Stokes equations, we have to adapt the notion of *suitable variations* in the context of the Navier-Stokes equations:

Definition 4 (Suitable variations for the Navier-Stokes equations)**.** *Let w solve the Navier-Stokes equations* (6)*. We call a function* $\mathrm{d}w : \Omega \to \mathbb{R}^4$ *a suitable variation to w, if $w + \mathrm{d}w$ fulfills the linearized version of the boundary conditions imposed on w. More precisely,*

$$U'_\Gamma(w)\mathrm{d}w = 0 \quad \forall x \in \partial\Omega. \tag{13}$$

Lemma 2. *Let* $\mathrm{d}w = (\mathrm{d}w_1, \mathrm{d}w_2, \mathrm{d}w_3, \mathrm{d}w_4) : \Omega \to \mathbb{R}^4$*, and recall that* $w = (\rho, \rho u, \rho v, E)$*. Then $\mathrm{d}w$ is a suitable variation in the sense of Def. 4, iff it fulfills for all $x \in \partial\Omega$:*

$$\begin{aligned} 0 &= \mathrm{d}w_2 = \mathrm{d}w_3 \\ 0 &= \begin{cases} \frac{1}{c_v\rho^2}\left(\rho n \cdot \nabla \mathrm{d}w_4 - En \cdot \nabla \mathrm{d}w_1\right) & \textit{adiabatic boundary} \\ \frac{1}{c_v\rho^2}\left(-E\mathrm{d}w_1 + \rho \mathrm{d}w_4\right) & \textit{isothermal boundary.} \end{cases} \end{aligned}$$

Proof. Explicitly computing the expression (13) with U_Γ as defined in (8) yields the desired result. □

We state the adjoint Navier-Stokes equations in the following lemma:

Lemma 3. *For J as defined in Def.* (3)*, the corresponding adjoint Navier-Stokes equations are given as*

$$-\frac{d}{dw} f(w)^T \nabla z + \left(\frac{d}{dw} B(w)\nabla w\right)^T \nabla z - \nabla \cdot \left(B(w)^T \nabla z\right) = 0, \quad \forall x \in \Omega \tag{14}$$

$$U^*_\Gamma(z, \nabla z) = 0, \quad \forall x \in \partial\Omega, \tag{15}$$

where U^*_Γ is defined for an isothermal boundary as

$$U^*_\Gamma(z, \nabla z) = (z_2 - \beta_1, z_3 - \beta_2, z_4)$$

and for an adiabatic boundary as

$$U^*_\Gamma(z, \nabla z) = (z_2 - \beta_1, z_3 - \beta_2, \nabla z_4 \cdot n).$$

Proof. The above theorem is known, see, e.g., [16, 14], and we leave out some tedious details in the proof. For a detailed version, see [24].

The linearization of (12), i.e., $\frac{d}{dw} J(w)\, \mathrm{d}w$, can be written as (recall that $\tau \equiv \tau(w, \nabla w)$!)

$$J'(w)(\mathrm{d}w) = \int_{\partial\Omega} \left(\frac{d}{dw} p(w)\mathrm{d}w\right) \beta \cdot n \, d\sigma - \int_{\partial\Omega} \frac{d}{dw}(\tau\beta \cdot n)\mathrm{d}w - \frac{d}{d\nabla w}(\tau\beta \cdot n)\nabla \mathrm{d}w \, d\sigma. \tag{16}$$

We proceed with the linearized version of (the primal form of) (6), meaning that we consider the directional derivative of equations (6) at point w in direction $\mathrm{d}w$, which results in the term

$$\nabla \cdot \left(\frac{d}{dw} f(w)\mathrm{d}w\right) - \nabla \cdot \left(\frac{d}{dw} f_v(w, \nabla w)\mathrm{d}w\right) - \nabla \cdot \left(\frac{d}{d\nabla w} f_v(w, \nabla w)\nabla \mathrm{d}w\right). \tag{17}$$

Integrating (17) versus a smooth test-function z, and equating the result to (16) leads to

$$J'(w)(\mathrm{d}w) = \int_\Omega z^T \left(\nabla \cdot \left(\frac{d}{dw} f(w)\mathrm{d}w - \frac{d}{dw} f_v(w, \nabla w)\mathrm{d}w - \frac{d}{d\nabla w} f_v(w, \nabla w)\nabla \mathrm{d}w \right) \right) dx \tag{18}$$

for all test functions dw not disturbing the boundary conditions in the sense of Def. 4. After integration by parts and careful treatment of the boundary terms one can show that (18) is in fact fulfilled, provided that z fulfills the adjoint Navier-Stokes equations as given in (14)-(15). □

For our adjoint consistency analysis to be presented below, we need the following lemma, which is due to the adjoint boundary conditions imposed on the adjoint energy variable z_4:

