Model Order Reduction Techniques I & II

RB: General Preliminaries, Elliptic Problems

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Strong Formulation – Dirichlet Problem

Find u such that

$$\boxed{\begin{aligned} -\nabla^2 u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma \end{aligned}}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and Ω is a domain in $\mathbb{R}^2$ with boundary Γ.

In general, we assume

- $\Omega \subset \mathbb{R}^d$, $d = 1, \ldots, 3$, has a Lipschitz continuous boundary Γ (or $\partial\Omega$)
- Ω is an open domain, e.g., $\Omega = (0, 1)$ $(\bar{\Omega} = [0, 1])$

Minimization Principle – Statement

Find

$$\boxed{u = \arg\min_{w \in X} J(w)}$$

where

$$X = \{v \in H^1(\Omega) \mid v|_\Gamma = 0\}$$

and

$$J(w) = \frac{1}{2}\int_\Omega \underbrace{\nabla w \cdot \nabla w}_{w_x^2 + w_y^2} d\Omega - \int_\Omega f w d\Omega$$

Minimization Principle – Proof (sketch)

Let $w = u + v$, $u \in X$, $v \in X$, apply Gauss Theorem and boundary conditions to obtain

$$J(\underbrace{u + v}_{w}) = J(u) + \underbrace{\frac{1}{2}\int_{\Omega} \nabla v \cdot \nabla v d\Omega,}_{>0 \text{ unless } v=0} \qquad \forall v \in X$$

$$\boxed{\begin{gathered} J(w) > J(u), \qquad \forall w \in X,\ w \neq u \\ \Updownarrow \\ u \text{ is } \textit{the} \text{ minimizer of } J(w) \end{gathered}}$$

Weak Formulation

Let BILINEAR FORM

$$\boldsymbol{a}(\boldsymbol{w},\boldsymbol{v}) = \int_{\boldsymbol{\Omega}} \boldsymbol{\nabla} \boldsymbol{w} \cdot \boldsymbol{\nabla} \boldsymbol{v}\, \boldsymbol{d\Omega}, \quad \forall \boldsymbol{w},\boldsymbol{v} \in \boldsymbol{X}$$

and LINEAR FORM

$$\boldsymbol{f}(\boldsymbol{v}) = \int_{\boldsymbol{\Omega}} \boldsymbol{f}\, \boldsymbol{v}\, \boldsymbol{d\Omega}, \quad \forall \boldsymbol{v} \in \boldsymbol{X}$$

Weak Statement

Find $\boldsymbol{u} \in \boldsymbol{X}$, such that

$$\boldsymbol{a}(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{f}(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{X}$$

Neumann Problem

Find u such that STRONG

$$\boxed{\begin{aligned} -\nabla^2 u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma^D \\ \frac{\partial u}{\partial n} &= g && \text{on } \Gamma^N \end{aligned}}$$

where $\bar{\Gamma} = \bar{\Gamma}^D \cup \bar{\Gamma}^N$, Γ^D non-empty.

Find MINIMIZATION

$$\boxed{u = \arg\min_{w \in X} J(w)}$$

where

$$X = \{v \in H^1(\Omega) \mid v|_{\Gamma^D} = 0\}$$

$$J(w) = \frac{1}{2}\int_{\Omega} \nabla w \cdot \nabla w d\Omega - \int_{\Omega} f w d\Omega - \int_{\Gamma^N} g w dS$$

Neumann Problem – Weak Formulation

Let BILINEAR FORM

$$a(w,v) = \int_{\Omega} \nabla w \cdot \nabla v \, d\Omega, \quad \forall w, v \in X$$

and LINEAR FORM

$$f(v) = \int_{\Omega} f \, v \, d\Omega + \int_{\Gamma^N} g \, v \, dS, \quad \forall v \in X$$

Weak Statement

Find $u \in X$, such that

$$a(u,v) = f(v), \quad \forall v \in X$$

Summary

- Minimization/weak formulations are defined by:
  - a space $\boldsymbol{X}$;
  - a bilinear form $\boldsymbol{a}$;
  - a linear form $\boldsymbol{\ell}$.
- Essential (Dirichlet) boundary conditions are imposed by (or reflected in) $\boldsymbol{X}$
- Natural (Neumann) boundary conditions are imposed by (or reflected in) $\boldsymbol{J}$ (or $\boldsymbol{a}$ and $\boldsymbol{f}$).
- Points of departure for the finite element method are
  - the weak form (more generally); or
  - the minimization statement (if $\boldsymbol{a}$ is SPD).

Definitions

Definition (Linear space)

A space $\boldsymbol{Z}$ is a linear or vector space if, for any $\boldsymbol{\alpha} \in \mathbb{R}$, $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{Z}$,

$$\boldsymbol{\alpha}\, \boldsymbol{w} + \boldsymbol{v} \in \boldsymbol{Z}$$

Note: $\mathbb{R}$ denotes the real numbers, and $\mathbb{N}$ and $\mathbb{C}$ shall denote the natural and complex numbers, respectively.

Definition (Inner Product Space)

An inner product space $\boldsymbol{Z}$ is a linear space equipped with

- an inner product $(\boldsymbol{w}, \boldsymbol{v})_{\boldsymbol{Z}}$, $\forall \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{Z}$, and
- induced norm $\|\boldsymbol{w}\|_{\boldsymbol{Z}} = \sqrt{(\boldsymbol{w}, \boldsymbol{w})_{\boldsymbol{Z}}}$, $\forall \boldsymbol{w} \in \boldsymbol{Z}$.

Definitions

Definition (Inner Product)

An inner product $w, v \in Z \to (w, v)_Z \in \mathbb{R}$ has to satisfy

- Bilinearity
  $$(\alpha w + v, z)_Z = \alpha(w, z)_Z + (v, z)_Z, \quad \forall \alpha \in \mathbb{R},\ w, v, z \in Z$$
  $$(z, \alpha w + v)_Z = \alpha(z, w)_Z + (z, v)_Z, \quad \forall \alpha \in \mathbb{R},\ w, v, z \in Z$$
- Symmetry
  $$(w, v)_Z = (v, w)_Z, \quad \forall w, v \in Z$$
- Positivity
  $$(w, w)_Z > 0, \quad \forall w \in Z,\ w \neq 0$$
  $$(w, w)_Z = 0, \quad \text{only if } w = 0$$

Cauchy-Schwarz inequality:

$$(w, v)_Z \leq \|w\|_Z \|v\|_Z, \qquad \forall w, v \in Z.$$

Definitions

Definition (Norm)

A norm is a map $\|\cdot\| : Z \to \mathbb{R}$ such that

$$\begin{aligned} \|w\|_Z &> 0 && \forall w \in Z,\ w \neq 0, \\ \|\alpha w\|_Z &= |\alpha|\, \|w\|_Z && \forall \alpha \in \mathbb{R},\ \forall w \in Z, \\ \|w+v\|_Z &\leq \|w\|_Z + \|v\|_Z && \forall w \in Z, \forall v \in Z. \end{aligned}$$

Equivalence of norms $\|\cdot\|_Z$ and $\|\cdot\|_Y$: there exist positive constants $C_1,\ C_2$ such that

$$C_1\|v\|_Z \leq \|v\|_Y \leq C_2\|v\|_Z.$$

Definitions

Definition (Cartesian Product Spaces)

Given two inner product spaces Z_1 and Z_2, we define

$$Z = Z_1 \times Z_2 \equiv \{(w_1, w_2) | w_1 \in Z_1, w_2 \in Z_2\}$$

and given $w = (w_1, w_2) \in Z$, $v = (v_1, v_2) \in Z$, we define

$$w + v \equiv (w_1 + v_1, w_2 + v_2).$$

We also equip Z with the inner product

$$(w, v)_Z = (w_1, v_1)_{Z_1} + (w_2, v_2)_{Z_2}$$

and induced norm

$$\|w\|_Z = \sqrt{(w, w)_Z}.$$

Linear Forms

Definition (Linear Functional)

A functional $g : Z \to \mathbb{R}$ is a linear functional if, for any $\alpha \in \mathbb{R}$, $w, v \in Z$

$$g(\alpha w + v) = \alpha g(w) + g(v)$$

A linear form is bounded, or continuous, over Z if

$$|g(v)| \leq C \, \|v\|_Z, \qquad \forall v \in Z,$$

for some finite real constant C.

