A parameter-dependent smoother for the multigrid method

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Parameter-dependent problems

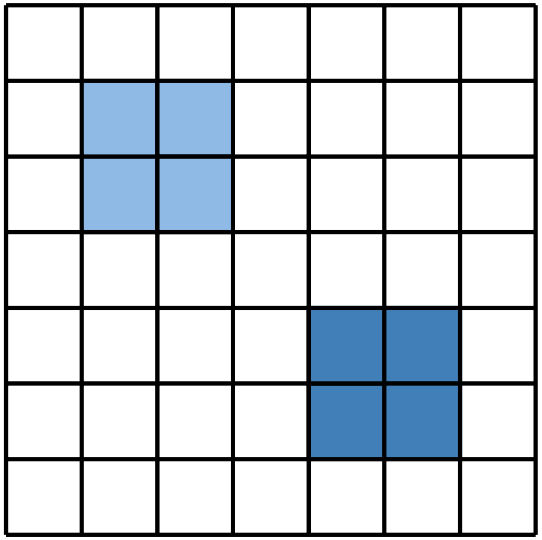
Example PDE:

$$-\nabla \cdot (\sigma(x,p)\nabla u(x,p)) = f(x,p) \quad \text{in } \Omega \times \mathcal{P},\ \Omega \subset \mathbb{R}^d,$$
$$u(x,p) = 0 \quad \text{on } \partial\Omega,$$

where $\sigma(x,p)$ is an uncertain diffusion coefficient depending on p.

Assume: $\sigma(x,p) := \sigma_0 + \sum_{\nu=1}^{d} p^{(\nu)}(x)$,

with $p^{(\nu)}$ uniform in $[0,1]$.



Geometry

1. *Grasedyck, Löbbert '18*
2. *Khoromskij, Oseledets '10*
3. *Khoromskij, Schwab '11*
4. *Kressner, Tobler '11*
5. *Matthies, Zander '12*
6. *Schwab, Todor '06*

Parameter-dependent problems

- Discretization with d different parameters leads to

$$\left(A^{(0)} + \sum_{\nu=1}^{d} p^{(\nu)} A^{(\nu)} \right) u(p) = f(p) \ ,$$

where each $A^{(\nu)}$ is parameter-independent.