Lemma 4. *Let $\varphi : \Omega \to \mathbb{R}^4$ be such that $\varphi_2 = \varphi_3 = 0$ on the boundary, and $-E\varphi_1 + \rho\varphi_4 = 0$ on an isothermal boundary, where ρ and E are first and fourth component of w, respectively. Then, there holds*

$$\varphi \cdot (B(w)^T \nabla z) n = 0.$$

Proof. A straightforward computation yields

$$\begin{aligned}
\varphi \cdot (B(w)^T \nabla z) n &= \varphi \cdot \left(\left(B_{11}^T z_{x_1} + B_{21}^T z_{x_2} \right) n_1 + \left(B_{12}^T z_{x_1} + B_{22}^T z_{x_2} \right) n_2 \right) \\
&= \frac{\mu}{\rho} \begin{pmatrix} -\frac{\gamma}{Pr} \frac{E}{\rho} \nabla z_4 \cdot n \\ \frac{4}{3} n_1 (z_2)_{x_1} - \frac{2n_1}{3} (z_3)_{x_2} + n_2 (z_2)_{x_2} + n_2 (z_3)_{x_1} \\ n_1 (z_2)_{x_2} + n_1 (z_3)_{x_1} - \frac{2n_2}{3} (z_2)_{x_1} + \frac{4n_2}{3} (z_3)_{x_2} \\ \frac{\gamma}{Pr} \nabla z_4 \cdot n \end{pmatrix} \cdot \varphi \\
&= \frac{\mu}{\rho} \frac{\gamma}{Pr} \nabla z_4 \cdot n \left(-\frac{E}{\rho} \varphi_1 + \varphi_4 \right) = \frac{\mu}{\rho^2} \frac{\gamma}{Pr} \nabla z_4 \cdot n \left(-E\varphi_1 + \rho\varphi_4 \right).
\end{aligned}$$

For an adiabatic boundary, where $\nabla z_4 \cdot n = 0$, the expression vanishes. Due to the requirements on φ, on an isothermal boundary, the expression vanishes also. □

4 Formulation of the Hybrid Mixed Method

In this section, we introduce both the discretization spaces and a class of Hybrid Mixed methods. The analysis has been motivated by the method as defined in [25], however, it extends to a broader class of methods which will be introduced in this section. We formulate the methods for both the convection-diffusion equation and the Navier-Stokes equations.

4.1 Preliminaries

Our domain Ω is assumed to be regularly triangulated as $\{\Omega_k\}_{k=1}^N$, where

$$\bigcup_{k=1}^N \overline{\Omega}_k = \overline{\Omega}, \qquad \Omega_k \cap \Omega_{k'} = \emptyset \quad \forall k \neq k'.$$

As our method operates on the skeleton of the mesh, we also need some definitions regarding edges: We define Γ as the set of both interior and boundary edges. Following standard nomenclature, we define an interior edge e as an intersection of two neighboring element boundaries $\partial\Omega_k \cap \partial\Omega_{k'}$ having a positive one-dimensional measure. A boundary edge e is defined as the intersection of an element boundary $\partial\Omega_k$ with the physical boundary $\partial\Omega$. Let us furthermore define $\Gamma_0 \subset \Gamma$ to be the set of all internal edges. $\Gamma =: \{\Gamma_k\}_{k=1}^{\widehat{N}}$, and the Γ_k are equipped with an orientation given by a normal vector n_k.

Remark 4. *Note that due to the definition of Ω_k, we implicitly assume that the physical domain Ω is such that the boundary edges align with the physical boundary $\partial\Omega$, more precisely:*

$$\Gamma \backslash \Gamma_0 = \partial\Omega.$$

We need this (rather standard) assumption in our analysis. However, as we allow arbitrary Ω_k, this is no restriction.

With this in mind, we can define our Ansatz spaces:

$$\begin{aligned}
H_h &:= \{\tau \in L^2(\Omega)^2 \quad |\tau_{|\Omega_k} \in H_{loc}(\Omega_k) \quad \forall k = 1, \dots N\}^d \\
V_h &:= \{\varphi \in L^2(\Omega) \quad |\varphi_{|\Omega_k} \in V_{loc}(\Omega_k) \quad \forall k = 1, \dots N\}^d \\
M_h &:= \{\mu \in L^2(\Gamma) \quad |\mu_{|e} \in M_{loc}(e) \quad \forall e \in \Gamma\}^d \\
\mathbb{X}_h &:= H_h \times V_h \times M_h,
\end{aligned}$$

where the local spaces differ for different methods. The spaces are needed as follows:

- The approximate solution w_h is a function in V_h.
- The approximate viscous flux σ_h is a function in H_h.
- The hybrid variable λ_h, which is an approximation of w on Γ, is a function in M_h.

Note that the definition of the spaces depends on the dimension of the system. We have $d = 1$ for the scalar convection-diffusion equation and $d = 4$ for the two-dimensional compressible Navier-Stokes equations.

Remark 5. *We wish to comment on the choices of the local spaces. Hybridized DG methods as presented by [19] use*

$$V_{loc}(\Omega_k) := \Pi^p(\Omega_k), \quad H_{loc}(\Omega_k) := \Pi^p(\Omega_k)^2, \qquad M_{loc}(e) := \Pi^p(e)$$

while the method in [25] relies on the choice

$$V_{loc}(\Omega_k) := \Pi^p(\Omega_k), \quad H_{loc}(\Omega_k) := \Pi^{p+1}(\Omega_k)^2, \quad M_{loc}(e) := \Pi^{p+1}(e).$$

Π^p denotes the space of polynomials up to degree p. Yet another method is obtained if one chooses, as in [12],

$$V_{loc}(\Omega_k) := \Pi^p(\Omega_k), \quad H_{loc}(\Omega_k) := RT_p(\Omega_k), \quad M_{loc}(e) := \Pi^p(e).$$

where RT_p is the Raviart-Thomas space of order p ([22]).

A last definition is concerned with the use of discontinuous functions on the edges:

Definition 5. *Let $x \in \Gamma_k$ and v be a function in V_h or H_h. We define*

$$v(x)^{\pm} := \lim_{\tau \to 0^+} v(x \pm \tau n_k),$$

and average and jump operators

$$\{v\} = \frac{v^+ + v^-}{2}, \quad [\![v]\!] = v^- n_k - v^+ n_k.$$

We define $v(x)^{\pm}$ on $\partial\Omega_k$ equivalently, however, instead of n_k, we then choose the outward facing unit normal.