Dual Spaces

Definition (Dual Space)

Given Z, we define the dual space Z' as the space of all *bounded* linear functionals over Z. We associate to Z' the *dual norm*

$$\|g\|_{Z'} = \sup_{v \in Z} \frac{g(v)}{\|v\|_Z}, \qquad \forall g \in Z'.$$

Theorem (Riesz representation)

For any $g \in Z'$, there exists a unique $w_g \in Z$ such that

$$(w_g, v)_Z = g(v), \qquad \forall v \in Z.$$

It directly follows that

$$\boxed{\|g\|_{Z'} = \|w_g\|_Z.}$$

Bilinear Forms

Definition (Bilinear Form)

A form $b : Z_1 \times Z_2 \to \mathbb{R}$ is bilinear if, for any $\alpha \in \mathbb{R}$,

$$b(\alpha w + v, z) = \alpha b(w, z) + b(v, z), \quad \forall w, v \in Z_1, \; z \in Z_2$$

$$b(z, \alpha w + v) = \alpha b(z, w) + b(z, v), \quad \forall z \in Z_1, \; w, v \in Z_2$$

The bilinear form $b : Z \times Z \to \mathbb{R}$ is

- symmetric, if
$$b(w, v) = b(v, w), \qquad \forall w, v \in Z;$$
- skew-symmetric, if
$$b(w, v) = -b(v, w), \qquad \forall w, v \in Z;$$
- positive definite, if
$$b(v, v) \geq 0, \qquad \forall v \in Z,$$
with equality only for $v = 0$.

Bilinear Forms

The bilinear form $b : Z \times Z \to \mathbb{R}$ is

- positive semidefinite, if
$$b(v, v) \geq 0, \qquad \forall v \in Z.$$

We also define, for a general bilinear form $b : Z \times Z \to \mathbb{R}$,

- the symmetric part as
$$b_S(w, v) = 1/2\ (b(w, v) + b(v, w))\,, \quad \forall w, v \in Z;$$
- the skew-symmetric part as
$$b_{SS}(w, v) = 1/2\ (b(w, v) - b(v, w))\,, \quad \forall w, v \in Z.$$

Bilinear Forms

The bilinear form $b : Z \times Z \to \mathbb{R}$ is

- coercive over Z if

  $$\alpha \equiv \inf_{w \in Z} \frac{b(w,w)}{\|w\|_Z^2}$$

  is positive;

- continuous over Z if

  $$\gamma \equiv \sup_{w \in Z} \sup_{v \in Z} \frac{b(w,v)}{\|w\|_Z \|v\|_Z}$$

  is finite.

Parametric Linear and Bilinear Forms

We introduce

- $\mathcal{D} \in \mathbb{R}^P$: closed bounded parameter domain;
- $\mu = (\mu_1, \ldots, \mu_P) \in \mathcal{D}$: parameter vector.

We shall say that

- $g : Z \times \mathcal{D} \to \mathbb{R}$ is a parametric linear form if,
  for all $\mu \in \mathcal{D}$, $g(\cdot; \mu) : Z \to \mathbb{R}$ is a linear form;
- $b : Z \times Z \times \mathcal{D} \to \mathbb{R}$ is a parametric bilinear form if,
  for all $\mu \in \mathcal{D}$, $b(\cdot, \cdot; \mu) : Z \times Z \to \mathbb{R}$ is a bilinear form.

Concepts of symmetry, . . . directly extend to the parametric case.

Parametric Linear and Bilinear Forms

The parametric bilinear form $b : Z \times Z \times \mathcal{D} \to \mathbb{R}$ is

- coercive over Z if

$$\alpha(\mu) \equiv \inf_{w \in Z} \frac{b(w, w; \mu)}{\|w\|_Z^2}$$

  is positive for all $\mu \in \mathcal{D}$;

- continuous over Z if

$$\gamma(\mu) \equiv \sup_{w \in Z} \sup_{v \in Z} \frac{b(w, v; \mu)}{\|w\|_Z \|v\|_Z}$$

  is finite for all $\mu \in \mathcal{D}$.

We also define

$$\begin{aligned} (0 <)\ \alpha_0 &\equiv \min_{\mu \in \mathcal{D}} \alpha(\mu), \\ \gamma_0 &\equiv \max_{\mu \in \mathcal{D}} \gamma(\mu)\ (< \infty). \end{aligned}$$

Coercivity Eigenproblem

We have

$$\alpha(\mu) \equiv \inf_{w \in Z} \frac{b_S(w, w; \mu)}{\|w\|_Z^2};$$

Associated generalized eigenproblem:

Given $\mu \in \mathcal{D}$, find $(\chi^{co}, \nu^{co})_i(\mu) \in Z \times \mathbb{R},\ 1 \leq i \leq \dim(Z)$, such that

$$b_S(\chi_i^{co}(\mu), v; \mu) = \nu_i^{co}(\mu)(\chi_i^{co}(\mu), v)_Z, \qquad \forall v \in Z,$$

and

$$\|\chi_i^{co}(\mu)\|_Z = 1.$$

Let $\nu_1^{co}(\mu) \leq \nu_2^{co}(\mu) \leq \ldots \leq \nu_{\dim(Z)}^{co}(\mu)$ and b coercive, then

$$\alpha(\mu) = \nu_1^{co}(\mu) > 0.$$

Affine Parameter Dependence

We assume

$$g(v;\mu) = \sum_{q=1}^{Q_g} \Theta_g^q(\mu)\, g^q(v), \quad \forall v \in Z,$$

where, for $1 \leq q \leq Q_g$ (finite),

- parameter-*dependent* functions $\Theta_g^q : \mathcal{D} \to \mathbb{R}$,
- parameter-*independent* forms $g^q : Z \to \mathbb{R}$;

and

$$b(w,v;\mu) = \sum_{q=1}^{Q_b} \Theta_b^q(\mu)\, b^q(w,v), \quad \forall w, v \in Z,$$

where, for $1 \leq q \leq Q_b$ (finite),

- parameter-*dependent* functions $\Theta_b^q : \mathcal{D} \to \mathbb{R}$,
- parameter-*independent* forms $b^q : Z \times Z \to \mathbb{R}$.

Parametric Coercivity

Definition (Parametric Coercivity)

The coercive bilinear form $b : Z \times Z \times \mathcal{D} \to \mathbb{R}$

$$b(w, v; \mu) = \sum_{q=1}^{Q_b} \Theta_b^q(\mu)\, b^q(w, v), \qquad \forall w, v \in Z$$

is parametrically coercive if $c \equiv b_S$ is affine

$$c(w, v; \mu) = \sum_{q=1}^{Q_c} \Theta_c^q(\mu)\, c^q(w, v), \qquad \forall w, v \in Z$$

and satisfies

$$\Theta_c^q(\mu) > 0, \qquad \forall \mu \in \mathcal{D},\ 1 \le q \le Q_c,$$

and

$$c^q(v, v) \ge 0, \qquad \forall v \in Z,\ 1 \le q \le Q_c.$$

Scalar and Vector Field

We consider (real)

- scalar-valued field variables (e.g., temperature, pressure)
$$w : \Omega \to \mathbb{R}^{d_v=1}$$
- vector-valued field variables (e.g., displacement, velocity)
$$w : \Omega \to \mathbb{R}^{d_v},$$
where $w(x) = (w_1(x), \ldots, w_{d_v}(x))$;

and

- $\Omega \in \mathbb{R}^d,\ d = 1,\ 2,$ or 3 is an open bounded; domain
- $x = (x_1, \ldots, x_d) \in \Omega$;
- Ω has Lipschitz continuous boundary $\partial\Omega$; and
- we define the canonical basis vectors as $e_i,\ 1 \leq i \leq d$.

Multi-index Derivatives

Given a scalar (or one component of a vector)

- field $w : \Omega \to \mathbb{R}$ SPATIAL DERIVATIVE

$$(D^{\sigma} w)(x) = \frac{\partial^{\sigma} w}{\partial x_1^{\sigma_1} \dots \partial x_d^{\sigma_d}}$$

- parametric field $w : \Omega \times \mathcal{D} \to \mathbb{R}$ SENSITIVITY DERIVATIVE

$$(D_{\sigma} w)(x) = \frac{\partial^{\sigma} w}{\partial \mu_1^{\sigma_1} \dots \partial \mu_d^{\sigma_d}}$$

where

- $\sigma = (\sigma_1, \dots, \sigma_d)$, σ_i, $1 \le i \le d$, non-negative integers;
- $|\sigma| = \sum_{j=1}^{d} \sigma_j$ is the order of the derivative; and
- $I^{d,n}$ is set of all index vectors $\sigma \in \mathbb{N}_0^d$ such that $|\sigma| \le n$.

Function Spaces

Definition (Spaces of Continuous Functions)

Let $m \in \mathbb{N}_0$, the space $C^m(\Omega)$ is defined as

$$C^m(\Omega) \equiv \{w \mid \; D^\sigma w \in C^0(\Omega), \; \forall \sigma \in I^{d,m}\},$$

and $C^0(\Omega)$ is the space of continuous functions over $\Omega \in \mathbb{R}^d$.