A *fast solver* is needed, e.g., the **multigrid method**

- With n choices for every parameter

$$\mathcal{O}(n^d) \text{ linear systems}$$

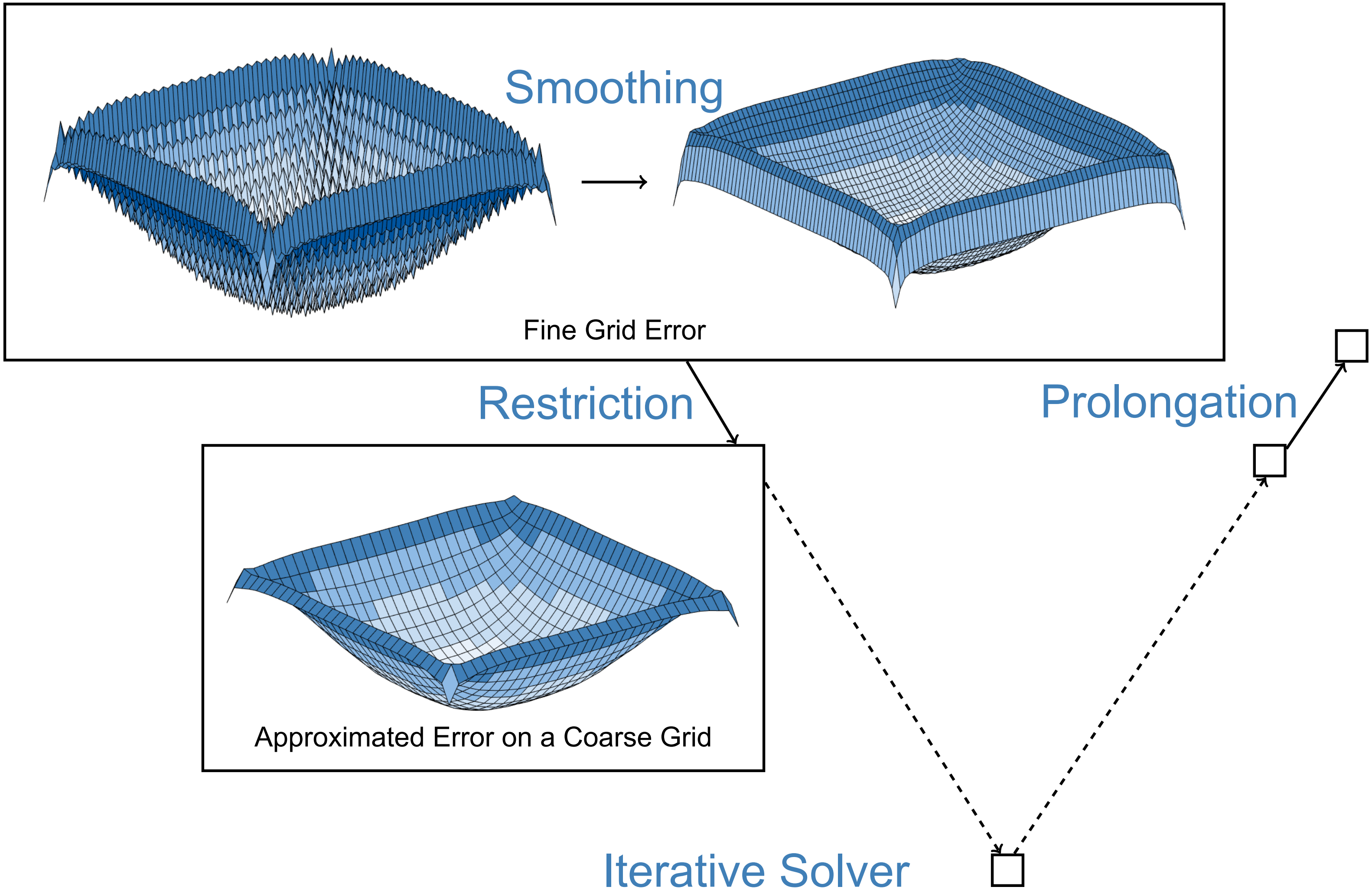
need to be solved.

Break *curse of dimensionality*, e.g., **parameter-dependent representations**

$\rightsquigarrow$ **parameter-dependent multigrid**

7. Ballani, Grasedyck '13
8. Dolgov, Savostyanov '14
9. Hackbusch '15
2. Khoromskij, Oseledets '10
4. Kressner, Tobler '11
10. Oseledets, Dolgov '12

The multigrid method



11. *Brandt '77*
12. *Hackbusch '85*
9. *Hackbusch '15*
13. *Xu '92*
14. *Yserentant '86*

Representation of parameter-dependent problems

The operator

Rewrite $A^{(0)} + \sum_{\nu=1}^{d} p^{(\nu)} A^{(\nu)}$ as block-diagonal system $\mathcal{A}(p)$, then

$$
\begin{aligned}
\mathcal{A}(p) = \mathrm{Id}_{n_d} \otimes \cdots \otimes \mathrm{Id}_{n_2} \otimes \mathrm{Id}_{n_1} \otimes A^{(0)} \\
+ \ \mathrm{Id}_{n_d} \otimes \cdots \otimes \mathrm{Id}_{n_2} \otimes \mathrm{diag}\left(p^{(1)}\right) \otimes A^{(1)} \\
+ \ \ldots \\
+ \ \mathrm{diag}\left(p^{(d)}\right) \otimes \cdots \otimes \mathrm{Id}_{n_2} \otimes \mathrm{Id}_{n_1} \otimes A^{(d)}
\end{aligned}
$$