4.2 Convection-Diffusion Equation

In [19, 25], Hybrid Mixed methods were proposed for the convection-diffusion equation. The starting point of these methods is the mixed formulation (5). We define a unified formulation of a hybrid-mixed method as follows:

Definition 6 (Hybrid Mixed Method for the convection-diffusion equation)**.** *Let $\boldsymbol{x}_h := (\sigma_h, w_h, \lambda_h) \in \mathbb{X}_h$ and $\boldsymbol{y}_h := (\tau_h, \varphi_h, \mu_h) \in \mathbb{X}_h$. We define a unified hybrid mixed method for eq.* (5) *as the task of finding $\boldsymbol{x}_h \in \mathbb{X}_h$, such that*

$$N(\boldsymbol{x}_h; \boldsymbol{y}_h) = \int_\Omega h\varphi_h \, dx \quad \forall \boldsymbol{y}_h \in \mathbb{X}_h, \tag{19}$$

where

$$N(\boldsymbol{x}_h; \boldsymbol{y}_h) := N_1(\boldsymbol{x}_h; \tau_h) + N_2(\boldsymbol{x}_h; \varphi_h) + N_3(\boldsymbol{x}_h; \mu_h),$$

and

$$N_1(\boldsymbol{x}_h; \tau_h) := \sum_{k=1}^{N} \left(\int_{\Omega_k} \sigma_h \cdot \tau_h + \varepsilon w_h \nabla \cdot \tau_h \, dx - \varepsilon \int_{\partial\Omega_k \setminus \partial\Omega} \lambda_h \tau_h^- \cdot n \, d\sigma - \varepsilon \int_{\partial\Omega_k \cap \partial\Omega} g \tau_h^- \cdot n \, d\sigma \right),$$

$$N_2(\boldsymbol{x}_h; \varphi_h) := \sum_{k=1}^{N} \left(- \int_{\Omega_k} f(w_h) \cdot \nabla \varphi_h \, dx + \int_{\partial\Omega_k \setminus \partial\Omega} \varphi_h^- \left(f(\lambda_h) \cdot n - \alpha(\lambda_h - w_h^-) \right) \, d\sigma \right.$$
$$\left. + \int_{\partial\Omega_k \cap \partial\Omega} \varphi_h^- \left(f(g) \cdot n - (\lambda_h - g) \right) \, d\sigma - \int_{\Omega_k} \nabla \cdot \sigma_h \varphi_h \, dx \right),$$

$$N_3(\boldsymbol{x}_h; \mu_h) := \int_{\Gamma_0} \mu_h \left(\sigma_h^- \cdot n - \sigma_h^+ \cdot n + \alpha(2\lambda_h - w_h^- - w_h^+) \right) \, d\sigma + \int_{\Gamma \setminus \Gamma_0} \mu_h (\lambda_h - g) \, d\sigma.$$

Remark 6 (Meaning of N_i)**.** *N_1 and N_2 are direct discretizations of* (5)*. The third line ensures that the combined numerical flux $f(\lambda_h) \cdot n - \alpha(\lambda_h - w_h^-) - \sigma_h^- \cdot n$ is continuous over the element boundaries.*

Remark 7 (Choice of α)**.** *The choice of the parameter α depends on the method at hand. For the method as defined in [25], it denotes a Lax-Friedrichs-type coefficient, while for the method proposed in [19], it denotes the sum of a Lax-Friedrichs-type coefficient and an LDG-stabilization parameter stemming from the viscous discretization, see [1]. While we treat it as constant, we note in Remark 10 that the analysis is not substantially changed if α depends on, for example, λ_h.*

Remark 8. *Note that in the third row, the term* $-(\lambda_h - g)$ *was not present in the original formulation in [25]. However, due to the fact that the last equation enforces* $\lambda_h = g$ *weakly on the boundary, this term does not change the original method. It is, however, needed for the adjoint consistency analysis.*

4.3 Navier-Stokes Equations

Similar to the previous subsection, we merely state the definition of the method for the compressible Navier-Stokes equations (6). In [21], the authors have extended their scheme for the convection-diffusion problem, presented in [19], to the compressible Navier-Stokes equations. The variable σ is defined there as $\sigma = \nabla w$, while we define $\sigma = f_v(w, \nabla w)$ (cf. eq. 6). However, using our unified formulation with the hybridized DG spaces, results in a hybridized DG scheme for the Navier-Stokes equations, very similar to that proposed in [21]. It should be pointed out that adjoint consistency does not depend on this choice. An analysis very similar to that shown below can be used to show adjoint consistency for a unified formulation defining $\sigma = \nabla w$.