- We denote by $C^\infty(\Omega)$ the space of functions w for which D^σ exists and is continuous for any order $|\sigma|$.

Lebesgue Spaces

Definition (Lebesgue Spaces)

We define, for $1 \le p < \infty$, the Lebesgue space $L^p(\Omega)$ as

$$L^p(\Omega) \equiv \left\{ w \text{ measurable} \mid \|w\|_{L^p(\Omega)} < \infty \right\}$$

where

$$\|w\|_{L^p(\Omega)} \equiv \left(\int_\Omega |w|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$$\|w\|_{L^\infty(\Omega)} \equiv \operatorname*{ess\,sup}_{x \in \Omega} |w(x)|, \qquad p = \infty.$$

Hilbert Space

Definition (Hilbert space)

Let $m \in \mathbb{N}_0$, the space $H^m(\Omega)$ is then defined as

$$H^m(\Omega) \equiv \{w \mid D^\sigma w \in L^2(\Omega),\ \forall \sigma \in I^{d,m}\},$$

with associated inner product

$$(w, v)_{H^m(\Omega)} \equiv \sum_{\sigma \in I^{d,m}} \int_\Omega D^\sigma w \, D^\sigma v \, dx,$$

and induced norm

$$\|w\|_{H^m(\Omega)} \equiv \sqrt{(w, w)_{H^m(\Omega)}}.$$

Special (Most Important) Cases

Since we only consider second-order PDEs, we require mostly

- $L^2(\Omega) = H^0(\Omega)$: Lebesgue Space $p = 2$

$$L^2(\Omega) \equiv \{w \text{ measurable} \mid \int_\Omega w^2 < \infty\}$$

with inner product and induced norm

$$(w, v)_{L^2(\Omega)} \equiv \int_\Omega w\, v, \qquad \forall w, v \in L^2(\Omega)$$

$$\|w\|_{L^2(\Omega)} \equiv \sqrt{(w, w)_{L^2(\Omega)}}, \quad \forall w \in L^2(\Omega),$$

$\Rightarrow$ Space of all functions $w : \Omega \to \mathbb{R}$ square-integrable over Ω.

Special (Most Important) Cases

Since we only consider second-order PDEs, we require mostly

- $H^1(\Omega)$

$$H^1(\Omega) \equiv \{w \in L^2(\Omega) \mid \frac{\partial w}{\partial x_i} \in L^2(\Omega),\ 1 \le i \le d\}$$

with inner product and induced norm

$$(w, v)_{H^1(\Omega)} \equiv \int_\Omega \nabla w \cdot \nabla v + wv, \quad \forall w, v \in H^1(\Omega),$$

$$\|w\|_{H^1(\Omega)} \equiv \sqrt{(w, w)_{H^1(\Omega)}}, \qquad \forall w \in H^1(\Omega),$$

and seminorm

$$|w|_{H^1(\Omega)} \equiv \int_\Omega \nabla w \cdot \nabla w, \qquad \forall w \in H^1(\Omega).$$

Special (Most Important) Cases

Since we only consider second-order PDEs, we require mostly

- and the space

$$H^1_0(\Omega) \equiv \{v \in H^1(\Omega) \mid v|_{\partial\Omega} = 0\}$$

  where $v = 0$ on the boundary $\partial\Omega$.

Note that, for any $v \in H^1_0(\Omega)$, we have

$$C_{PF}\|v\|_{H^1(\Omega)} \leq |v|_{H^1(\Omega)} \leq \|v\|_{H^1(\Omega)},$$

and thus

$$|v|_{H^1(\Omega)} = 0 \text{ implies } v = 0.$$

$\Rightarrow$ $|v|_{H^1(\Omega)}$ constitutes a norm for $v \in H^1_0(\Omega)$.

Projection

Definition (Projection)

Given Hilbert Spaces Y and $Z \subset Y$, the projection, $\Pi : Y \to Z$, of $y \in Y$ onto Z is defined as

$$(\Pi y, v)_Y = (y, v)_Y, \qquad \forall v \in Z$$

Properties

- Orthogonality: $(y - \Pi y, v)_Y = 0, \qquad \forall v \in Z$
- Idempotence: $\Pi(\Pi y) = \Pi y$
- Best Approximation
$$\|y - \Pi y\|_Y^2 = \inf_{v \in Z} \|y - v\|_Y^2$$
- Given an orthonormal basis $\{\phi_i\}_{i=1}^N$, $N = \dim(Z)$, then
$$\Pi y = \sum_{i=1}^{N} (\phi_i, y)_Y \, \phi_i, \qquad \forall y \in Y$$

Problem Statement

Given $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}} \subset \mathbb{R}^{\boldsymbol{P}}$, evaluate — $\boldsymbol{P}$-vector INPUT

$$\boldsymbol{s}^{\mathrm{e}}(\boldsymbol{\mu}) = \boldsymbol{\ell}(\boldsymbol{u}^{\mathrm{e}}(\boldsymbol{\mu}); \boldsymbol{\mu}) \qquad \text{OUTPUT}$$

where $\boldsymbol{u}^{\mathrm{e}}(\boldsymbol{x}; \boldsymbol{\mu}) \in \boldsymbol{X}^{\mathrm{e}}(\boldsymbol{\Omega})$ satisfies — FIELD $(\boldsymbol{x}; \boldsymbol{\mu})$

$$\boldsymbol{a}(\boldsymbol{u}^{\mathrm{e}}(\boldsymbol{\mu}), \boldsymbol{v}; \boldsymbol{\mu}) = \boldsymbol{f}(\boldsymbol{v}; \boldsymbol{\mu}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{X}^{\mathrm{e}}(\boldsymbol{\Omega}). \qquad \boldsymbol{\mu}\text{PDE}$$

- Here $^{\mathrm{e}}$ refers to "exact," i.e., the exact (or analytic) solution to the problem for the prescribed mathematical model.
- To begin, we consider elliptic problems which are linear, coercive, affine, and compliant — explanations to follow ...

Definitions

μ: input parameter - $\mu = (\mu_1, \mu_2, \ldots, \mu_P)$; P-tuple

$\mathcal{D}$: parameter domain in $\mathbb{R}^P$;

Ω: spatial domain in $\mathbb{R}^d$;

s^{e}: output;

ℓ: output functional;

u^{e}: field variable;

X^{e}: function space $(H_0^1(\Omega))^\nu \subset X^{\mathrm{e}} \subset (H^1(\Omega))^\nu$ †,
with inner product $(w, v)_{X^{\mathrm{e}}}, \ \forall w, v \in X^{\mathrm{e}}$,
and induced norm $\|w\|_{X^{\mathrm{e}}} = \sqrt{(w, w)_{X^{\mathrm{e}}}}, \ \forall w \in X^{\mathrm{e}}$.

† For simplicity we first assume $\nu = 1$.

Reference Geometry

Note Ω is parameter-independent:

- the reduced basis requires a common spatial configuration, i.e., a reference domain Ω_{ref}
- Introduce a piecewise affine mapping $\mathcal{T}(\cdot;\mu) : \Omega \to \Omega_o(\mu)$

$$a_o(w_o, v_o; \mu) \text{ over } \Omega_o(\mu)$$
$$\Downarrow$$
$$\mathcal{T}(\cdot;\mu)^{-1} : \Omega_o(\mu) \to \Omega_{\text{ref}} \equiv \Omega \qquad (\Omega_{\text{ref}} = \Omega_o(\mu_{\text{ref}}))$$
$$\Downarrow$$
$$a(w, v; \mu) \text{ over } \Omega$$

where $a(w, v; \mu) = a_o(w_o \circ \mathcal{T}_\mu, v_o \circ \mathcal{T}_\mu; \mu)$

We will discuss this issue in detail later on.

Hypotheses – Continuity, Stability, Compliance

$$\left.\begin{array}{rl} a(\cdot,\cdot;\mu): & \text{bilinear,} \\ & \text{symmetric,} \\ & X^{\mathrm{e}}\text{-continuous,} \\ & X^{\mathrm{e}}\text{-coercive form,} \quad \forall \mu \in \mathcal{D}; \\ f(\cdot;\mu),\ \ell(\cdot;\mu): & \text{linear,} \\ & X^{\mathrm{e}}\text{-bounded,} \quad \forall \mu \in \mathcal{D} \end{array}\right\} \mu\text{PDE}$$

We first consider the compliant case:

- $\ell(\cdot;\mu) = f(\cdot;\mu),\ \forall \mu \in \mathcal{D}$
- a symmetric

Hypotheses – Affine parameter dependence

Require also $f(v;\mu)$, $\ell(v;\mu)$

$$a(w,v;\mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\ a^q(w,v),$$

where for $q = 1,\dots,Q_a$

$\Theta_a^q : \mathcal{D} \to \mathbb{R}$, μ-dependent functions;
$a^q : X^{\mathrm{e}} \times X^{\mathrm{e}} \to \mathbb{R}$, μ-independent forms.

