Contents

1 General Preliminaries

Contents

1.1 The Poisson Problem

Strong Formulation – Dirichlet Problem

Find $\boldsymbol{u}$ such that

$$\boxed{\begin{aligned} -\nabla^2 u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma \end{aligned}}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and Ω is a domain in $\mathbb{R}^2$ with boundary Γ.

In general, we assume

- $\Omega \subset \mathbb{R}^d,\ d = 1, \ldots, 3$, has a Lipschitz continuous boundary Γ (or $\partial\Omega$)
- Ω is an open domain, e.g., $\Omega = (0, 1)$ ($\bar{\Omega} = [0, 1]$)

Minimization Principle – Statement

Find

$$\boxed{\boldsymbol{u} = \arg\min_{\boldsymbol{w}\in\boldsymbol{X}} \boldsymbol{J}(\boldsymbol{w})}$$

where

$$\boldsymbol{X} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\boldsymbol{\Omega}) \mid \boldsymbol{v}|_{\boldsymbol{\Gamma}} = \boldsymbol{0}\}$$

and

$$\boldsymbol{J}(\boldsymbol{w}) = \frac{1}{2}\int_{\boldsymbol{\Omega}} \underbrace{\boldsymbol{\nabla w}\cdot\boldsymbol{\nabla w}}_{w_x^2+w_y^2}\, d\boldsymbol{\Omega} - \int_{\boldsymbol{\Omega}} \boldsymbol{f w}\, d\boldsymbol{\Omega}$$

Minimization Principle – Proof (sketch)

Let $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$, $\boldsymbol{u} \in \boldsymbol{X}$, $\boldsymbol{v} \in \boldsymbol{X}$, apply Gauss Theorem and boundary conditions to obtain

$$\boldsymbol{J}(\underbrace{\boldsymbol{u}+\boldsymbol{v}}_{\boldsymbol{w}}) = \boldsymbol{J}(\boldsymbol{u}) + \underbrace{\frac{1}{2}\int_{\boldsymbol{\Omega}} \boldsymbol{\nabla v}\cdot\boldsymbol{\nabla v}\, d\boldsymbol{\Omega}}_{>0 \text{ unless } \boldsymbol{v}=\boldsymbol{0}}, \qquad \forall \boldsymbol{v} \in \boldsymbol{X}$$

$$\boxed{\begin{array}{c} \boldsymbol{J}(\boldsymbol{w}) > \boldsymbol{J}(\boldsymbol{u}), \qquad \forall \boldsymbol{w} \in \boldsymbol{X},\ \boldsymbol{w} \neq \boldsymbol{u} \\ \Updownarrow \\ \boldsymbol{u} \text{ is } \textit{the} \text{ minimizer of } \boldsymbol{J}(\boldsymbol{w}) \end{array}}$$

Weak Formulation

Let — BILINEAR FORM

$$\boldsymbol{a}(\boldsymbol{w},\boldsymbol{v}) = \int_{\boldsymbol{\Omega}} \boldsymbol{\nabla w}\cdot\boldsymbol{\nabla v}\, d\boldsymbol{\Omega}, \quad \forall \boldsymbol{w},\boldsymbol{v} \in \boldsymbol{X}$$

and — LINEAR FORM

$$\boldsymbol{f}(\boldsymbol{v}) = \int_{\boldsymbol{\Omega}} \boldsymbol{f}\,\boldsymbol{v}\, d\boldsymbol{\Omega}, \quad \forall \boldsymbol{v} \in \boldsymbol{X}$$

Weak Statement

Find $\boldsymbol{u} \in \boldsymbol{X}$, such that

$$\boldsymbol{a}(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{f}(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{X}$$

Neumann Problem

Find u such that STRONG

$$\boxed{\begin{aligned} -\nabla^2 u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma^D \\ \frac{\partial u}{\partial n} &= g && \text{on } \Gamma^N \end{aligned}}$$

where $\bar{\Gamma} = \bar{\Gamma}^D \cup \bar{\Gamma}^N,\ \Gamma^D$ non-empty.

Find $\boxed{u = \arg\min_{w \in X} J(w)}$ MINIMIZATION

where

$$X = \{v \in H^1(\Omega) \mid v|_{\Gamma^D} = 0\}$$

$$J(w) = \frac{1}{2} \int_{\Omega} \nabla w \cdot \nabla w d\Omega - \int_{\Omega} f w d\Omega - \int_{\Gamma^N} g w dS$$

Neumann Problem – Weak Formulation

Let BILINEAR FORM

$$a(w, v) = \int_{\Omega} \nabla w \cdot \nabla v \, d\Omega, \quad \forall w, v \in X$$

and LINEAR FORM

$$f(v) = \int_{\Omega} f \, v \, d\Omega + \int_{\Gamma^N} g \, v \, dS, \quad \forall v \in X$$

Weak Statement

Find $u \in X$, such that

$$a(u, v) = f(v), \quad \forall v \in X$$

Summary

- Minimization/weak formulations are defined by:
  - a space X;
  - a bilinear form a;
  - a linear form ℓ.
- Essential (Dirichlet) boundary conditions are imposed by (or reflected in) X
- Natural (Neumann) boundary conditions are imposed by (or reflected in) J (or a and f).
- Points of departure for the finite element method are
  - the weak form (more generally); or
  - the minimization statement (if a is SPD).