Definition 7 (Hybrid Mixed Method for the Navier-Stokes equations)**.** *Let* $\boldsymbol{x}_h := (\sigma_h, w_h, \lambda_h) \in \mathbb{X}_h$*, and* $\boldsymbol{y}_h := (\tau_h, \varphi_h, \mu_h) \in \mathbb{X}_h$*. We define a unified hybrid mixed method for the compressible Navier-Stokes equations* (6) *as*

$$N(\boldsymbol{x}_h; \boldsymbol{y}_h) = 0 \quad \forall \boldsymbol{y}_h \in \mathbb{X}_h. \tag{20}$$

where

$$N(\boldsymbol{x}_h; \boldsymbol{y}_h) := N_1(\boldsymbol{x}_h; \tau_h) + N_2(\boldsymbol{x}_h; \varphi_h) + N_3(\boldsymbol{x}_h; \mu_h),$$

with

$$\begin{aligned}
N_1(\boldsymbol{x}_h; \tau_h) := & \int_\Omega \sigma_h \cdot \tau_h \; dx + \int_\Omega w_h \nabla \cdot (B(w_h)^T \tau_h) \, dx - \int_{\Gamma_0} \lambda_h \cdot [\![B(w_h)^T \tau_h]\!] \, d\sigma \\
& - \int_{\Gamma \setminus \Gamma_0} w_{\partial\Omega}(\lambda_h) \cdot (B(w)^T \tau_h) n \, d\sigma \\
N_2(\boldsymbol{x}_h; \varphi_h) := & - \int_\Omega f(w_h) \nabla \varphi_h \; dx - \int_\Omega \nabla \cdot \sigma_h \varphi_h \, dx \\
& + \int_{\Gamma_0} \left(\varphi_h^- - \varphi_h^+ \right) f(\lambda_h) \cdot n \; d\sigma - \int_{\Gamma_0} \varphi_h^- \left(\alpha(\lambda_h - w_h^-) \right) + \varphi_h^+ \left(\alpha(\lambda_h - w_h^+) \right) d\sigma \\
& + \int_{\Gamma \setminus \Gamma_0} \varphi_h^- \left(f(w_{\partial\Omega}(\lambda_h)) \cdot n - \alpha(\lambda_h - w_{\partial\Omega}(w_h^-)) \right) \; d\sigma \\
N_3(\boldsymbol{x}_h; \mu_h) := & + \int_{\Gamma_0} \mu_h \alpha(2\lambda_h - w_h^- - w_h^+) + \int_{\Gamma_0} \mu_h (\sigma_h^- \cdot n - \sigma_h^+ \cdot n) \, d\sigma \\
& + \int_{\Gamma \setminus \Gamma_0} \mu_h \alpha(\lambda_h - w_{\partial\Omega}(w_h^-)) \, d\sigma + \int_{\Gamma \setminus \Gamma_0} \mu_h (\sigma_h^- \cdot n - \sigma_{\partial\Omega}(\sigma_h^-) \cdot n) \, d\sigma.
\end{aligned}$$

The boundary operators $w_{\partial\Omega} \equiv w_{\partial\Omega}(w_h^-)$ *and* $\sigma_{\partial\Omega} \equiv \sigma_{\partial\Omega}(\sigma_h^-)$ *are defined for* adiabatic *boundary conditions as*

$$\begin{aligned}
w_{\partial\Omega}(w_h) \equiv w_{\partial\Omega}((\rho_h, \rho_h u_h, \rho_h v_h, E_h)) = (\rho_h, 0, 0, E_h), \\
\sigma_{\partial\Omega}(\sigma_h) \cdot n = (0, \sigma_{h,2} \cdot n, \sigma_{h,3} \cdot n, 0),
\end{aligned}$$

while for isothermal *boundary conditions, one defines*

$$\begin{aligned}
w_{\partial\Omega}(w_h) \equiv w_{\partial\Omega}((\rho_h, \rho_h u_h, \rho_h v_h, E_h)) = (\widehat{\rho}, 0, 0, \widehat{E}), \\
\sigma_{\partial\Omega}(\sigma_h) \cdot n = (0, \sigma_{h,2} \cdot n, \sigma_{h,3} \cdot n, \sigma_{h,4} \cdot n),
\end{aligned}$$

where $\widehat{\rho}$ and $\widehat{E}$ are such that $T \equiv T(\widehat{\rho}, \widehat{E}) = T_{wall}$, thereby incorporating the isothermal boundary conditions.

5 Adjoint Consistency Analysis

In this section, we show that the class of Hybrid Mixed methods as defined in Sec. 4 is in fact adjoint consistent. Before briefly introducing the concept of adjoint consistency, we note the following definition regarding *consistent* functionals:

Definition 8. *A functional J_h is consistent with a functional J, if*

$$J_h(\boldsymbol{x}) = J(\boldsymbol{x})$$

for the solution vector $\boldsymbol{x} = (\sigma, w, w_{|\Gamma})$.

In a straightforward manner, as we have seen in the introduction, we formulate the discrete adjoint, as in the continuous case, just replacing the weak formulation by the discrete Galerkin formulation. This procedure yields the *discrete adjoint equations*. The definition of discrete adjoint is independent of whether one discretizes Navier-Stokes or convection-diffusion equations.

Definition 9 (Discrete Adjoint Equations)**.** *The discrete adjoint equations for the hybrid mixed methods as given in* (19) *and* (20)*, respectively, is the solution $\boldsymbol{z}_h^d := (\tau_h^z, \varphi_h^z, \mu_h^z) \in \mathbb{X}_h$ such that*

$$N'(\boldsymbol{x}_h)(\mathbf{d}\boldsymbol{y}_h; \boldsymbol{z}_h^d) = J_h'(\boldsymbol{x}_h)(\mathbf{d}\boldsymbol{y}_h) \quad \forall \mathbf{d}\boldsymbol{y}_h \in \mathbb{X}_h, \tag{21}$$

where J_h is a functional that is consistent with J, and $\boldsymbol{x}_h = (\sigma_h, w_h, \lambda_h)$ is the approximate solution computed in the primal problem ((19) *or* (20)*, respectively)*

The tuple $\boldsymbol{x}_h = (\sigma_h, w_h, \lambda_h)$ represents the approximation to the viscous flux σ, the unknown solution w and the unknown solution on the edges of the triangulation $w_{|\Gamma}$. The tuple $\boldsymbol{z}_h^d := (\tau_h^z, \varphi_h^z, \mu_h^z)$ denotes approximations to quantities related to the adjoint solution z. Our analysis will show that they are, provided that (21) is well-posed, approximations to $(-\nabla z, z, z_{|\Gamma})$.