This assumption

- is broadly applicable to many instances of property and geometry parametric variations, but
- can also be relaxed (see lectures on "nonaffine problems").

Inner Products and Norms

- Energy inner product and induced norm (parameter-dependent)

$$\begin{aligned} (((w,v)))_{\mu} &= a(w,v;\mu), && \forall w,v \in X^{\mathrm{e}} \\ |||w|||_{\mu} &= \sqrt{a(w,w;\mu)}, && \forall w \in X^{\mathrm{e}} \end{aligned}$$

- X^{e}-inner product and induced norm (parameter-independent)

$$\begin{aligned} (w,v)_{X^{\mathrm{e}}} &\equiv (((w,v)))_{\bar{\mu}} \, (= a(w,v;\bar{\mu})), && \forall w,v \in X^{\mathrm{e}} \\ \|w\|_{X^{\mathrm{e}}} &\equiv |||w|||_{\bar{\mu}} \, (= \sqrt{a(w,w;\bar{\mu})}), && \forall w \in X^{\mathrm{e}} \end{aligned}$$

Coercivity and Continuity constants

Recall

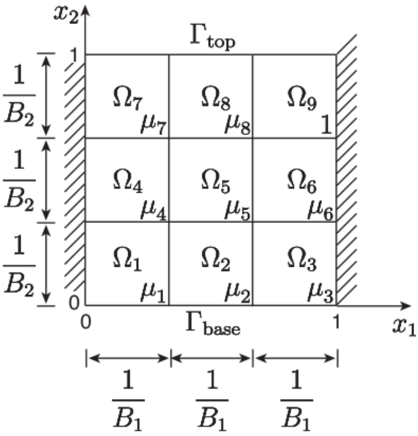
- Coercivity constant

$$(0 <)\ \alpha^{\mathrm{e}}(\mu) \equiv \inf_{w \in X^{\mathrm{e}}} \frac{a(w, w; \mu)}{\|w\|^2_{X^{\mathrm{e}}}};$$

- Continuity constant

$$\gamma^{\mathrm{e}}(\mu) \equiv \sup_{w \in X^{\mathrm{e}}} \sup_{v \in X^{\mathrm{e}}} \frac{a(w, v; \mu)}{\|w\|_{X^{\mathrm{e}}} \|v\|_{X^{\mathrm{e}}}}\ (< \infty).$$

Example: ThermalBlock



$$\overline{\Omega} = \cup_{i=1}^{B_1 B_2} \overline{\Omega}_i$$

Example: ThermalBlock

Problem Statement

Given $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_P) \subset \mathcal{D} \equiv [\mu^{\min}, \mu^{\max}]^P$, evaluate

$$s^{\mathrm{e}}(\boldsymbol{\mu}) = f(u^{\mathrm{e}}(\boldsymbol{\mu})) \qquad\qquad \ell = f$$

where $u^{\mathrm{e}}(\boldsymbol{\mu}) \in X^{\mathrm{e}} \equiv \{v \in H^1(\Omega) \mid v|_{\Gamma_{\mathrm{top}}} = 0\}$ satisfies

$$a(u^{\mathrm{e}}(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = f(v), \quad \forall\, v \in X^{\mathrm{e}}.$$

Here, $P = B_1 B_2 - 1$, and we assume that

$$\mu^{\min} = 1/\sqrt{\mu_r},\ \mu^{\max} = \sqrt{\mu_r}, \text{ for } 1 < \mu_r < \infty,$$

such that $\mu^{\max}/\mu^{\min} = \mu_r$.

Example: ThermalBlock

Here, $\forall v \in X^{\mathrm{e}}$, compliant $\ell = f$

$$f(v) \equiv \int_{\Gamma_{\mathrm{base}}} v,$$

and, $\forall w, v \in X^{\mathrm{e}}$,

$$a(w, v; \mu) = \sum_{i=1}^{P} \mu_i \int_{\Omega_i} \nabla w \cdot \nabla v + \int_{\Omega_{P+1}} \nabla w \cdot \nabla v,$$

where $\overline{\Omega} = \cup_{i=1}^{P+1} \overline{\Omega}_i$.

Example: ThermalBlock

Inner product, $w, v \in X^{\mathrm{e}}$,

$$(w, v)_{X^{\mathrm{e}}} = \sum_{i=1}^{P} \overline{\mu}_i \int_{\Omega_i} \nabla w \cdot \nabla v + \int_{\Omega_{P+1}} \nabla w \cdot \nabla v,$$

where $\overline{\mu}_i$ is the reference parameter.

The bilinear form a is

- symmetric,
- (parametrically) coercive,
  $$0 < \frac{1}{\sqrt{\mu_r}} \leq \mathrm{Min}(\mu_1/\overline{\mu}_1, \ldots, \mu_P/\overline{\mu}_P, 1) \leq \alpha^{\mathrm{e}}(\mu),$$
- and continuous,
  $$\gamma^{\mathrm{e}}(\mu) < \mathrm{Max}(\mu_1/\overline{\mu}_1, \ldots, \mu_P/\overline{\mu}_P, 1) \leq \sqrt{\mu_r} < \infty.$$

The linear form f is bounded.

Example: ThermalBlock

We obtain

$$a(w, v; \mu) = \sum_{q=1}^{Q_a = P+1} \Theta^q(\mu)\, a^q(w, v):$$

for

$$\begin{aligned}
\Theta^1(\mu) &= \mu_1, & a^1(w, v) &= \int_{\Omega_1} \nabla w \cdot \nabla v\ ; \\
&\vdots \\
\Theta^P(\mu) &= \mu_P, & a^P(w, v) &= \int_{\Omega_P} \nabla w \cdot \nabla v\ ; \\
\Theta^{P+1}(\mu) &= 1, & a^{P+1}(w, v) &= \int_{\Omega_{P+1}} \nabla w \cdot \nabla v\ .
\end{aligned}$$

$\Rightarrow$ affine assumption is honored

Approximation Space

We replace the "exact" statement by a "truth" approximation, i.e., we introduce

- an approximation space $X^{\mathcal{N}} \subset X^{\mathrm{e}}$ of dimension $\dim(X^{\mathcal{N}}) = \mathcal{N}$, and
- an associated set of basis functions $\varphi_k^{\mathcal{N}} \in X^{\mathcal{N}}$, $1 \leq k \leq \mathcal{N}$.

We also equip $X^{\mathcal{N}}$ with inner product and induced norm

$$\begin{aligned} (w, v)_{X^{\mathcal{N}}} &\equiv (w, v)_{X^{\mathrm{e}}} \equiv a(w, v; \bar{\mu}), \qquad \forall w, v \in X^{\mathcal{N}} \\ \|w\|_{X^{\mathcal{N}}} &\equiv \|w\|_{X^{\mathrm{e}}} \equiv \sqrt{a(w, w; \bar{\mu})}, \quad \forall w \in X^{\mathcal{N}} \end{aligned}$$

Approximation Space

- Coercivity constant of a over $X^{\mathcal{N}}$

$$\alpha^{\mathcal{N}}(\mu) \equiv \inf_{w \in X^{\mathcal{N}}} \frac{a(w, w; \mu)}{\|w\|_{X^{\mathcal{N}}}^2};$$

- Continuity constant of a over $X^{\mathcal{N}}$

$$\gamma^{\mathcal{N}}(\mu) \equiv \sup_{w \in X^{\mathcal{N}}} \sup_{v \in X^{\mathcal{N}}} \frac{a(w, v; \mu)}{\|w\|_{X^{\mathcal{N}}} \|v\|_{X^{\mathcal{N}}}}.$$

Note: since $X^{\mathcal{N}} \subset X^{\mathrm{e}}$,

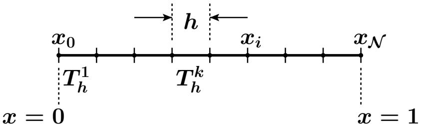
$$\begin{aligned} \alpha^{\mathcal{N}}(\mu) &\geq \alpha^{\mathrm{e}}(\mu)\ (> 0), \quad && \forall \mu \in \mathcal{D}, \\ \gamma^{\mathcal{N}}(\mu) &\leq \gamma^{\mathrm{e}}(\mu)\ (< \infty), \quad && \forall \mu \in \mathcal{D}. \end{aligned}$$

