Exercise sheet 6 for Friday, Jun 2, 2017

To be handed in before Thursday, Jun 1, 2017, 4:00 p.m.
at the drop box in front of room 149.

Exercise 19: (Nested iteration)

a) Assume that the following routines are given.

Res$[u_h, \mathbb{U}_h, \mathbb{S}_h] \to r_h \in \mathbb{S}_h$ such that for $w_h \in \mathbb{U}_h$

$$(r_h, v_h)_{\mathbb{V}} = B(u_h, v_h) - f(v_h), \quad v_h \in \mathbb{S}_h. \tag{1}$$

Proj$[\tilde{u}_h, z_h, \mathbb{U}_h, \mathbb{S}_h] \to u_h$ such that for $\tilde{u}_h \in \mathbb{U}_h$, $z_h \in \mathbb{S}_h$,

$$(u_h, w_h)_{\mathbb{U}} = (\tilde{u}_h, w_h)_{\mathbb{U}} + B(w_h, z_h), \quad w_h \in \mathbb{U}_h. \tag{2}$$

Exp$[\mathbb{U}_h, \mathbb{S}_h] \to (\tilde{\mathbb{U}}_h, \tilde{\mathbb{S}_h})$ such that $\mathbb{U}_h \subset \tilde{\mathbb{U}}_h$, $\mathbb{S}_h \subset \tilde{\mathbb{S}_h} \subset \mathbb{V}$ such that for some fixed $\delta < 1$ the space $\tilde{\mathbb{S}_h}$ is δ-proximal for $\tilde{\tilde{\mathbb{U}}}_h$ and for some constant $\zeta < 1$ one has

$$\|r_h(\tilde{u}_h, f)\|_{\mathbb{V}} \le \zeta \|r_h(u_h, f)\|_{\mathbb{V}}, \tag{3}$$

where $r_h(u_h, f)$, $r_h(\tilde{u}_h, f)$ are the solution components in $\mathbb{V}$ of the saddle-point problems (4.8.1) with respect to $\mathbb{U}_h, \mathbb{S}_h$, $\tilde{\mathbb{U}}_h, \tilde{\mathbb{S}_h}$, respectively.

Employ the above routines to formulate a skeleton of an algorithm

Solve$[B, f, \epsilon] \to (\mathbb{U}_\epsilon, \mathbb{V}_\epsilon, u_\epsilon)$ such that $\mathbb{U}_\epsilon \subset \mathbb{U}$, $\mathbb{V}_\epsilon \subset \mathbb{V}$ are finite dimensional trial- and test-search-spaces with the following properties:

(i) $\mathbb{V}_\epsilon$ is δ-proximal for $\mathbb{U}_\epsilon$;

(ii) the corresponding PG solution u_ϵ satisfies

$$\|u - u_\epsilon\|_{\mathbb{U}} \le \epsilon,$$

where u is the exact solution of

$$B(u, v) = f(v), \quad v \in \mathbb{V}.$$

b) Estimate the computational work in terms of the number of calls of the routines **Res**, **Proj**;

c) Let #**Proj**$[\cdot,\cdot,\cdot,\cdot]$ denote the number flops required for executing **Proj** for the given parameters (and analogously for the other routines). Estimate the total number $N = N(\epsilon) =$ #**Solve**$[B, f, \epsilon] \to (\mathbb{U}_\epsilon, \mathbb{V}_\epsilon, u_\epsilon)$ under the following assumptions:

- #**Proj**$[\tilde{u}_h, z_h, \mathbb{U}_h, \mathbb{S}_h] \le C(\dim \mathbb{U}_h + \dim \mathbb{S}_h)$;
- #**Res**$[u_h, \mathbb{U}_h, \mathbb{S}_h] \le C(\dim \mathbb{U}_h + \dim \mathbb{S}_h)$;
- for **Exp**$[\mathbb{U}_h, \mathbb{S}_h] \to (\tilde{\mathbb{U}}_h, \tilde{\mathbb{S}_h})$ one has $\dim \tilde{\mathbb{U}}_h \le C \dim \mathbb{U}_h$ and $\dim \tilde{\mathbb{S}}_h \le C \dim \mathbb{S}_h$,

where C is an absolute constant. Assume that the constants $C_{\mathcal{B}}, c_{\mathcal{B}}, \delta$ are known to you.

6+2+2=10 points

Exercise 20: (Convection-Diffusion)

Let $c \in L_\infty(\Omega)$, $b\colon \Omega \to \mathbb{R}^d$ such that

$$c - \frac{1}{2}\operatorname{div} b \geq 0, \quad \text{in } \Omega, \quad \epsilon > 0, \quad \operatorname{div} b \in L_\infty(\Omega)$$

A possible weak formulation of the boundary value problem

$$-\Delta u + b \cdot \nabla u + cu = f \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0,$$

is based on

$$\mathbb{U} = \mathbb{V} = H_0^1(\Omega),$$

and the bilinear form

$$B(u,v) = \int_\Omega \epsilon \nabla u \cdot \nabla v + (b \cdot \nabla u + cu)v dx.$$

Show that

a) Under the above assumptions, for $f \in H_0^1(\Omega)'$, the problem

$$B(u,v) = f(v), \quad v \in H_0^1(\Omega)$$

is well posed.

b) Give an upper bound for $\kappa_{H_0^1(\Omega),H^{-1}(\Omega)}(\mathcal{B})$. What happens for $\epsilon \to 0$?

4+2=6 points

Exercise 21: (Optimal Norm)

Represent the Riesz-map $\mathcal{R}_{\mathbb{V}_{\text{opt}}}$ in terms of $\mathcal{B}$. 2 points

Exercise 22: (Optimal Norm)

Considering $B(\cdot,\cdot)\colon \mathbb{U}_{\text{opt}} \times \mathbb{V} \to \mathbb{R}$, show that one has again

$$\inf_{w\in\mathbb{U}_{\text{opt}}} \sup_{v\in\mathbb{V}} \frac{B(w,v)}{\|w\|_{\mathbb{U}_{\text{opt}}}\|v\|_{\mathbb{V}}} = 1 = \sup_{w\in\mathbb{U}_{\text{opt}}} \sup_{v\in\mathbb{V}} \frac{B(w,v)}{\|w\|_{\mathbb{U}_{\text{opt}}}\|v\|_{\mathbb{V}}}, \tag{4}$$

i.e., we have again that

$$\bar{C}_{\mathcal{B}} = \bar{c}_{\mathcal{B}} = 1 \tag{5}$$

and $\mathcal{B} \in \mathcal{L}(\mathbb{U}_{\text{opt}}, \mathbb{V}')$ is an isometry. 2 points