Exercise sheet 2 for Friday, May 5, 2017

To be handed in at the beginning of the exercise session, or before May 5, 12:00 p.m. at the drop box in front of room 149.

Exercise 6: (Weak Formulations)

We would like to derive weak formulations for some of the examples from the lecture.

a) Use the Green's formulas to show that the weak formulation of the biharmonic equation (3.1.4) is: find $u \in \mathbb{U} := H_0^2(\Omega)$ such that

$$B(u,v) := \int_\Omega \Delta u \Delta v \,\mathrm{d}x = \int_\Omega v f \,\mathrm{d}x =: f(v), \qquad v \in H_0^2(\Omega),$$

and that

$$|B(u,v)| \le d\|u\|_{2,\Omega}\|v\|_{2,\Omega}, \qquad u,v \in H^2(\Omega).$$

b) Defining $H(\mathrm{curl};\Omega)$ as in (3.1.9) with $C_0^\infty(\Omega;\mathbb{R}^3)$ replaced by $C^\infty(\Omega;\mathbb{R}^3)$, show that

$$\int_\Omega (\mathrm{curl}\, w) \cdot v \,\mathrm{d}x = \int_\Omega w \cdot (\mathrm{curl}\, v) \,\mathrm{d}x + \int_{\partial\Omega} w \cdot (v \wedge n) \,\mathrm{d}s$$

for all $w, v \in H(\mathrm{curl};\Omega)$.

c) Show that for two sufficiently smooth vector fields v, w vanishing on $\partial\Omega$ one has

$$-\int_\Omega \Delta v \cdot w \,\mathrm{d}x = \int_\Omega \nabla v : \nabla w \,\mathrm{d}x := \sum_{i=1}^d \int_\Omega \nabla v_i \cdot \nabla w_i \,\mathrm{d}x.$$

This is needed to derive a weak formulation of the Stokes system.

$3 + 3 + 2 = 8$ points

Exercise 7: (Inf-Sup-Condition)

Let $\mathcal{B} \in \mathbb{R}^{d\times d}$, i. e. $\mathcal{B}$ induces a bilinear form $B : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R},\ B(w,v) = v^T\mathcal{B}w$. Then

$$\inf_{w\in\mathbb{R}^d\setminus\{0\}} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \frac{v^T\mathcal{B}w}{|w||v|} = \sigma_d(\mathcal{B})$$

where $\sigma_d(\mathcal{B})$ is the smallest singular value of $\mathcal{B}$.

3 points

Exercise 8: (Galerkin Case)

Assume that $B(\cdot,\cdot)$ is a continuous and coercive bilinear form on $\mathbb{U} \times \mathbb{U}$ ($\mathbb{U}$ a Hilbert space), i.e.,

$$|B(w,v)| \leq C_{\mathcal{B}} \|w\|_{\mathbb{U}} \|v\|_{\mathbb{U}}, \quad B(w,w) \geq c_{\mathcal{B}} \|w\|_{\mathbb{U}}^2, \quad v, w \in \mathbb{U}. \tag{1}$$

a) Show that $\mathcal{B} : \mathbb{U} \to \mathbb{U}'$, defined by $(\mathcal{B}w)(v) = B(w,v)$, $w, v \in \mathbb{U}$, has a bounded condition

$$\kappa_{\mathbb{U},\mathbb{U}'}(\mathcal{B}) \leq \frac{C_{\mathcal{B}}}{c_{\mathcal{B}}}.$$

b) For any subspace $\mathbb{U}_h \subset \mathbb{U}$, the Galerkin scheme:
find $u_h \in \mathbb{U}_h$ such that

$$B(u_h, v_h) = f(v_h), \quad v_h \in \mathbb{U}_h$$

has a unique solution $u_h \in \mathbb{U}_h$ and the induced discrete operator

$$(\mathcal{B}_h w_h)(v_h) = f(v_h), \quad w_h, v_h \in \mathbb{U}_h$$

has the same condition bound

$$\kappa_{\mathbb{U},\mathbb{U}'}(\mathcal{B}_h) \leq \frac{C_{\mathcal{B}}}{c_{\mathcal{B}}},$$

independent of $\mathbb{U}_h$.

c) (Ceà-Lemma) One has the best approximation property (BAP)

$$\|u - u_h\|_{\mathbb{U}} \leq \kappa_{\mathbb{U},\mathbb{U}'}(\mathcal{B}) \inf_{\bar{u}_h \in \mathbb{U}_h} \|u - \bar{u}_h\|_{\mathbb{U}}.$$

d) Show that when $B(\cdot,\cdot)$ is in addition symmetric one even has

$$\|u - u_h\|_{\mathbb{U}} \leq \sqrt{\kappa_{\mathbb{U},\mathbb{U}'}(\mathcal{B})} \inf_{\bar{u}_h \in \mathbb{U}_h} \|u - \bar{u}_h\|_{\mathbb{U}}.$$

$2 + 1 + 2 + 2 = 7$ points