Approximation Space – Finite Element

We consider linear ($\mathbb{P}_1$) or quadratic ($\mathbb{P}_2$) Finite Element Spaces

Example: Consider the domain $\Omega \equiv]0,1[\subset \mathbb{R}^{d=1}$

- Triangulation $\mathcal{T}_h$ with segments T_h^j, $1 \leq j \leq \mathcal{N}$

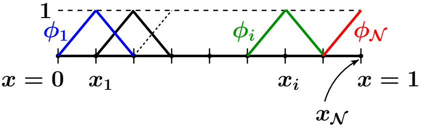


Properties

- T_h^j are open (closure $\overline{T}_h^j$)
- $T_h^j \cap T_h^i = \emptyset$ for $j \neq i$
- $\overline{\Omega} = \bigcup_{T_h \in \mathcal{T}_h} \overline{T}_h$

Approximation Space – Finite Element

- Nodal basis functions $\varphi_i(x)$, $1 \le i \le \mathcal{N}$



Properties

  - Compact support in Ω (i.e., φ_i nonzero only on $\overline{T}_h^i \bigcup \overline{T}_h^{i+1}$)
  - For all $v \in X^{\mathcal{N}}$, $\exists$ unique $v_n \in \mathbb{R}$, such that $v(x) = \sum_{n=1}^{\mathcal{N}} v_n \varphi_n(x)$

- "Truth" finite element approximation space
  $X^{\mathcal{N}} = \{v \in X^{\mathrm{e}} \mid v|_{T_h^j} \in \mathbb{P}_1(T_h^j),\ 1 \le j \le \mathcal{N}\};$
  (Space of functions which are linear over each element)

"Truth" Problem Statement

Given $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}} \subset \mathbb{R}^{P}$, evaluate

Galerkin projection

$$s^{\mathcal{N}}(\boldsymbol{\mu}) = f(u^{\mathcal{N}}(\boldsymbol{\mu}); \boldsymbol{\mu})$$

where $u^{\mathcal{N}}(x; \boldsymbol{\mu}) \in X^{\mathcal{N}}(\Omega)$ satisfies

$$a(u^{\mathcal{N}}(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = f(v; \boldsymbol{\mu}), \quad \forall\, v \in X^{\mathcal{N}}(\Omega).$$

Assumption:
No variational crimes in Galerkin approximation, i.e.,

- truth approximation represents exact geometry,
- all quadratures are exact.

"Truth" Problem Statement

Galerkin projection convergent:

- $u^{\mathcal{N}}(\mu) \to u^{\mathrm{e}}(\mu)$ as $\mathcal{N} \to \infty$; and (f is bounded)
- $s^{\mathcal{N}}(\mu) \to s^{\mathrm{e}}(\mu)$ as $\mathcal{N} \to \infty$.

We choose $\mathcal{N}$ such that $\mathcal{N}$ *conservative*

$$\varepsilon^{\mathcal{N}} = \max_{\mu \in \mathcal{D}} \|u^{\mathrm{e}}(\mu) - u^{\mathcal{N}}(\mu)\|_{X^{\mathrm{e}}} \ll \epsilon_{\mathrm{tol,\,min}},$$

and hence we require that our RB formulation be

(a) numerically stable as $\mathcal{N} \to \infty$; and

(b) computationally efficient as $\mathcal{N} \to \infty$.

Truth Approximation – Role

We shall

(*i*) *build* our reduced basis approximation upon "truth" solutions $\boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu}) \in \boldsymbol{X}^{\mathcal{N}}$;

(*ii*) *measure* the error in the reduced basis approximation relative to the "truth" solution $\boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu}) \in \boldsymbol{X}^{\mathcal{N}}$ (and $\boldsymbol{s}^{\mathcal{N}}(\boldsymbol{\mu})$);

$\Rightarrow \boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu})$ is a *calculable surrogate* for $\boldsymbol{u}^{\mathrm{e}}(\boldsymbol{\mu})$.

$$\|\boldsymbol{u}^{\mathrm{e}}(\boldsymbol{\mu}) - \boldsymbol{u}_N(\boldsymbol{\mu})\|_{X^{\mathrm{e}}} \leq \underbrace{\|\boldsymbol{u}^{\mathrm{e}}(\boldsymbol{\mu}) - \boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu})\|_{X^{\mathrm{e}}}}_{\leq \varepsilon_{\mathbf{exact}}} + \underbrace{\|\boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu}) - \boldsymbol{u}_N(\boldsymbol{\mu})\|_{X^{\mathrm{e}}}}_{\leq \varepsilon_{\mathbf{tol,min}}}$$

with $\varepsilon_{\mathbf{exact}} \ll \varepsilon_{\mathbf{tol,min}}$.

Algebraic Equations

Expand $$u^{\mathcal{N}}(x;\mu) = \sum_{j=1}^{\mathcal{N}} u_j^{\mathcal{N}}(\mu)\varphi_j^{\mathcal{N}}(x)$$

Given $\mu \in \mathcal{D}$, evaluate

$$s^{\mathcal{N}}(\mu) = (\underline{F}^{\mathcal{N}}(\mu))^T\, \underline{u}^{\mathcal{N}}(\mu)$$

where $\underline{u}^{\mathcal{N}}(\mu) \equiv [u_1^{\mathcal{N}}(\mu)\; u_2^{\mathcal{N}}(\mu) \ldots u_{\mathcal{N}}^{\mathcal{N}}(\mu)]^T \in \mathbb{R}^{\mathcal{N}}$ satisfies

$$\underline{A}^{\mathcal{N}}(\mu)\, \underline{u}^{\mathcal{N}}(\mu) = \underline{F}^{\mathcal{N}}(\mu).$$

- Stiffness matrix $\underline{A}^{\mathcal{N}}(\mu) \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$
$$A_{ij}^{\mathcal{N}}(\mu) = a(\varphi_j^{\mathcal{N}}, \varphi_i^{\mathcal{N}}; \mu), \qquad 1 \le i,j \le \mathcal{N}$$
- Load/source vector $\underline{F}^{\mathcal{N}}(\mu) \in \mathbb{R}^{\mathcal{N}}$
$$F_i^{\mathcal{N}}(\mu) = f(\varphi_i^{\mathcal{N}}; \mu), \qquad 1 \le i \le \mathcal{N}$$

Algebraic Equations

Affine Assumption

- Stiffness matrix: $\underline{\boldsymbol{A}}^{\mathcal{N}}(\boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \boldsymbol{\Theta}_a^q(\boldsymbol{\mu}) \underline{\mathbb{A}}^{\mathcal{N}\,q}$, where
  $\underline{\mathbb{A}}^{\mathcal{N}\,q} \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}},\ 1 \le q \le Q_a$, and
  $\mathbb{A}_{ij}^{\mathcal{N}\,q} = a^q(\varphi_j^{\mathcal{N}}, \varphi_i^{\mathcal{N}}), \quad 1 \le i,j \le \mathcal{N},\ 1 \le q \le Q_a.$
- Load/source vector: $\underline{\boldsymbol{F}}^{\mathcal{N}}(\boldsymbol{\mu}) = \sum_{q=1}^{Q_f} \boldsymbol{\Theta}_f^q(\boldsymbol{\mu}) \underline{\mathbb{F}}^{\mathcal{N}\,q}$, where
  $\underline{\mathbb{F}}^{\mathcal{N}\,q} \in \mathbb{R}^{\mathcal{N}},\ 1 \le q \le Q_f$, and
  $\mathbb{F}_i^{\mathcal{N}\,q} = f^q(\varphi_i^{\mathcal{N}}), \quad 1 \le i \le \mathcal{N},\ 1 \le q \le Q_f.$

Note: the $\underline{\mathbb{A}}^{\mathcal{N}\,q}$ and $\underline{\mathbb{F}}^{\mathcal{N}\,q}$ are parameter-independent.