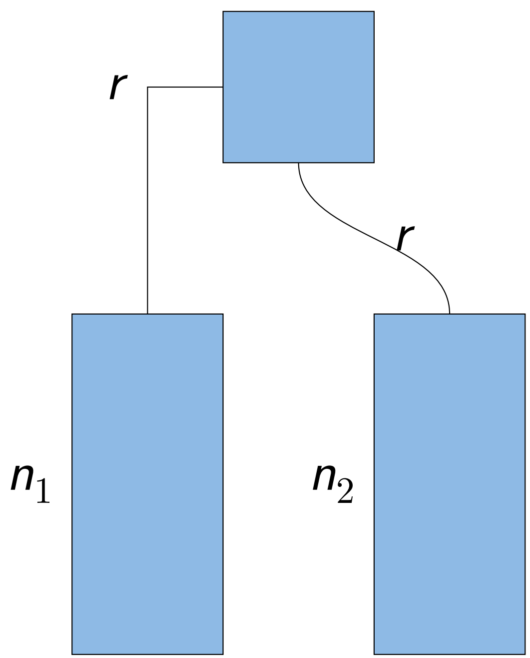
$$
= \sum_{\nu=0}^{d} \bigotimes_{\mu=0}^{d} \begin{cases} A^{(\nu)} & \text{if } \mu = d, \\ \mathrm{diag}\left(p^{(\nu)}\right) & \text{if } \mu + \nu = d \text{ and } \nu \neq 0, \\ \mathrm{Id}_{n_\nu} & \text{otherwise.} \end{cases}
$$

$\Rightarrow$ Low-rank representation

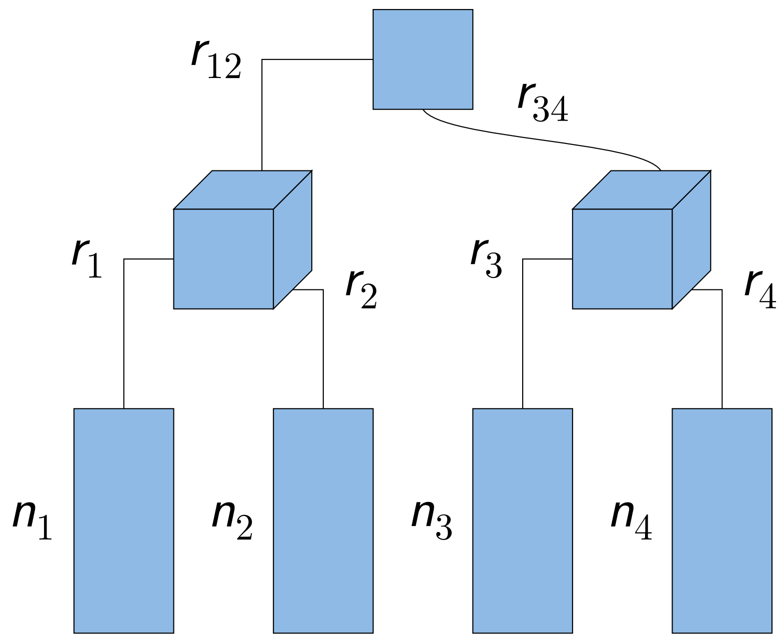
15. *Lathauwer, De Moor, Vandewalle '00*
16. *Grasedyck '10*
17. *Hackbusch '12*
18. *Hackbusch, Kühn '09*
19. *Kolda, Bader '09*
20. *Oseledets '11*

Representation of parameter-dependent problems

Hierarchical singular value decomposition



(a) $d = 2$



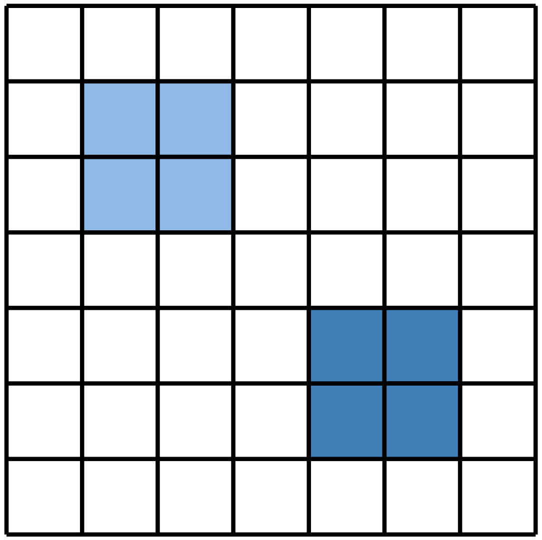
(b) $d = 4$

$\Rightarrow$ Storage: $\mathcal{O}\left(n^d\right) \rightsquigarrow \mathcal{O}\left(dnr + dr^3\right)$