Adjoint Consistency is defined in a way similar to 'normal' consistency: Substituting the exact solutions into (21), meaning to substitute $\boldsymbol{x}_h$ by $\boldsymbol{x} := (\sigma, w, w_{|\Gamma})$, and substituting $\boldsymbol{z}_h^d$ by $\boldsymbol{z} := (-\nabla z, z, z_{|\Gamma})$, (21) should be fulfilled:

Definition 10. *A method N is said to be* adjoint consistent *if*

$$N'(\boldsymbol{x})(\mathbf{d}\boldsymbol{y}_h; \boldsymbol{z}) = J_h'(\boldsymbol{x})(\mathbf{d}\boldsymbol{y}_h) \quad \forall \mathbf{d}\boldsymbol{y}_h \in \mathbb{X}_h, \tag{22}$$

for a functional J_h consistent with J. Here, we have defined

$$\boldsymbol{x} := (\sigma, w, w_{|\Gamma}), \quad \boldsymbol{z} := (-\nabla z, z, z_{|\Gamma}). \tag{23}$$

Remark 9. *Computing not only w as in a standard DG scheme, but also derived quantities σ and λ, makes it not so obvious which quantities related to the adjoint* *solution z are approximated by the discrete adjoint approach* (21)*. This is why for the convection-diffusion case, we start with the expression*

$$N'(\boldsymbol{x})(\mathbf{d}\boldsymbol{y}_h; \boldsymbol{z}) = J'_h(\boldsymbol{x})(\mathbf{d}\boldsymbol{y}_h) \tag{24}$$

for a generic $\boldsymbol{z} := (\tau, \varphi, \mu)$ and derive conditions on the quantites τ, φ and μ, such that (24) holds for all quantities $\mathbf{d}\boldsymbol{y}_h \in \mathbb{X}_h$. This will then justify the choice of $\boldsymbol{z}$ in (23).

The section is subdivided in the usual way, treating the conceptually simple convection-diffusion equation first, and then extending the analysis to the compressible Navier-Stokes equations.

5.1 Convection-Diffusion Equation

Noting that $\sigma := \varepsilon \nabla w$, a consistent discretization of (10) is

$$J_h(\boldsymbol{x}_h) := \int_\Omega \zeta w_h \, dx + \int_{\partial\Omega} \xi(\sigma_h \cdot n) \, d\sigma, \tag{25}$$

with the derivative

$$J'_h(\boldsymbol{x}_h)(\mathbf{d}\boldsymbol{y}_h) = \int_\Omega \zeta \mathrm{d}w \, dx + \int_{\partial\Omega} \xi(\mathrm{d}\sigma \cdot n) \, d\sigma, \tag{26}$$

where $\mathbf{d}\boldsymbol{y}_h := (\mathrm{d}\sigma, \mathrm{d}w, \mathrm{d}\lambda) \in \mathbb{X}_h$. Adjoint consistency means that we have to show that (24) holds for all $\mathbf{d}\boldsymbol{y}_h$ for a suitable choice of $\boldsymbol{z} := (\tau, \varphi, \mu)$, meaning that

$$J'_h(\boldsymbol{x})(\mathbf{d}\boldsymbol{y}_h) = N'(\boldsymbol{x})(\mathbf{d}\boldsymbol{y}_h; \boldsymbol{z}) \quad \forall \mathbf{d}\boldsymbol{y}_h \in \mathbb{X}_h \tag{27}$$

where τ, φ and μ should in some way relate to the exact adjoint z as given implicitly by (11). We will derive conditions on these quantities in the course of the analysis. Let us begin with a very first assumption, which is present in all adjoint consistency analysis: We assume that w, τ, φ and μ are smooth.

Theorem 1. *The hybrid mixed method as defined in Def. 6 is adjoint consistent.*

Proof. To show (27), we note that it is proved as soon as

$$\begin{aligned} N'(\boldsymbol{x})(\mathbf{d}\boldsymbol{y}_h; \boldsymbol{z}) &= N'(\boldsymbol{x})(\mathrm{d}\sigma, 0, 0; \boldsymbol{z}) + N'(\boldsymbol{x})(0, \mathrm{d}w, 0; \boldsymbol{z}) + N'(\boldsymbol{x})(0, 0, \mathrm{d}\lambda; \boldsymbol{z}) \\ &= J'_h(\boldsymbol{x})(\mathrm{d}\sigma, 0, 0) \quad + J'_h(\boldsymbol{x})(0, \mathrm{d}w, 0) \quad + 0, \end{aligned} \tag{28}$$

holds, which clearly follows from the linearity of the derivative.