Algebraic Equations

We also introduce $\underline{\mathbb{X}}^{\mathcal{N}} \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$, such that

$$\mathbb{X}^{\mathcal{N}}_{ij} = (\varphi_j^{\mathcal{N}}, \varphi_i^{\mathcal{N}})_{X^{\mathcal{N}}}, \quad 1 \leq i, j \leq \mathcal{N}$$

Then, given $w, v \in X^{\mathcal{N}}$, we expand

$$w = \sum_{j=1}^{\mathcal{N}} w_j \varphi_j^{\mathcal{N}} \text{ and } v = \sum_{j=1}^{\mathcal{N}} v_j \varphi_j^{\mathcal{N}},$$

and thus

$$(w, v)_{X^{\mathcal{N}}} = \sum_{j=1}^{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} w_j v_i (\varphi_j^{\mathcal{N}}, \varphi_i^{\mathcal{N}})_{X^{\mathcal{N}}} = \underline{w}^T \underline{\mathbb{X}}^{\mathcal{N}} \underline{v},$$

where

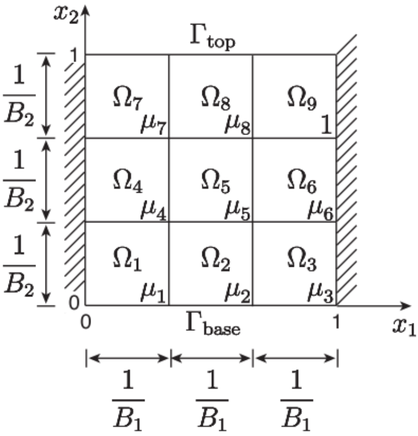
$$\underline{w} \equiv [w_1 \ w_2 \ldots w_{\mathcal{N}}]^T \in \mathbb{R}^{\mathcal{N}} \text{ and } \underline{v} \equiv [v_1 \ v_2 \ldots v_{\mathcal{N}}]^T \in \mathbb{R}^{\mathcal{N}}.$$

Nomenclature

For the rest of this course we will use the following notation

$()^{\mathrm{e}}$	exact problem statement $\boldsymbol{X}^{\mathrm{e}}$, $\boldsymbol{u}^{\mathrm{e}}$, $\boldsymbol{s}^{\mathrm{e}}$, ...
$() = ()^{\mathcal{N}}$	truth approximation e.g. $\boldsymbol{X} = \boldsymbol{X}^{\mathcal{N}}$, $\boldsymbol{u} = \boldsymbol{u}^{\mathcal{N}}$, $\boldsymbol{s} = \boldsymbol{s}^{\mathcal{N}}$, ...
$()_{N} = ()_{N}^{\mathcal{N}}$	reduced basis approximation e.g. $\boldsymbol{X}_N = \boldsymbol{X}_N^{\mathcal{N}}$, $\boldsymbol{u}_N = \boldsymbol{u}_N^{\mathcal{N}}$, $\boldsymbol{s}_N = \boldsymbol{s}_N^{\mathcal{N}}$, ...

Example: thermal block



$$\overline{\boldsymbol{\Omega}} = \cup_{i=1}^{B_1 B_2} \overline{\boldsymbol{\Omega}}_i$$

Example: thermal block

"Truth" Problem Statement

Given $\mu = (\mu_1, \ldots, \mu_P) \subset \mathcal{D} \equiv [\mu^{\min}, \mu^{\max}]^P$, evaluate

$$s^{\mathcal{N}}(\mu) = f(u^{\mathcal{N}}(\mu)) \qquad \ell = f$$

where $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}}$ satisfies

$$a(u^{\mathcal{N}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathcal{N}}.$$

Here, $P = B_1 B_2 - 1$ and we require $0 < \mu^{\min} < \mu^{\max} < \infty$.

Example: thermal block

"Truth" Problem Statement

Given $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_P) \subset \mathcal{D} \equiv [\mu^{\min}, \mu^{\max}]^P$, evaluate

$$s(\boldsymbol{\mu}) = f(u(\boldsymbol{\mu})) \qquad\qquad \ell = f$$

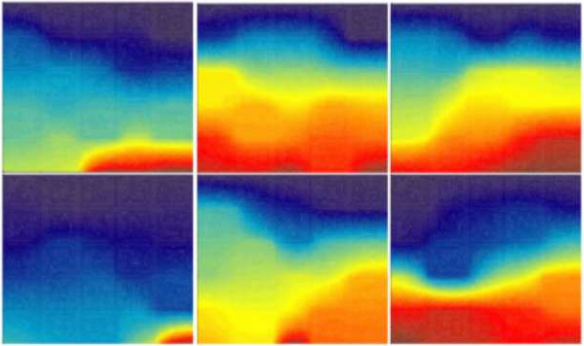
where $u(\boldsymbol{\mu}) \in X$ satisfies

$$a(u(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = f(v), \quad \forall\, v \in X.$$

Here, $P = B_1 B_2 - 1$ and we require $0 < \mu^{\min} < \mu^{\max} < \infty$.

Example: thermal block

Representative Solution ($P = 24$)



Source: S. Sen, NHT Part B

Reduced Basis (RB) Objective

For any $\varepsilon_{\mathbf{des}} > \mathbf{0}$, evaluate ACCURACY

$$\mu \in \mathcal{D} \to s_N(\mu) \; (\approx s^{\mathcal{N}}(\mu)) \text{ and } \Delta_N^s(\mu)$$

that provably achieves desired accuracy RELIABILITY

$$|s^{\mathcal{N}}(\mu) - s_N(\mu)| \leq \Delta_N^s(\mu) \leq \varepsilon_{\mathbf{des}}$$

but at (very low) marginal cost $\partial t_{\mathbf{comp}}$ EFFICIENCY

independent of $\mathcal{N}$ as $\mathcal{N} \to \infty$.

Here, $\partial t_{\mathbf{comp}}$ is the time to perform one additional certified evaluation $\mu \to (s_N(\mu), \Delta_N^s(\mu))$.

RB Objective – Rapid Convergence

Find a reduced basis approximation

$$s_N(\mu) \in \mathbb{R} \text{ and } u_N(\mu) \in X_N \subset X^{\mathcal{N}}$$

for all $\mu \in \mathcal{D}$, such that

$$s_N(\mu) \to s^{\mathcal{N}}(\mu) \text{ and } u_N(\mu) \to u^{\mathcal{N}}(\mu)$$

rapidly as $N = \dim(X_N) \to \infty (= 10 - 200)$.

Note that the convergence rate should be independent of $\mathcal{N}$.

RB Objective – Rigor & Certainty

A posteriori error bounds $\mathbf{\Delta}_N(\boldsymbol{\mu})$ and $\mathbf{\Delta}_N^s(\boldsymbol{\mu})$:

$$\mathbf{1} \text{ (rigor)} \leq \frac{\mathbf{\Delta}_N(\boldsymbol{\mu})}{\|\boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu}) - \boldsymbol{u}_N(\boldsymbol{\mu})\|_X} \leq \boldsymbol{C} \text{ (sharpness)}$$

and

$$\mathbf{1} \text{ (rigor)} \leq \frac{\mathbf{\Delta}_N^s(\boldsymbol{\mu})}{|\boldsymbol{s}^{\mathcal{N}}(\boldsymbol{\mu}) - \boldsymbol{s}_N(\boldsymbol{\mu})|} \leq \boldsymbol{C} \text{ (sharpness)}$$

for all $N \in \mathbb{N} \equiv \{1, \ldots, N_{\max}\}$ and all $\boldsymbol{\mu} \in \mathcal{D}$.

RB Objective – Computational Efficiency

Offline-Online computational strategies:

Offline: expensive preprocessing

$$t_{\mathbf{comp}}^{\mathbf{Offline}} \gg \mathbf{cost}\{\mu \to s^{\mathcal{N}}(\mu)\}$$

BUT

Online: very rapid certified RB input-output evaluation

$$\partial t_{\mathbf{comp}} \equiv \mathbf{marginal\ cost}\{\mu \to s_N(\mu), \Delta_N^s(\mu)\}$$

depends only on Q and N – not on $\mathcal{N}$.

$\Rightarrow$ we may thus choose $\mathcal{N}$ very conservatively.

RB Objective – Relevance

Real-Time Context (parameter estimation, ...):

$$\begin{array}{ccc} \boldsymbol{\mu} & \rightarrow & s_N(\boldsymbol{\mu}),\ \Delta_N^s(\boldsymbol{\mu}). \\ t_0 \text{ (“need”)} & & t_0 + \partial t_{\text{comp}} \text{ (“response”)} \end{array}$$

Many-Query Context (design, ...):

$$\begin{array}{ccc} \boldsymbol{\mu}_j & \rightarrow & (s_N(\boldsymbol{\mu}_j),\ \Delta_N^s(\boldsymbol{\mu}_j)),\ j = 1, \ldots, J. \\ t_0 & & t_0 + \partial t_{\text{comp}}\, J \text{ as } J \rightarrow \infty \end{array}$$

$\Rightarrow$ Low marginal (real-time) and/or low average (many-query) cost.

