# Exercise sheet 3 for Friday, May 12, 2017

To be handed in at the beginning of the exercise session, or before May 12, 12:00 p.m. at the drop box in front of room 149.

## Exercise 9: (Petrov-Galerkin)

Assume that for $B(\cdot,\cdot)$, $\mathbb{U}$, $\mathbb{V}$, the conditions (3.2.2), (3.2.5), and (3.2.6) from the lecture notes hold with continuity constant $\|\mathcal{B}\| = C_{\mathcal{B}}$. Show that for a given pair of finite-dimensional subspaces $\mathbb{U}_h \subset \mathbb{U}$, $\mathbb{V}_h \subset \mathbb{V}$ with $\dim(\mathbb{U}_h) = \dim(\mathbb{V}_h)$

$$\kappa_{\mathbb{U},\mathbb{V}'}(\mathcal{B}_h) \leq \frac{C_{\mathcal{B}}}{c_{\mathcal{B}_h}}, \tag{1}$$

holds if and only if

$$\inf_{w_h \in \mathbb{U}_h} \sup_{v_h \in \mathbb{V}_h} \frac{B(w_h, v_h)}{\|w_h\|_{\mathbb{U}} \|v_h\|_{\mathbb{V}}} \geq c_{\mathcal{B}_h} \tag{2}$$

holds for some positive $c_{\mathcal{B}_h} > 0$. Here, the induced finite-dimensional operator $\mathcal{B}_h$ is again defined by

$$(\mathcal{B}_h w_h)(v_h) = B(w_h, v_h), \quad w_h \in \mathbb{U}_h,\ v_h \in \mathbb{V}_h.$$

2 points

## Exercise 10: (Discrete inf-sup Constant)

Assume that for $B(\cdot,\cdot)$, $\mathbb{U}$, $\mathbb{V}$, we have

$$\inf_{w \in \mathbb{U}} \sup_{v \in \mathbb{V}} \frac{B(w,v)}{\|w\|_{\mathbb{U}} \|v\|_{\mathbb{V}}} =: c_{\mathcal{B}} > 0 \qquad \text{and} \qquad |B(w,v)| \leq C_{\mathcal{B}} \|w\|_{\mathbb{U}} \|v\|_{\mathbb{V}}\,.$$

Furthermore, let $(\mathbb{U}_n \times \mathbb{V}_n)_{n\in\mathbb{N}}$ with $\mathbb{U}_n \subset \mathbb{U}$, $\mathbb{V}_n \subset \mathbb{V}$ and $\dim(\mathbb{U}_n) = \dim(\mathbb{V}_n)$, $n \in \mathbb{N}$ be a family of finite-dimensional subspaces with discrete inf-sup constants

$$\inf_{w_n \in \mathbb{U}_n} \sup_{v_n \in \mathbb{V}_n} \frac{B(w_n, v_n)}{\|w_n\|_{\mathbb{U}} \|v_n\|_{\mathbb{V}}} =: c_{\mathcal{B}_n} \geq 0\,.$$

Let $P_{\mathbb{U}_n}$ denote the $\mathbb{U}-$orthogonal projector into $\mathbb{U}_n$ (i.e., $(u - P_{\mathbb{U}_n} u, u_n)_{\mathbb{U}} = 0$, for all $u_n \in \mathbb{U}_n$).

a) Show that if for all $u \in \mathbb{U}$

$$\lim_{n\to\infty} \|u - P_{\mathbb{U}_n} u\|_{\mathbb{U}} = 0\,, \tag{3}$$

then $\limsup_{n\to\infty} c_{\mathcal{B}_n} \leq c_{\mathcal{B}}$.

b) Let $\bigcup_{n\in\mathbb{N}} \mathbb{U}_n$ be dense in $\mathbb{U}$ and $c_{\mathcal{B}_n} \geq \epsilon > 0$ for all $n \in \mathbb{N}$. Show that $\bigcup_{n\in\mathbb{N}} \mathbb{V}_n$ is dense in $\mathbb{V}$.

c) Find an example for $\mathbb{U}, \mathbb{V}, B(\cdot,\cdot)$, $\mathbb{U}_n$ and $\mathbb{V}_n$ for which $c_{\mathcal{B}_n} > 0$ for all $n \in \mathbb{N}$ and 3 holds but $\limsup_{n\to\infty} c_{\mathcal{B}_n} < c_{\mathcal{B}}$.

d) What does this mean for Petrov-Galerkin methods for general problems in contrast to Galerkin methods for coercive problems?

3+2+2+1=8 points

## Exercise 11: (Linear Projector)

Assume that under the hypotheses of Exercise 9 the discrete inf-sup condition (2) holds for the pair $\mathbb{U}_h \subset \mathbb{U}$, $\mathbb{V}_h \subset \mathbb{V}$. Let $u, u_h$ denote the solutions of (4.0.1), (4.1.2) from the lecture notes, respectively. Then, the mapping

$$\Pi_{\mathbb{U}_h,\mathbb{V}_h} : u \to u_h \tag{4}$$

is a well defined linear projector, i.e., in particular,

$$\Pi_{\mathbb{U}_h,\mathbb{V}_h}(\Pi_{\mathbb{U}_h,\mathbb{V}_h} u) = \Pi_{\mathbb{U}_h,\mathbb{V}_h} u.$$

**Hint:** Note first that

$$B(\Pi_{\mathbb{U}_h,\mathbb{V}_h} u, v_h) = B(u, v_h), \quad v_h \in \mathbb{V}_h.$$

3 points

## Exercise 12: ($\mathbb{V}'$-orthogonal Projector)

Suppose that $\mathbb{V}_h \subset \mathbb{V}$ is a closed subspace and let $P_{\mathbb{V}_h} = P_{\mathbb{V},\mathbb{V}_h}$ denote the $\mathbb{V}$-orthogonal projector into $\mathbb{V}_h$ (i.e., $(v - P_{\mathbb{V}_h} v, v_h)_{\mathbb{V}} = 0$, for all $v_h \in \mathbb{V}_h$). Show that

$$Q := \mathcal{R}_{\mathbb{V}'} P_{\mathbb{V}_h} \mathcal{R}_{\mathbb{V}}$$

is a $\mathbb{V}'$-orthogonal projector into $\mathcal{R}_{\mathbb{V}'}(\mathbb{V}_h) \subset \mathbb{V}'$.

Can you improve the estimate (4.5.12) from the lecture notes somewhat, i.e., which portion of the data $f$ is actually “seen” by the method? 5 points