We start with the terms in (28) that correspond to $\mathrm{d}\sigma$, yielding

$$\begin{aligned} N'(\boldsymbol{x})(\mathrm{d}\sigma, 0, 0; \boldsymbol{z}) &= \int_\Omega \mathrm{d}\sigma \cdot \tau \, dx - \int_\Omega \nabla \cdot \mathrm{d}\sigma \varphi \, dx + \int_{\Gamma_0} \mu [\![\mathrm{d}\sigma]\!] \, d\sigma \\ &= \int_\Omega \mathrm{d}\sigma \cdot (\tau + \nabla\varphi) \, dx - \int_{\Gamma_0} [\![\mathrm{d}\sigma]\!] \varphi \, d\sigma - \int_{\Gamma \setminus \Gamma_0} (\mathrm{d}\sigma \cdot n) \varphi \, d\sigma + \int_{\Gamma_0} \mu [\![\mathrm{d}\sigma]\!] \, d\sigma \\ &= \int_\Omega \mathrm{d}\sigma \cdot (\tau + \nabla\varphi) \, dx - \int_{\Gamma \setminus \Gamma_0} (\mathrm{d}\sigma \cdot n) \varphi \, d\sigma + \int_{\Gamma_0} (\mu - \varphi) [\![\mathrm{d}\sigma]\!] \, d\sigma \end{aligned}$$

We assume that $\tau = -\nabla\varphi$, $\varphi_{|\Gamma_0} = \mu$ and $\varphi_{|\partial\Omega} = z_{\partial\Omega}$ $(= -\xi)$. Under this assumption, we can conclude

$$N'(\boldsymbol{x})(\mathrm{d}\sigma, 0, 0; \boldsymbol{z}) = J_h'(\boldsymbol{x})(\mathrm{d}\sigma, 0, 0).$$

We continue with the terms depending on $\mathrm{d}w$, exploiting the assumptions we have already made:

$$\begin{aligned} N'(\boldsymbol{x})(0, \mathrm{d}w, 0; \boldsymbol{z}) &= \int_\Omega -\varepsilon \mathrm{d}w \nabla\cdot\nabla\varphi - f'(w)\mathrm{d}w\cdot\nabla\varphi\, dx + \sum_{k=1}^N \int_{\partial\Omega_k\setminus\partial\Omega} \alpha\varphi \mathrm{d}w^-\, d\sigma - \int_{\Gamma_0} \alpha\mu(\mathrm{d}w^- + \mathrm{d}w^+)\, d\sigma \\ &= \int_\Omega \mathrm{d}w \left(-\varepsilon\Delta\varphi - f'(w)^T\nabla\varphi\right) dx + \int_{\Gamma_0} \alpha(\varphi-\mu)(\mathrm{d}w^- + \mathrm{d}w^+)\, dx \\ &= \int_\Omega \mathrm{d}w \left(-\varepsilon\Delta\varphi - f'(w)^T\nabla\varphi\right) dx. \end{aligned}$$

We assume that $\varphi = z$. Together with the previous assumptions, this yields the identities $\tau = -\nabla z$ and $\mu = z_{|\Gamma}$. With this, we can conclude

$$N'(\boldsymbol{x})(0, \mathrm{d}w, 0; \boldsymbol{z}) = \int_\Omega \mathrm{d}w\ \zeta\, dx = J_h'(\boldsymbol{x})(0, \mathrm{d}w, 0),$$

which is due to the definition of the adjoint equation (11). The last part involves those terms containing $\mathrm{d}\lambda$:

$$\begin{aligned} N'(\boldsymbol{x})(0, 0, \mathrm{d}\lambda; \boldsymbol{z}) &= \sum_{k=1}^N \left(\varepsilon \int_{\partial\Omega_k\setminus\partial\Omega} \mathrm{d}\lambda\nabla z\cdot n + z\left(f'(w)\cdot n\mathrm{d}\lambda - \alpha\mathrm{d}\lambda\right) d\sigma - \int_{\partial\Omega_k\cap\partial\Omega} z\mathrm{d}\lambda\, d\sigma\right) \\ &\quad + \int_{\Gamma_0} 2z\alpha\mathrm{d}\lambda\, d\sigma + \int_{\Gamma\setminus\Gamma_0} z\mathrm{d}\lambda\, d\sigma \\ &= \int_{\Gamma_0} \varepsilon\mathrm{d}\lambda [\![\nabla z\cdot n]\!] + \mathrm{d}\lambda [\![f'(w)^T z]\!] - 2\alpha z\mathrm{d}\lambda\, d\sigma - \int_{\Gamma\setminus\Gamma_0} z\mathrm{d}\lambda\, d\sigma \\ &\quad + \int_{\Gamma_0} 2z\alpha\mathrm{d}\lambda\, d\sigma + \int_{\Gamma\setminus\Gamma_0} z\mathrm{d}\lambda\, d\sigma \\ &= 0 \end{aligned}$$

This, together with (28) proves that the method is adjoint consistent. We have thus proven that (27) holds with τ, φ, μ as defined by the assumptions. To summarize, we have proved that

$$J_h'(\boldsymbol{x})(\mathbf{d}\boldsymbol{y}_h) = N'(\boldsymbol{x})(\mathbf{d}\boldsymbol{y}_h; \boldsymbol{z}) \quad \forall \mathbf{d}\boldsymbol{y}_h \in \mathbb{X}_h, \tag{29}$$

with $\boldsymbol{z} := (-\nabla z, z, z_{|\Gamma})$. □

Remark 10. *Usually, α is not a constant, but a nonlinear function of λ_h. (For the method defined in [25], it is a Lax-Friedrichs-type constant, and could therefore be defined as $\alpha = \max\{|c|\}$, where c is an eigenvalue of $f'(\lambda_h)\cdot n$.) Such a choice does not destroy the adjoint consistency property. This can be easily seen when considering those terms in (19) where α appears, i.e. terms of the form $\alpha(\lambda_h - w_h^-)$. Upon differentiation, and using the product rule, one obtains $\alpha'(\lambda_h - w_h^-) + \alpha(\lambda_h - w_h^-)'$. The second term has been treated in our analysis, and the first term is zero, if one substitutes the exact solution for λ_h and w_h. The same observation holds true for the discretization of the Navier-Stokes equations.*