Parametric Manifold $\mathcal{M}^{\mathcal{N}}$

We assume

- the form a is continuous and coercive (or inf-sup stable); and
- affine μ-dependence; and
- the $\Theta^q(\mu)$, $1 \leq q \leq Q$, are smooth (i.e., $\Theta^q \in C^\infty(\mathcal{D})$);

then

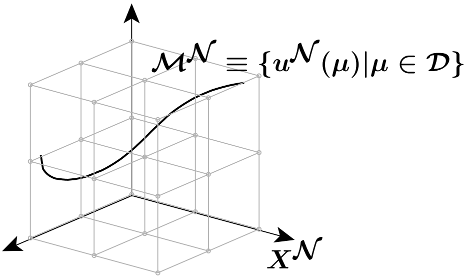
$$\mathcal{M}^{\mathcal{N}} \equiv \{u^{\mathcal{N}}(\mu) \,|\, \forall \mu \in \mathcal{D}\}$$

is a smooth P-dimensional manifold in $X^{\mathcal{N}}$, since

$$\|D_\sigma y^{\mathcal{N}}(\mu)\|_X \leq C_\sigma, \ \forall \mu \in \mathcal{D}, \text{ for any order } |\sigma| \in \mathbb{N}_{+0}.$$

Parametric Manifold $\mathcal{M}^{\mathcal{N}}$

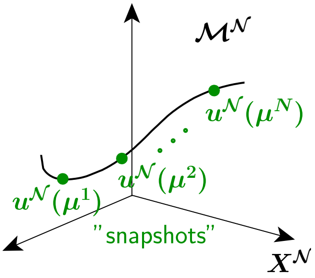
To approximate $u^{\mathcal{N}}(\mu)$, and hence $s^{\mathcal{N}}(\mu)$,
we need not represent every possible function in $X^{\mathcal{N}}$.



Early work: [ASB], [NP], [FR], [Po], [Pe], [G], [IR], . . .

Parametric Manifold $\mathcal{M}^{\mathcal{N}}$

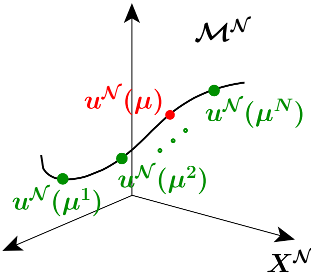
To approximate $u^{\mathcal{N}}(\mu)$, and hence $s^{\mathcal{N}}(\mu)$,
we need not represent every possible function in $X^{\mathcal{N}}$.



LOCALIZATION

Parametric Manifold $\mathcal{M}^{\mathcal{N}}$

To approximate $u^{\mathcal{N}}(\mu)$, and hence $s^{\mathcal{N}}(\mu)$,
we need not represent every possible function in $X^{\mathcal{N}}$.



SMOOTHNESS

Spaces & Bases

We define the RB approximation space

$$X_N = \mathrm{span}\{\xi^n,\ 1 \le n \le N\}, \quad 1 \le N \le N_{\max},$$

with linearly independent basis functions

$$\xi^n \in X, \quad 1 \le n \le N_{\max}.$$

We thus obtain

$$X_N \subset X, \quad \dim(X_N) = N, \quad 1 \le N \le N_{\max},$$

and "nested" (hierarchical) spaces

$$X_1 \subset X_2 \subset \ldots \subset X_{N_{\max}-1} \subset X_{N_{\max}} (\subset X).$$

We denote non-hierarchical RB spaces as X_N^{nh}, $1 \le N \le N_{\max}$,

$$X_N^{\mathrm{nh}} \subset X, \quad \dim(X_N^{\mathrm{nh}}) = N, \quad 1 \le N \le N_{\max}.$$

Spaces & Bases – Lagrangian

Parameter samples:

$$S_N = \{\mu^1 \in \mathcal{D}, \ldots, \mu^N \in \mathcal{D}\}, \quad 1 \leq N \leq N_{\max},$$

with nested

$$S_1 \subset S_2 \subset \ldots \subset S_{N_{\max}-1} \subset S_{N_{\max}} \subset \mathcal{D}.$$

Lagrangian reduced basis spaces: hierarchical

$$W_N = \text{span}\{ \underbrace{u(\mu^n)}_{\text{"snapshots"}}, 1 \leq n \leq N\}, \quad 1 \leq N \leq N_{\max},$$

with

$$W_1 \subset W_2 \subset \ldots \subset W_{N_{\max}-1} \subset W_{N_{\max}} (\subset X).$$

Spaces & Bases – Taylor & Hermite

Taylor reduced basis spaces: hierarchical

$$W_N^{\text{Taylor}} = \text{span}\{D_\sigma u(\mu),\ \forall \sigma \in I^{P,N-1}\}, \quad 1 \leq N \leq N_{\max},$$

field variable and sensitivity derivatives at one point in $\mathcal{D}$.

Hermite reduced basis spaces: hierarchical

$$W_N^{\text{Hermite}} \text{ ``}{=}\text{'' } W_N^{\text{Lagrangian}} \cup W_N^{\text{Taylor}}$$

field variable and sensitivity derivatives at several points in $\mathcal{D}$

Note: We will exclusively use Lagrangian RB spaces in this course.

Spaces & Bases – Orthogonal Basis

Given $\xi^n = u(\mu^n)$, $1 \leq n \leq N_{\max}$ (Lagrange case) we construct the basis set $\{\zeta^n\}$, $1 \leq n \leq N_{\max}$, from

Gram-Schmidt Orthogonalization

$\zeta^1 = \xi^1/\|\xi^1\|_X;$

for $n = 2 : N_{\max}$

$$z^n = \xi^n - \sum_{m=1}^{n-1} (\xi^n, \zeta^m)_X \, \zeta^m;$$

$$\zeta^n = z^n/\|z^n\|_X;$$

end.

Note: $(\zeta^n, \zeta^m)_X = \delta_{nm}, \quad 1 \leq n, m \leq N_{\max}$

Spaces & Bases – Orthogonal Basis

Given reduced basis space

$$X_N = \text{span}\{\zeta^n,\ 1 \le n \le N\}, \quad 1 \le N \le N_{\max}.$$

we can express any $w_N \in X_N$ as

$$w_N = \sum_{n=1}^{N} w_{N\,n} \zeta^n$$

for unique $w_{N\,n} \in \mathbb{R},\ 1 \le n \le N.$

Reduced basis "matrices" $\underline{\mathbb{Z}}_N \in \mathbb{R}^{\mathcal{N} \times N},\ 1 \le N \le N_{\max}$:

$$\underline{\mathbb{Z}}_N = [\zeta^1\ \zeta^2 \ldots \zeta^N], \quad 1 \le N \le N_{\max},$$

where, from orthogonality, $\underline{\mathbb{Z}}_{N_{\max}}^T \underline{\mathbb{X}}\, \underline{\mathbb{Z}}_{N_{\max}}^T = \underline{\mathbb{I}}_{N_{\max}}$, and $\underline{\mathbb{I}}_M$ is the Identity matrix in $\mathbb{R}^{M \times M}$.

Galerkin Projection

Given $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}} \subset \mathbb{R}^{P}$, evaluate optimality

$$s_N(\boldsymbol{\mu}) = f(u_N(\boldsymbol{\mu}); \boldsymbol{\mu})$$

where $u_N(x; \boldsymbol{\mu}) \in X_N \subset X^{\mathcal{N}}$ satisfies

$$a(u_N(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = f(v; \boldsymbol{\mu}), \quad \forall\, v \in X_N.$$

Note:

- Lagrangian reduced basis space: $X_N = W_N$
- Solution $u_N(\boldsymbol{\mu})$ unique (from coercivity, continuity, linear independence)

Galerkin Optimality

Proposition

For any $\mu \in \mathcal{D}$, we have

$$\begin{aligned} |||u(\mu) - u_N(\mu)|||_\mu &= \inf_{w_N \in X_N} |||u(\mu) - w_N(\mu)|||_\mu, \\ \|u(\mu) - u_N(\mu)\|_X &\leq \sqrt{\frac{\gamma^e(\mu)}{\alpha^e(\mu)}} \inf_{w_N \in X_N} \|u(\mu) - w_N(\mu)\|_X, \end{aligned}$$

and furthermore

$$\begin{aligned} s(\mu) - s_N(\mu) &= |||u(\mu) - u_N(\mu)|||_\mu^2 \\ &= \inf_{w_N \in X_N} |||u(\mu) - w_N(\mu)|||_\mu^2, \end{aligned}$$

as well as

$$0 \leq s(\mu) - s_N(\mu) \leq \gamma^e(\mu) \inf_{w_N \in X_N} ||u(\mu) - w_N(\mu)||_X^2.$$

