Finite Volume and Finite Element Methods I&II

Summer term 19

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1. Exercise

Deadline: April 9th 2019.

Task 1: (Divergence, gradient and integration by parts)

Verify the following statements:

a) Let $f : \mathbb{R}^n \to \mathbb{R}$, $\underline{g} : \mathbb{R}^n \to \mathbb{R}^n$ be continuous differentiable. Then it holds

$$\text{div}(f\underline{g}) = \underline{g}^T \text{grad} f + f\ \text{div}\underline{g}.$$

b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous differentiable. Then there holds

$$\text{div}(\text{grad} f) = \Delta f := \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

c) Rewrite the statements in a) and b) by means of the ∇-operator.

d) (*Integration by parts*) Let $\Omega \subset \mathbb{R}^n$ be a compact domain with piecewise smooth boundary, furthermore let $u \in C^1(\overline{\Omega})$ and $\underline{w} \in C^1(\overline{\Omega}, \mathbb{R}^n)$, then it holds

$$\int_\Omega \nabla u \cdot \underline{w} \text{dx} = -\int_\Omega u \nabla \cdot \underline{w} \text{dx} + \int_{\partial\Omega} u\underline{w} \cdot n \text{ds}.$$

Task 2: (Temporal derivative of the Jacobian)

Let $\underline{\Phi}(\underline{x}, t)$ be the trajectory which originates in $\underline{x}$ at $t = 0$. The determinant of the Jacobian with respect to the spatial derivative is defined by

$$\mathcal{J}(\underline{x}, t) := \det \left(\frac{\partial \phi_j(\underline{x}, t)}{\partial x_i} \right)_{i,j=1,\dots,d}.$$

Furthermore we define $\underline{\underline{A}}(t) := \left(\frac{\partial \phi_j(\underline{x},t)}{\partial x_i} \right)_{i,j=1,\dots,d} = (\underline{a}_1(t), \dots, a_d(t))$.

Verify that for the temporal derivative of $\mathcal{J}$ is given by

$$\frac{d}{dt}\left(\mathcal{J}(\underline{x}, t)\right) = \frac{\partial}{\partial t}\mathcal{J}(\underline{x}, t) = \mathcal{J}(\underline{x}, t)\ \text{div}\ \underline{u}(\underline{\Phi}(\underline{x}, t), t)$$

Initially show that

$$\frac{d}{dt}\underline{a}_i(t) = \sum_{l=1}^{d} \frac{\partial u_i}{\partial x_l}(\underline{\Phi}(\underline{x}, t), t)\underline{a}_l(t).$$

In order to prove this use the telescope sum

$$\det \underline{\underline{A}}(t+h) - \det \underline{\underline{A}}(t) = \sum_{i=1}^{d} \Big(\det(\underline{a}_1(t), \ldots, \underline{a}_{i-1}(t), \underline{a}_i(t+h), \ldots, \underline{a}_d(t+h)) - \det(\underline{a}_1(t), \ldots, \underline{a}_i(t), \underline{a}_{i+1}(t+h), \ldots, \underline{a}_d(t+h))\Big)$$

and

$$\frac{d}{dt}\det \underline{\underline{A}}(t) = \lim_{h\to 0} \frac{1}{h}[\det \underline{\underline{A}}(t+h) - \det \underline{\underline{A}}(t)].$$

Task 3: (Incompressible flow)

A flow is called incompressible if for each $V \subseteq \Omega$ there holds:

$$|V_t| := \int_{V_t} dV = \text{const}$$

with respect to the time t.

Show that the following statements are equivalent:

a) A flow is incompressible

b) $\operatorname{div} \underline{u} = 0$

c) $\mathcal{J}(\underline{x}, t) = 1$

d) $\frac{D\rho}{Dt} = 0$, i.e., the density of mass is constant along the stream lines

Hint: Show: $a \Leftrightarrow b$, $a \Leftrightarrow c$, $b \Leftrightarrow d$