Exercise sheet 1 for Friday, Apr 28, 2017

To be handed in at the beginning of the exercise session, or before Apr 28, 12:00 p.m. at the drop box in front of room 149.

Exercise 1: (Weak formulation of the Laplace equation)

Prove (1.3.14) from the lecture notes, i. e. show for any sequence $(v_j)_{j\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ with limit v^* w.r.t. $\|\cdot\|_{1,\Omega}$, that for any given $f \in L^2(\Omega)$, the solution u of

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \qquad \forall v \in C_0^\infty(\Omega)$$

satisfies

$$\int_\Omega \nabla u \cdot \nabla v^* \,\mathrm{d}x = \int_\Omega f v^* \,\mathrm{d}x.$$

Remark: This result still holds for f in the larger space $H^{-1}(\Omega)$, which is the dual space to the Sobolev space $H_0^1(\Omega)$. 2 points

Exercise 2: (Weak Derivative)

Solve exercise 2.3.1 of the lecture notes: The function $v(x) := |x| \in L_p((-1,1))$ has the weak derivative

$$D^1 v(x) = \begin{cases} -1, & x \in (-1,0), \\ 1, & x \in (0,1), \end{cases}$$

in $L_p((-1,1))$ for any $1 < p < \infty$. Show that v has no second derivative in $L_p(\Omega)$. 3 points

Exercise 3: (Embedding in 1D)

Prove the following:

a) Remark 2.4.1 of the lecture: $H^1((a,b)) \subset C((a,b))$. To this end, first show the estimate

$$\|u\|_{C((a,b))} = \sup_{x\in(a,b)} |u(x)| \le c\|u\|_{H^1((a,b))}$$

for any $u \in C^1((a,b))$ with a constant c depending only on a and b.
Hint: $C((a,b))$ is complete.

b) The space $C^1((a,b))$ equipped with the norm $\|u\|_{C^1((a,b))} = \|u\|_{C((a,b))} + \|u'\|_{C((a,b))}$ is compactly embedded into $C((a,b))$.
Hint: Use the Arzelà-Ascoli theorem.

c) Example 2.4.1 of the lecture: let $\Omega := \{x \in \mathbb{R}^2 \mid |x| < 1\}$, then the function $u(x) = \log(\log(2/|x|))$ is in $H^1(\Omega)$ but not bounded in Ω.

$3 + 2 + 4 = 9$ points

Exercise 4: (Poincaré inequality for $\tilde{\mathbf{H}}(\Omega)$)

Prove Corollary 2.5.2 of the lecture: Assume that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. Show that there exists a constant $C_\Omega < \infty$, depending only on Ω such that

$$\left\| v - |\Omega|^{-1} \int_\Omega v \,\mathrm{d}x \right\|_{0,\Omega} \leq C_\Omega |v|_{1,\Omega}, \qquad \forall v \in H^1(\Omega).$$

3 points

Exercise 5: (Trace theorem)

Solve exercise 2.6.1 from the lecture notes.

4 points