5.2 Navier-Stokes Equations

A consistent modification of the target functional given in (12) is achieved by considering

$$J_h(\boldsymbol{x}_h) := \int_{\partial\Omega} p(w_{\partial\Omega}(\lambda_h))\beta \cdot n - (\sigma_{h,2} \cdot n, \sigma_{h,3} \cdot n)\beta \, d\sigma. \tag{30}$$

Theorem 2. *The hybrid mixed method as defined in Def. 7 is adjoint consistent.*

Proof. In order to show adjoint consistency, we have to show that the following statement is true:

$$N'(\boldsymbol{x})(\mathbf{d}\boldsymbol{y}_h; \boldsymbol{z}) = J_h'(\mathbf{d}\boldsymbol{y}_h) \quad \forall \mathbf{d}\boldsymbol{y}_h \in \mathbb{X}_h, \tag{31}$$

for $\boldsymbol{x} = (\sigma, w, w_{|\Gamma})$ and $\boldsymbol{z} = (-\nabla z, z, z_{|\Gamma})$. The procedure is similar to that in the previous section, so let us begin with the expression

$$\begin{aligned} N'(\boldsymbol{x})(\mathrm{d}\sigma, 0, 0; \boldsymbol{z}) = &- \int_\Omega \mathrm{d}\sigma \nabla z \, dx - \int_\Omega \nabla \cdot \mathrm{d}\sigma \; z \, dx + \int_{\Gamma_0} z [\![\mathrm{d}\sigma]\!] \, d\sigma \\ &+ \int_{\Gamma \setminus \Gamma_0} z \mathrm{d}\sigma \cdot n - z \cdot \frac{d}{d\sigma}(0, \sigma_2 \cdot n, \sigma_3 \cdot n, (\sigma_{\partial\Omega})_4 \cdot n) \mathrm{d}\sigma \, d\sigma \\ = &- \int_{\Gamma \setminus \Gamma_0} z \cdot \frac{d}{d\sigma}(0, \sigma_2 \cdot n, \sigma_3 \cdot n, (\sigma_{\partial\Omega})_4 \cdot n) \mathrm{d}\sigma \, d\sigma \\ = &J_h'(\boldsymbol{x})(\mathrm{d}\sigma, 0, 0) \end{aligned}$$

Note that the term $z_4 (\sigma_{\partial\Omega})_4 \cdot n$ is always zero, as either $(\sigma_{\partial\Omega})_4 \cdot n = 0$ (adiabatic boundary) or $z_4 = 0$ (isothermal boundary). The remaining steps follow from the boundary conditions imposed on z.

To simplify matters, we split

$$N'(\boldsymbol{x})(0, \mathrm{d}w, 0; \boldsymbol{z}) = N_\Omega'(\boldsymbol{x})(0, \mathrm{d}w, 0; \boldsymbol{z}) + N_\Gamma'(\boldsymbol{x})(0, \mathrm{d}w, 0; \boldsymbol{z})$$

into two parts, one consisting of all volume integrals, and one consisting of all face integrals. For the volume part, there holds

$$\begin{aligned} N_\Omega'(\boldsymbol{x})(0, \mathrm{d}w, 0; \boldsymbol{z}) = &- \int_\Omega \mathrm{dw} \nabla \cdot (B(w)^T \nabla z) dx - \int_\Omega w \nabla \cdot (\frac{d}{dw} B(w)^T \mathrm{dw} \nabla z) dx - \int_\Omega f'(w) \mathrm{dw} \nabla z \, dx \\ = &\int_\Omega \mathrm{dw} \left((\frac{d}{dw} B(w) \nabla w)^T \nabla z - \nabla \cdot (B(w)^T \nabla z) - f'(w)^T \nabla z \right) dx \\ &- \int_{\Gamma_0} w \left(\left(\frac{d}{dw} B(w)^T \mathrm{dw}^- - \frac{d}{dw} B(w)^T \mathrm{dw}^+ \right) \nabla z n \right) d\sigma - \int_{\Gamma \setminus \Gamma_0} w \left(\frac{d}{dw} B(w)^T \mathrm{dw} \nabla z n \right) d\sigma \\ = &- \int_{\Gamma_0} w \left(\left(\frac{d}{dw} B(w)^T \mathrm{dw}^- - \frac{d}{dw} B(w)^T \mathrm{dw}^+ \right) \nabla z n \right) d\sigma - \int_{\Gamma \setminus \Gamma_0} w \left(\frac{d}{dw} B(w)^T \mathrm{dw} \nabla z n \right) d\sigma, \end{aligned}$$

while the face part can be written as

$$
\begin{aligned}
N'_\Gamma(\boldsymbol{x})(0,\mathrm{d}w,0;\boldsymbol{z}) = &\int_{\Gamma_0} w\left(\frac{d}{dw}B(w)^T\mathrm{dw}^- - \frac{d}{dw}B(w)^T\mathrm{dw}^+\right)\nabla z n\,d\sigma + \int_{\Gamma\setminus\Gamma_0} w\cdot(\frac{d}{dw}B(w)^T\mathrm{dw}\nabla z)n\,d\sigma \\
&+\int_{\Gamma_0} z\alpha(\mathrm{dw}^- + \mathrm{dw}^+)\,d\sigma + \int_{\Gamma\setminus\Gamma_0} z\alpha w'_{\partial\Omega}(w)\mathrm{d}w\,d\sigma \\
&-\int_{\Gamma_0} z\alpha(\mathrm{dw}^- + \mathrm{dw}^+)\,d\sigma - \int_{\Gamma\setminus\Gamma_0} z\alpha w'_{\partial\Omega}(w)\mathrm{d}w\,d\sigma \\
=&\int_{\Gamma_0} w\left(\frac{d}{dw}B(w)^T\mathrm{dw}^- - \frac{d}{dw}B(w)^T\mathrm{dw}^+\right)\nabla z n\,d\sigma + \int_{\Gamma\setminus\Gamma_0} w\cdot(\frac{d}{dw}B(w)^T\mathrm{dw}\nabla z)n\,d\sigma
\end{aligned}
$$

