Exercise sheet 5 for Friday, May 26, 2017

To be handed in at the beginning of the exercise session, or before May 26, 12:00 p.m. at the drop box in front of room 149.

Exercise 16: (Fortin's Criterion)

Assume that there exists a projector $\Pi_h : \mathbb{V} \to \mathbb{S}_h$ such that

$$B(w_h, \Pi_h v) = B(w_h, v), \quad v \in \mathbb{V}.$$

Show that then

$$\inf_{w_h \in \mathbb{U}_h} \sup_{v_h \in \mathbb{S}_h} \frac{B(w_h, v_h)}{\|w_h\|_{\mathbb{U}} \|v_h\|_{\mathbb{V}}} \geq \frac{c_{\mathcal{B}}}{\|\Pi_h\|_{\mathcal{L}(\mathbb{V},\mathbb{V})}}.$$

3 points

Exercise 17: (Saddle-point formulation)

a) Formulate a saddle-point problem which is equivalent to the PG-scheme (4.5.11) with the optimal test-space $\mathcal{T}(\mathbb{U}_h)$.

b) Formulate a saddle-point problem which is equivalent to the infinite dimensional weak formulation (4.5.1). What is the solution for the auxiliary variable r in this case?

2+3=5 points

Exercise 18: (Nested iteration)

Either complete the following exercise or hand in a sketch of your ideas for a solution where you point out your questions:

Assume that the following routines are given.

$\mathbf{Res}[u_h, \mathbb{U}_h, \mathbb{S}_h] \to r_h \in \mathbb{S}_h$ such that for $w_h \in \mathbb{U}_h$

$$(r_h, v_h)_{\mathbb{V}} = B(u_h, v_h) - f(v_h), \quad v_h \in \mathbb{S}_h. \tag{1}$$

$\mathbf{Proj}[\tilde{u}_h, z_h, \mathbb{U}_h, \mathbb{S}_h] \to u_h$ such that for $\tilde{u}_h \in \mathbb{U}_h$, $z_h \in \mathbb{S}_h$,

$$(u_h, w_h)_{\mathbb{U}} = (\tilde{u}_h, w_h)_{\mathbb{U}} + B(w_h, z_h), \quad w_h \in \mathbb{U}_h. \tag{2}$$

$\mathbf{Exp}[\mathbb{U}_h, \mathbb{S}_h] \to (\tilde{\mathbb{U}}_h, \tilde{\mathbb{S}}_h)$ such that $\mathbb{U}_h \subset \tilde{\mathbb{U}}_h$, $\mathbb{S}_h \subset \tilde{\mathbb{S}}_h \subset \mathbb{V}$ such that for some fixed $\delta < 1$ the space $\tilde{\mathbb{S}}_h$ is δ-proximal for $\tilde{\mathbb{U}}_h$ and for some constant $\zeta < 1$ one has

$$\|r_h(\tilde{u}_h, f)\|_{\mathbb{V}} \leq \zeta \|r_h(u_h, f)\|_{\mathbb{V}}, \tag{3}$$

where $r_h(u_h, f)$, $r_h(\tilde{u}_h, f)$ are the solution components in $\mathbb{V}$ of the saddle-point problems (4.8.1) with respect to $\mathbb{U}_h, \mathbb{S}_h$, $\tilde{\mathbb{U}}_h, \tilde{\mathbb{S}}_h$, respectively.

Employ the above routines to formulate a skeleton of an algorithm

Solve$[B, f, \epsilon] \to (\mathbb{U}_\epsilon, \mathbb{V}_\epsilon, u_\epsilon)$ such that $\mathbb{U}_\epsilon \subset \mathbb{U}$, $\mathbb{V}_\epsilon \subset \mathbb{V}$ are finite dimensional trial- and test-search-spaces with the following properties:

(i) $\mathbb{V}_\epsilon$ is δ-proximal for $\mathbb{U}_\epsilon$;

(ii) the corresponding PG solution u_ϵ satisfies

$$\|u - u_\epsilon\|_{\mathbb{U}} \leq \epsilon,$$

where u is the exact solution of

$$B(u, v) = f(v), \quad v \in \mathbb{V}.$$

6 points

Preview to Exercise sheet 6 (incomplete):

Exercise 19: (Nested iteration - due date: Friday, June 2)

a) Complete exercise 18.

b) Estimate the computational work in terms of the number of calls of the routines **Res**, **Proj**;

c) Estimate the total number $N = N(\epsilon)$ of flops if the number of flops for **Proj**$[\tilde{u}_h, z_h, \mathbb{U}_h, \mathbb{S}_h]$ and **Res**$[u_h, \mathbb{U}_h, \mathbb{S}_h]$ stays uniformly proportional to $\dim \mathbb{U}_h, \dim \mathbb{S}_h$, respectively, and if $\dim \mathbb{S}_h \leq C \dim \mathbb{U}_h$, uniformly.

6+2+2=10 points