Offline-Online Decomposition

We expand $$u_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\zeta^j$$

and obtain $$v = \zeta^i, 1 \le i \le N$$

$$a(u_N(\mu), v; \mu) = f(v; \mu)$$

$$\sum_{j=1}^{N} u_{N\,j}(\mu) a(\zeta^j, \zeta^i; \mu) = f(\zeta^i; \mu)$$

$$\underbrace{\sum_{j=1}^{N} u_{N\,j}(\mu) \underbrace{\sum_{q=1}^{Q_a} \Theta_a^q(\mu) \underbrace{a^q(\zeta^j, \zeta^i)}_{\text{OFFLINE: } O(\mathcal{N})}}_{\text{ONLINE: } O(Q_a N^2)} = \underbrace{\sum_{q=1}^{Q_f} \Theta_f^q(\mu) \underbrace{f^q(\zeta^i)}_{\text{OFFLINE: } O(\mathcal{N})}}_{\text{ONLINE: } O(Q_f N)}}_{\text{ONLINE: } O(N^3)}$$

Offline-Online Decomposition

We expand $$u_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\zeta^j$$

and obtain $$v = \zeta^i, 1 \le i \le N$$

$$a(u_N(\mu), v; \mu) = f(v; \mu)$$

$$\sum_{j=1}^{N} u_{N\,j}(\mu) a(\zeta^j, \zeta^i; \mu) = f(\zeta^i; \mu)$$

$$\underbrace{\sum_{j=1}^{N} u_{N\,j}(\mu) \underbrace{\sum_{q=1}^{Q_a} \Theta_a^q(\mu) \underbrace{a^q(\zeta^j, \zeta^i)}_{\text{OFFLINE: } O(\mathcal{N})}}_{\text{ONLINE: } O(Q_a N^2)} = \underbrace{\sum_{q=1}^{Q_f} \Theta_f^q(\mu) \underbrace{f^q(\zeta^i)}_{\text{OFFLINE: } O(\mathcal{N})}}_{\text{ONLINE: } O(Q_f N)}}_{\text{ONLINE: } O(N^3)}$$

Offline-Online Decomposition

We expand $u_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\zeta^j$

and obtain $v = \zeta^i, 1 \leq i \leq N$

$$a(u_N(\mu), v; \mu) = f(v; \mu)$$

$$\sum_{j=1}^{N} u_{N\,j}(\mu) a(\zeta^j, \zeta^i; \mu) = f(\zeta^i; \mu)$$

$$\underbrace{\sum_{j=1}^{N} u_{N\,j}(\mu) \underbrace{\sum_{q=1}^{Q_a} \Theta_a^q(\mu) \underbrace{a^q(\zeta^j, \zeta^i)}_{\text{OFFLINE: } O(\mathcal{N})}}_{\text{ONLINE: } O(Q_a N^2)}}_{\text{ONLINE: } O(N^3)} = \underbrace{\sum_{q=1}^{Q_f} \Theta_f^q(\mu) \underbrace{f^q(\zeta^i)}_{\text{OFFLINE: } O(\mathcal{N})}}_{\text{ONLINE: } O(Q_f N)}$$

Offline-Online Decomposition

Given $u_{N\,j}(\mu), 1 \leq j \leq N$, we evaluate the output from

$$s_N(\mu) = f(u_N(\mu); \mu) = \sum_{j=1}^{N} u_{N\,j}(\mu) f(\zeta^j; \mu)$$

$$= \underbrace{\sum_{j=1}^{N} u_{N\,j}(\mu) \underbrace{\sum_{q=1}^{Q_f} \Theta_f^q(\mu) \underbrace{f^q(\zeta^j)}_{\text{OFFLINE: } O(\mathcal{N})}}_{\text{ONLINE: } O(Q_f N)}}_{\text{ONLINE: } O(N)}$$

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Offline-Online Decomposition

Given $u_{N\,j}(\mu), 1 \leq j \leq N$, we evaluate the output from

$$
\begin{aligned}
s_N(\mu) = f(u_N(\mu);\mu) &= \sum_{j=1}^{N} u_{N\,j}(\mu)\, f(\zeta^j;\mu) \\
&= \underbrace{\sum_{j=1}^{N} u_{N\,j}(\mu) \underbrace{\sum_{q=1}^{Q_f} \Theta_f^q(\mu) \quad \underbrace{f^q(\zeta^j)}_{\text{OFFLINE: } O(\mathcal{N})}}_{\text{ONLINE: } O(Q_f N)}}_{\text{ONLINE: } O(N)}
\end{aligned}
$$

Offline-Online Decomposition

Summary computational cost: $(Q = Q_a + Q_f)$

OFFLINE — once, parameter *independent*

$$O(N_{\max}\mathcal{N}^\bullet) \quad + \quad O(QN_{\max}^2\mathcal{N}) \quad ;$$

solve for ζ_n — form μ-independent quantities

ONLINE — many times, parameter *dependent* μ^{new}

$$O(QN^2) \quad + \quad O(N^3) \quad + \quad O(N) \quad ;$$

form RB matrices — solve for $u_{Nj}(\mu)$ — evaluate output

Online cost is independent of $\mathcal{N}$.

Algebraic Equations

Evaluation of RB Stiffness Matrix $\underline{A}_N \in \mathbb{R}^{N \times N}$:

Parameter-independent matrices $\underline{\mathbb{A}}_N^q \in \mathbb{R}^{N \times N}$, $1 \leq q \leq Q_a$:

$$\begin{aligned} \mathbb{A}^q_{N\,n\,m} &= a^q(\zeta^m, \zeta^n) \\ &= \sum_{i=1}^{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} \zeta_i^m \, a^q(\varphi_i^{\mathcal{N}}, \varphi_j^{\mathcal{N}}) \, \zeta_j^n, \quad 1 \leq n, m \leq N, \end{aligned}$$

thus

$$\underline{\mathbb{A}}_N^q = \mathbb{Z}_N^T \, \underline{\mathbb{A}}^{\mathcal{N} q} \, \mathbb{Z}_N.$$

We finally assemble

$$\underline{A}_N = \sum_{q=1}^{Q_a} \Theta_a^q(\mu) \, \underline{\mathbb{A}}_N^q.$$

Algebraic Equations

Evaluation of RB Load/Source/Output Vector $\underline{F}_N \in \mathbb{R}^N$:

Parameter-independent vectors $\underline{\mathbb{F}}_N^q \in \mathbb{R}^N,\ 1 \le q \le Q_f$:

$$\begin{aligned} \mathbb{F}^q_{N\,n} &= f^q(\zeta^n) \\ &= \sum_{i=1}^{\mathcal{N}} \zeta_i^m\, f^q(\varphi_i^{\mathcal{N}}), \quad 1 \le n \le N, \end{aligned}$$

thus

$$\underline{\mathbb{F}}_N^q = \mathbb{Z}_N^T\, \underline{\mathbb{F}}^{\mathcal{N}q}.$$

We finally assemble

$$\underline{F}_N = \sum_{q=1}^{Q_f} \Theta_f^q(\mu)\, \underline{\mathbb{F}}_N^q.$$

Algebraic Equations

We expand $$u_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\zeta^j$$

and define the vector of RB coefficients

$$\underline{u}_N(\mu) = [u_{N\,1}(\mu)\ u_{N\,2}(\mu) \ldots u_{N\,N}(\mu)] \in \mathbb{R}^N.$$

Then, given $\mu \in \mathcal{D} \subset \mathbb{R}^P$, we evaluate

$$s_N(\mu) = \underline{F}_N^T(\mu)\,\underline{u}_N(\mu)$$

where $\underline{u}_N(\mu) \in \mathbb{R}^N$ satisfies

$$\underline{A}_N(\mu)\,\underline{u}_N(\mu) = \underline{F}_N(\mu).$$

Algebraic Equations

Proposition

The condition number of $\underline{\boldsymbol{A}}_N(\boldsymbol{\mu})$ is bounded from above by $\gamma^{\mathrm{e}}(\boldsymbol{\mu})/\alpha^{\mathrm{e}}(\boldsymbol{\mu})$, the ratio of the continuity and coercivity constants for the continuous problem.

Sketch of proof:

- Rayleigh Quotient

$$\frac{\underline{\boldsymbol{w}}_N^T \underline{\boldsymbol{A}}_N(\boldsymbol{\mu}) \underline{\boldsymbol{w}}_N}{\underline{\boldsymbol{w}}_N^T \underline{\boldsymbol{w}}_N}, \quad \forall \underline{\boldsymbol{w}}_N \in \boldsymbol{R}^N$$

- Express

$$\underline{\boldsymbol{w}}_N = \sum_{m=1}^{N} w_{N\,m} \zeta^m$$

- Invoke coercivity (resp. continuity) and orthogonality.