16. *Grasedyck '10*
21. *Hackbusch, Khoromskij, Tyrtyshnikov '08*
18. *Hackbusch, Kühn '09*

Representation of parameter-dependent problems

The smoother



Geometry

Damped Richardson method:

$$\mathcal{S}_{\mathsf{R}}(p) = \mathsf{Id} - \omega(p)\mathcal{A}(p)$$

Damped Jacobi method:

$$\mathcal{S}_{\mathsf{J}}(p) = \mathsf{Id} - \omega(p)\mathsf{diag}(\mathcal{A}(p))^{-1}\mathcal{A}(p)$$

$\Rightarrow$ We need a low-rank approximation of $\mathsf{diag}(\mathcal{A}(p))^{-1}$

22. Bachmayr, Dahmen '15
23. Bachmayr, Schneider '17
3. Khoromskij, Schwab '11
4. Kressner, Tobler '11
24. Kressner, Uschmajew '16

The smoother

Let $M \in \mathbb{R}^{m\times m}$ with $\sigma(M) \subseteq \{a+ib \in \mathbb{C} \mid a<0\}$, then $M^{-1} = -\int_0^\infty \exp(tM)\mathrm{d}t$

Now it holds

$$\left\| \int_0^\infty \exp(tM)\,\mathrm{d}t - \sum_{m=1}^{k} \alpha_m \exp(\beta_m M) \right\| \leq \mathcal{O}(\exp(-\pi\sqrt{k}))$$

Application to $x, y \in \mathbb{R}$ yields

$$\begin{aligned}(x+y)^{-1} &= \sum_{m=1}^{k} \alpha_m \exp\Big(-\beta_m\big(x+y\big)\Big)\\ &= \sum_{m=1}^{k} \alpha_m \exp\Big(-\beta_m x\Big) \cdot \exp\Big(-\beta_m y\Big)\end{aligned}$$

25. *Braess, Hackbusch '05*
26. *Grasedyck '04*
27. *Hackbusch '19*

Representation of parameter-dependent problems

The smoother

Find low-rank representation of the inverse diagonal, e.g.,

$$\mathrm{diag}(\mathcal{A}(p))^{-1} = \left(\mathrm{Id}_{n_1} \otimes \mathrm{diag}(A^{(0)}) + \mathrm{diag}(p^{(1)}) \otimes \mathrm{diag}(A^{(1)})\right)^{-1} \tag{1}$$

Because

- all summands commute, as diagonal matrices, and
- for s.p.d. operator all eigenvalues are positive and bounded from 0

use approximation through exponential sums

$$(1) \approx \sum_{m=1}^{k} \alpha_m \exp\left(-\beta_m\left(\mathrm{Id}_{n_1} \otimes \mathrm{diag}(A^{(0)}) + \mathrm{diag}(p^{(1)}) \otimes \mathrm{diag}(A^{(1)})\right)\right)$$

$$= \sum_{m=1}^{k} \alpha_m \underbrace{\left(\mathrm{Id}_{n_1} \otimes \exp\left(-\beta_m \mathrm{diag}(A^{(0)})\right)\right)}_{\text{Low-rank structure}} \cdot \underbrace{\exp\left(-\beta_m \mathrm{diag}(p^{(1)}) \otimes \mathrm{diag}(A^{(1)})\right)}_{\textit{Problem}}$$

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Representation of parameter-dependent problems

The smoother

Problem: $\exp\left(-\beta_m \mathrm{diag}(p^{(1)}) \otimes \mathrm{diag}(A^{(1)})\right) =: (2)$ not separable

