Exercise sheet 4 for Friday, May 19, 2017

To be handed in at the beginning of the exercise session, or before May 19, 12:00 p.m. at the drop box in front of room 149.

Exercise 13: (Riesz-map)

Let $\mathbb{H}$ be a Hilbert-space.

a) Give a formular for the scalar product $(\cdot,\cdot)_{\mathbb{H}'}$ on $\mathbb{H}'$ which induces the norm $\|\ell\|_{\mathbb{H}'} = \sup_{0\neq v\in\mathbb{H}} \frac{|\ell(v)|}{\|v\|_{\mathbb{H}}}$ without using the Riesz-map.

b) Show that one has for any $\ell,\ell' \in \mathbb{H}'$

$$(\ell,\ell')_{\mathbb{H}'} = \ell(\mathcal{R}_{\mathbb{H}}\ell') = \ell'(\mathcal{R}_{\mathbb{H}}\ell) = (\mathcal{R}_{\mathbb{H}}\ell, \mathcal{R}_{\mathbb{H}}\ell')_{\mathbb{H}}\,.$$

c) Show that

$$\mathcal{R}_{\mathbb{H}}^{-1} = \mathcal{R}_{\mathbb{H}'}\,.$$

2+2+1=5 points

Exercise 14: (inf-sup)

Let $\mathbb{X},\mathbb{Y}$ be finite-dimensional subspaces of a Hilbert space $\mathbb{H}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|^2 = (\cdot,\cdot)$. Define

$$\beta(\mathbb{X},\mathbb{Y}) := \inf_{0\neq v\in\mathbb{X}} \sup_{0\neq z\in\mathbb{Y}} \frac{(v,z)_{\mathbb{H}}}{\|v\|_{\mathbb{H}}\|z\|_{\mathbb{H}}}.$$

Prove the following properties:

a) Let $P_{\mathbb{Y}}$ denote the $\mathbb{H}$-orthogonal projector onto $\mathbb{Y}$, then

$$\beta(\mathbb{X},\mathbb{Y}) = \inf_{0\neq v\in\mathbb{X}} \frac{\|P_{\mathbb{Y}}v\|}{\|v\|}.$$

b) One has

$$\sup_{0\neq v\in\mathbb{X}} \inf_{0\neq z\in\mathbb{Y}} \frac{\|v-z\|}{\|v\|} \leq \sqrt{1-\beta(\mathbb{X},\mathbb{Y})^2}.$$

c) Let $\Phi = \{x_1,\ldots,x_n\}$ and $\Psi = \{y_1,\ldots,y_m\}$ be orthonormal bases of $\mathbb{X}$ and $\mathbb{Y}$ (formally viewed as column vectors), respectively and let

$$\mathbf{G} := (\Phi,\Psi^T) := \Big((x_i,y_k)\Big)_{i=1,k=1}^{n,m}$$

be the corresponding *cross-Gramian* of the two bases. Then

$$\beta(\mathbb{X},\mathbb{Y}) = \sigma_{\min}(\mathbf{G})$$

is the smallest singular value of $\mathbf{G} := (\Phi,\Psi^T)$.

d) Interprete $\beta(\mathbb{X},\mathbb{Y})$ geometrically in terms of angles.

2+3+2+2=9 points

Exercise 15: (least-squares)

a) Show that (4.7.1) is equivalent to

$$(\mathcal{T}_{\mathbb{S}_h} u_h, \mathcal{T}_{\mathbb{S}_h} w_h)_{\mathbb{V}} = f(\mathcal{T}_{\mathbb{S}_h} w_h), \quad w_h \in \mathbb{U}_h.$$

b) What does this mean for the PG system matrix?

c) Formulate the least-squares problem which is equivalent to the optimal PG-scheme (4.5.11)

$$B(u_h, v_h) = f(v_h), \quad v_h \in \mathcal{T}(\mathbb{U}_h).$$

1+1+3=5 points