Summarizing, one obtains

$$
N'(\boldsymbol{x})(0,\mathrm{d}w,0;\boldsymbol{z}) = N'_\Omega(0,\mathrm{d}w,0;\boldsymbol{z}) + N'_\Gamma(0,\mathrm{d}w,0;\boldsymbol{z}) = 0 = J'_h(\boldsymbol{x})(0,\mathrm{d}w,0).
$$

The remaining term is $N'(\boldsymbol{x})(0,0,\mathrm{d}\lambda;\boldsymbol{z})$. To simplify the notational workload, let us note that due to the (assumed) smoothness of the quantities z and w, both $B(w^-)\nabla z^- - B(w^+)\nabla z^+$ and $z^- - z^+$ vanish. This leaves us with

$$
\begin{aligned}
N'(\boldsymbol{x})(0,0,\mathrm{d}\lambda;\boldsymbol{z}) = &\int_{\Gamma_0} -2\alpha\mathrm{d}\lambda z\,d\sigma + \int_{\Gamma_0} z\alpha 2\mathrm{d}\lambda\,dx + \int_{\Gamma\setminus\Gamma_0} -\alpha\mathrm{d}\lambda z\,d\sigma + \int_{\Gamma\setminus\Gamma_0} z\alpha\mathrm{d}\lambda\,dx \\
&+\int_{\Gamma\setminus\Gamma_0} w'_{\partial\Omega}(w)\mathrm{d}\lambda\cdot(B(w)^T\nabla z)n\,d\sigma + \int_{\Gamma\setminus\Gamma_0} z\frac{d}{dw}f(w_{\partial\Omega}(w))n\mathrm{d}\lambda\,d\sigma \\
=&\int_{\Gamma\setminus\Gamma_0} z\frac{d}{dw}(0,p(w_{\partial\Omega}(w))n_1,p(w_{\partial\Omega}(w))n_2,0)\mathrm{d}\lambda\,d\sigma \\
=&J'_h(\boldsymbol{x})(0,0,\mathrm{d}\lambda)
\end{aligned}
$$

We have thus shown that

$$
N'(\boldsymbol{x})(\mathbf{d}\boldsymbol{y}_h;\boldsymbol{z}) = J'_h(\boldsymbol{x})(\mathbf{d}\boldsymbol{y}_h) \quad \forall \mathbf{d}\boldsymbol{y}_h \in \mathbb{X}_h,
$$

which is clearly the adjoint consistency property. □

The second to last step in the proof requires some explanation:

- The quantity $f(w_{\partial\Omega}(w))$ is, for all w, equal to the expression $(0, pn_1, pn_2, 0)$, which is due to the fact that $w_{\partial\Omega}(w)_2 = w_{\partial\Omega}(w)_3 = 0$. Evaluating boundary fluxes in this manner is a modification that has already been done by [18] and [14]. (The quantity p is of course evaluated with the discrete quantities $w_{\partial\Omega}(w)$.)
- Adiabatic boundary: Due to La. 4, the quantity $\varphi\cdot(B(w)^T\nabla z)n$ is zero given that $\varphi_2 = \varphi_3 = 0$. It is easily seen that setting $\varphi = w'_{\partial\Omega}(w)\mathrm{d}\lambda$ fulfills this claim.
- Isothermal boundary: The quantity $\varphi = w'_{\partial\Omega}(w)\mathrm{d}\lambda$ still fulfills $\varphi_2 = \varphi_3 = 0$. Furthermore, due to the fact that $T(w_{\partial\Omega}(w)) = T_{wall}$ as claimed in Sec. 4.3, one easily computes

$$
\begin{aligned}
0 &= \frac{d}{dw}T(w_{\partial\Omega}(w))\mathrm{d}\lambda = T'(w_{\partial\Omega}(w))w'_{\partial\Omega}(w)\mathrm{d}\lambda = T'(w_{\partial\Omega}(w))\varphi \\
&= T'(w)\varphi = \frac{1}{c_v}(-\frac{E}{\rho^2},0,0,\frac{1}{\rho})\varphi \\
\Leftrightarrow 0 &= -E\varphi_1 + \varphi_4.
\end{aligned}
$$

As stated in La. 4, for isothermal walls this is precisely the condition for $\varphi \cdot (B(w)^T \nabla z)n = 0$.

6 Conclusions

We have presented an adjoint consistency analysis for a class of Hybrid Mixed methods, including the Hybridized Discontinuous Galerkin method presented in [19] and the method developed in [25]. In contrast to [14], we do not need to include additional terms in the functional to make the method adjoint consistent.

Adjoint methods are standard tools in the context of Discontinuous Galerkin methods. Future work should show that adjoint methods in the context of Hybrid Mixed methods work equally well in practice.

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