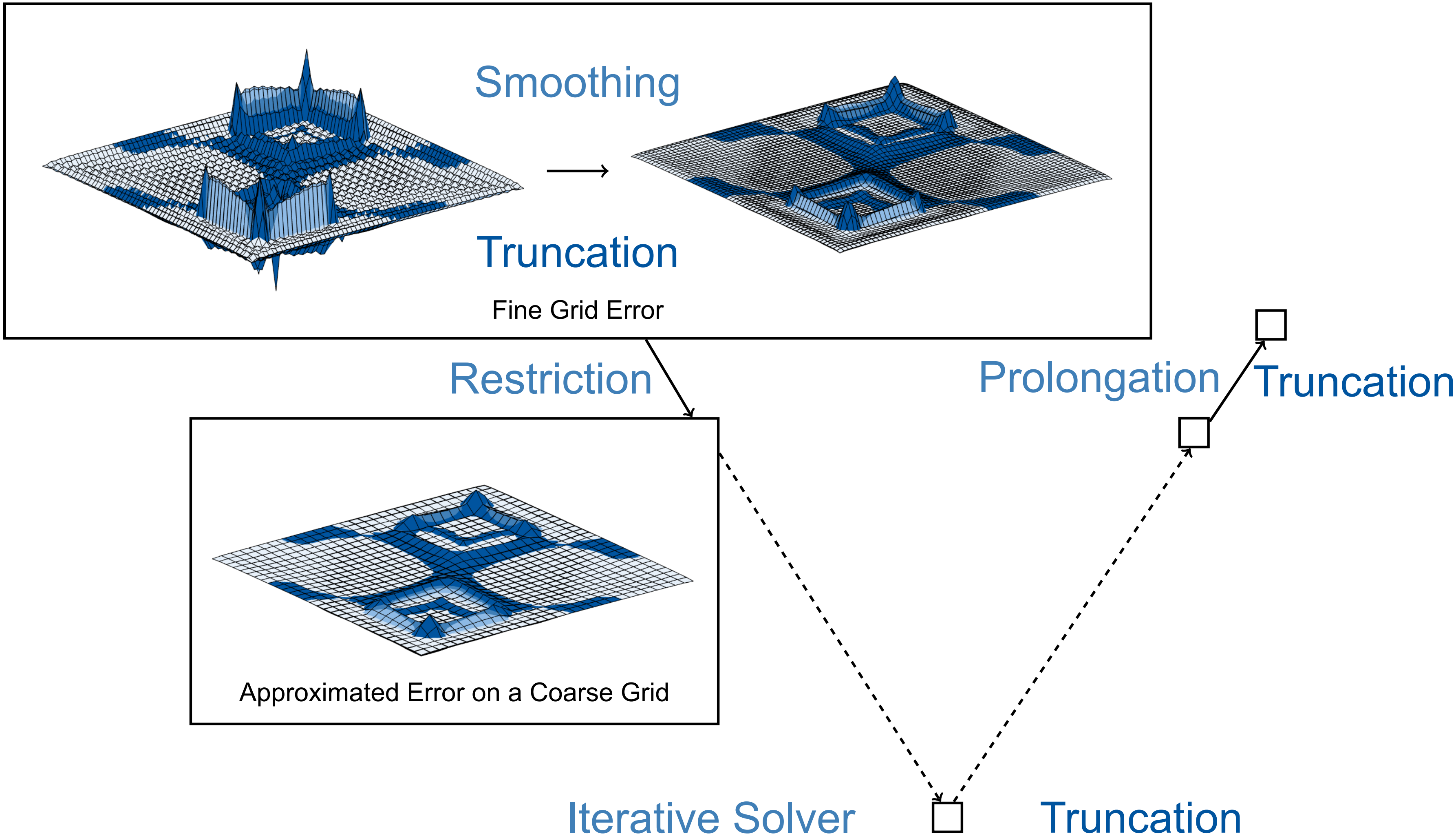
Assume $\mathrm{diag}(A^{(1)}) = \mathrm{diag}(0, \dots, 0, c_1, c_1, \dots, c_1, 0, \dots, 0) = c_1 \widetilde{\mathrm{Id}}_1$, then with

$$
\begin{aligned}
(2) &= \exp\left(-\beta_m c_1 \mathrm{diag}(p^{(1)}) \otimes \widetilde{\mathrm{Id}}_1\right) = \sum_{j=0}^{\infty} \frac{(-\beta_m c_1 \mathrm{diag}(p^{(1)}) \otimes \widetilde{\mathrm{Id}}_1)^j}{j!} \\
&= (\mathrm{Id}_{n_1} \otimes \mathrm{Id}_{n_0}) + \left(\sum_{j=1}^{\infty} \frac{(-\beta_m c_1 \mathrm{diag}(p^{(1)}))^j}{j!} \otimes \widetilde{\mathrm{Id}}_1\right) + (\mathrm{Id}_{n_1} \otimes \widetilde{\mathrm{Id}}_1) - (\mathrm{Id}_{n_1} \otimes \widetilde{\mathrm{Id}}_1)
\end{aligned}
$$

