Finite Volumen und Finite Elemente Verfahren I - SS 18

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1. Exercise sheet

Until Tuesday the 24th of April!

Ex. 1: (integration by parts)

An important tool of the course will be the *divergence theorem by Gauss*:

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$ and an outwards oriented normal-field n. Further on let $f = (f_1, \dots, f_n)$, $f_j : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. So follows:*

$$\int_\Omega \nabla \cdot f(x) \Delta x = \int_{\partial\Omega} f(x) \cdot n \mathrm{ds} \tag{1}$$

where the divergence of a function is defined as $\nabla \cdot f(x) := \sum_{j=1}^n \frac{\partial f_j}{\partial x_j}(x)$.

Prove, with the help of the divergence theorem, the following statements for the bounded domain $\Omega \subset \mathbb{R}^n$ with a piecewise smooth boundary:

a) (*Partial Integration*) Let $u, w \in C^1(\overline{\Omega})$ be scalar functions, so follows

$$\int_\Omega \frac{\partial u}{\partial x_j} w \Delta x = -\int_\Omega u \frac{\partial w}{\partial x_j} \Delta x + \int_{\partial\Omega} u w n_j \mathrm{ds}. \tag{2}$$

b) (*Green's Identity*) Let $u, w \in C^2(\overline{\Omega})$ be scalar functions, so follows

$$\int_\Omega (u \Delta w - w \Delta u) \Delta x = \int_{\partial\Omega} (u \nabla w \cdot n - w \nabla u \cdot n) \mathrm{ds}. \tag{3}$$

where $\Delta w(x) := \sum_{j=1}^n \frac{\partial^2 w}{\partial x_j^2}(x)$ denotes the Laplace-Operator. Note, if you're not sure write it down in one space dimension.

Points: 10

Download and install Matlab on your personal laptop. If you do not own a copy of Matlab, you can get a free Campus License by the RWTH software portal. There are more information on

`http://www.matlab.rwth-aachen.de/index.php?id=57`

and a detailed installation guide is given on

`http://www.matlab.rwth-aachen.de/index.php?id=27`.

Ex. 2: [Matlab] (characteristics - Burgers equation)

Consider *initial conditions* of the form

$$u_0(x) = \sin(2\pi x)$$

for $x \in [0, 1]$ and a function

$$f(u) = \frac{1}{2}u^2.$$

a) Plot the characteristic lines defined by

$$x(t) = x_0 + f'(u_0(x_0)) \cdot t$$

in an $x-t$ diagram with the help of Matlab for different values of x_0, e.g., $x_0 \in \{0, \frac{1}{n}, \dots, \frac{2n-1}{n}, 2\}$ for $n \in \mathbb{N}_0$, up to $t = 0.2$.

b) Compute the critical points where two characteristic lines hit. Can you give an analytical expression for computing this point (**Hint:** $n \to \infty$)?

Points: 10

Ex. 3: (critical time for breakdown of the continuity of the solution)

Consider the conservation law $u_t+f(u)_x = 0$. Let $x(t)$ be the characteristics starting at $x(0) = x_0$ with the initial datum $u(x_0, 0) = u_0(x_0)$. Then, as long as the characteristic lines do not hit, the initial datum and the solution are related as

$$\begin{aligned} x_0(x, t) &= x - t\, f'(u(x, t)), \\ u(x, t) &= u_0(x_0(x, t)). \end{aligned}$$

Show that the first intersection of the characteristic lines, so the breakdown of the continuity of the solution, happens at time $t^* > 0$ such that

$$t^* = -\frac{1}{\min\limits_{x\in\mathbb{R}} \left(f'(u_0(x))_x\right)}.$$

(**Hint:** You can get some idea from what you have observed in Ex. 2, Also, note that the loss of continuity means that u_x should not exist.) **Points: 10**