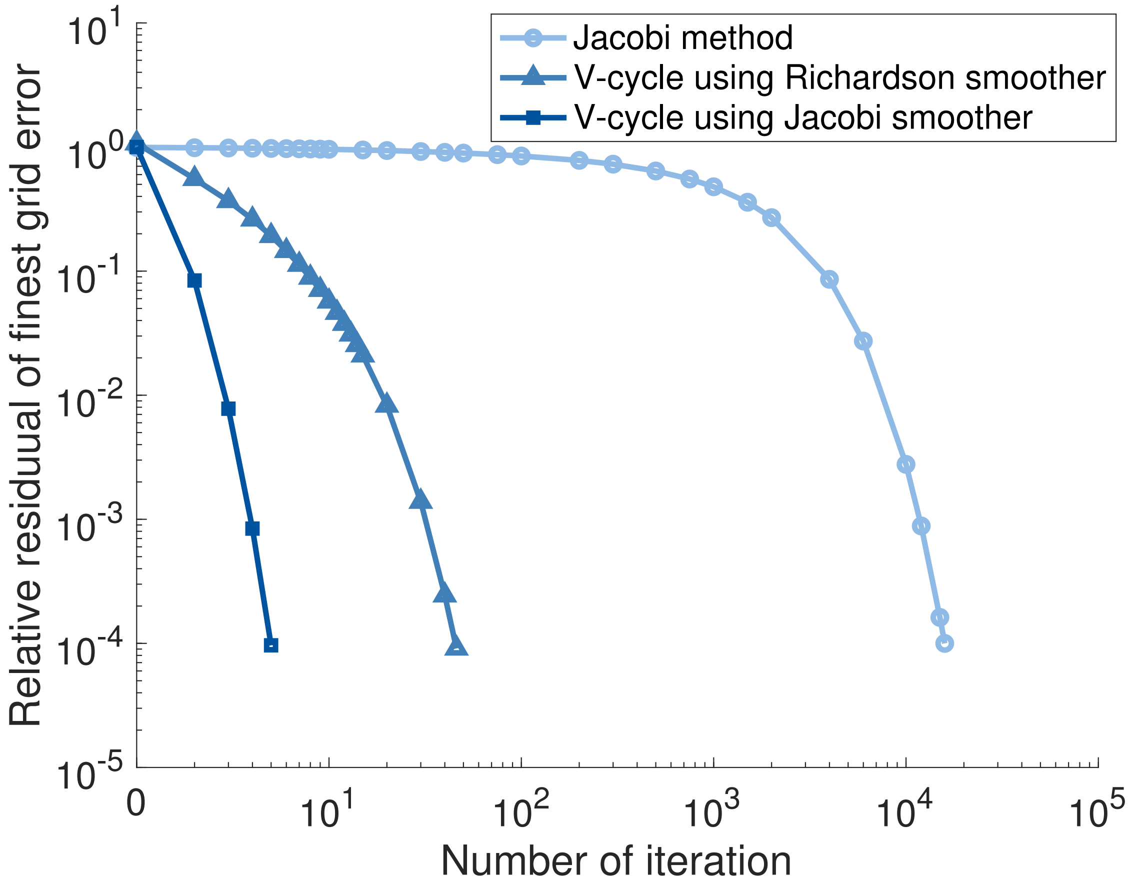
a low-rank representation of the inverse diagonal follows:

$$
\sum_{m=1}^{k} \alpha_m \left(\mathrm{Id}_{n_1} \otimes \exp\left(-\beta_m \mathrm{diag}(A^{(0)})\right)\right) \left(\exp\left(-\beta_m c_1 \mathrm{diag}(p^{(1)})\right) \otimes \widetilde{\mathrm{Id}}_1 + \mathrm{Id}_{n_1} \otimes (\mathrm{Id}_{n_0} - \widetilde{\mathrm{Id}}_1)\right)
$$

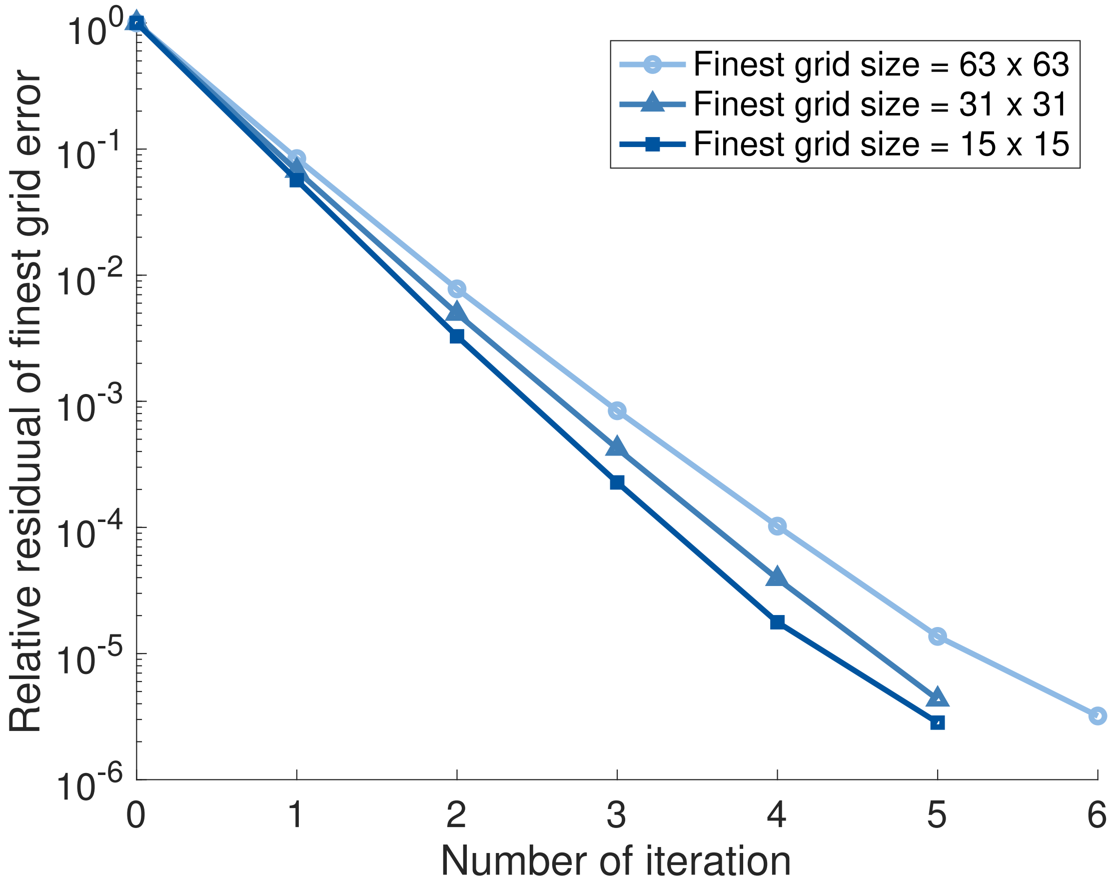
The parameter-dependent multigrid method



Numerical experiments
Comparison of parameter-dependent iterative solvers



Numerical experiments
Multigrid method using Jacobi smoother



A parameter-dependent smoother for the multigrid method

- Multigrid theory is valid for parameter-dependent problems
- Parameter-dependent representations of the linear system, smoother, prolongation and restriction through low-rank tensor formats
- Smoothing property holds for low-rank approximation of the parameter-dependent damped Jacobi method derived through exponential sums
- Numerical experiments suggest grid size independent convergence rate

Feel free to contact me for any kind of discussion

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Slides are available on my website

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