1.2 Inner Product Spaces

[Inner Product Spaces]

Definitions

Definition 1 (Linear space)**.** A space $\boldsymbol{Z}$ is a linear or vector space if, for any $\boldsymbol{\alpha} \in \mathbb{R}$, $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{Z}$,

$$\boldsymbol{\alpha}\, \boldsymbol{w} + \boldsymbol{v} \in \boldsymbol{Z}$$

Note: $\mathbb{R}$ denotes the real numbers, and $\mathbb{N}$ and $\mathbb{C}$ shall denote the natural and complex numbers, respectively.

Definition 2 (Inner Product Space)**.** An inner product space $\boldsymbol{Z}$ is a linear space equipped with

- an inner product $(\boldsymbol{w}, \boldsymbol{v})_{\boldsymbol{Z}}$, $\forall \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{Z}$, and
- induced norm $\|\boldsymbol{w}\|_{\boldsymbol{Z}} = \sqrt{(\boldsymbol{w}, \boldsymbol{w})_{\boldsymbol{Z}}}$, $\forall \boldsymbol{w} \in \boldsymbol{Z}$.

Definitions

Definition 3 (Inner Product)**.** An inner product $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{Z} \to (\boldsymbol{w}, \boldsymbol{v})_{\boldsymbol{Z}} \in \mathbb{R}$ has to satisfy

- Bilinearity
$$(\boldsymbol{\alpha w} + \boldsymbol{v}, \boldsymbol{z})_{\boldsymbol{Z}} = \boldsymbol{\alpha}(\boldsymbol{w}, \boldsymbol{z})_{\boldsymbol{Z}} + (\boldsymbol{v}, \boldsymbol{z})_{\boldsymbol{Z}}, \quad \forall \boldsymbol{\alpha} \in \mathbb{R},\ \boldsymbol{w}, \boldsymbol{v}, \boldsymbol{z} \in \boldsymbol{Z}$$
$$(\boldsymbol{z}, \boldsymbol{\alpha w} + \boldsymbol{v})_{\boldsymbol{Z}} = \boldsymbol{\alpha}(\boldsymbol{z}, \boldsymbol{w})_{\boldsymbol{Z}} + (\boldsymbol{z}, \boldsymbol{v})_{\boldsymbol{Z}}, \quad \forall \boldsymbol{\alpha} \in \mathbb{R},\ \boldsymbol{w}, \boldsymbol{v}, \boldsymbol{z} \in \boldsymbol{Z}$$
- Symmetry
$$(\boldsymbol{w}, \boldsymbol{v})_{\boldsymbol{Z}} = (\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{Z}}, \quad \forall \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{Z}$$
- Positivity
$$(\boldsymbol{w}, \boldsymbol{w})_{\boldsymbol{Z}} > \boldsymbol{0}, \quad \forall \boldsymbol{w} \in \boldsymbol{Z},\ \boldsymbol{w} \neq \boldsymbol{0}\ (\boldsymbol{w}, \boldsymbol{w})_{\boldsymbol{Z}} = \boldsymbol{0}, \quad \text{only if } \boldsymbol{w} = \boldsymbol{0}$$

Cauchy-Schwarz inequality:

$$(\boldsymbol{w}, \boldsymbol{v})_{\boldsymbol{Z}} \leq \|\boldsymbol{w}\|_{\boldsymbol{Z}} \|\boldsymbol{v}\|_{\boldsymbol{Z}}, \qquad \forall \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{Z}.$$

Definitions

Definition 4 (Norm)**.** A norm is a map $\|\cdot\| : \boldsymbol{Z} \to \mathbb{R}$ such that

$$\begin{aligned} \|\boldsymbol{w}\|_{\boldsymbol{Z}} &> \boldsymbol{0} && \forall \boldsymbol{w} \in \boldsymbol{Z},\ \boldsymbol{w} \neq \boldsymbol{0}, \\ \|\boldsymbol{\alpha w}\|_{\boldsymbol{Z}} &= |\boldsymbol{\alpha}|\, \|\boldsymbol{w}\|_{\boldsymbol{Z}} && \forall \boldsymbol{\alpha} \in \mathbb{R},\ \forall \boldsymbol{w} \in \boldsymbol{Z}, \\ \|\boldsymbol{w} + \boldsymbol{v}\|_{\boldsymbol{Z}} &\leq \|\boldsymbol{w}\|_{\boldsymbol{Z}} + \|\boldsymbol{v}\|_{\boldsymbol{Z}} && \forall \boldsymbol{w} \in \boldsymbol{Z}, \forall \boldsymbol{v} \in \boldsymbol{Z}. \end{aligned}$$

Equivalence of norms $\|\cdot\|_{\boldsymbol{Z}}$ and $\|\cdot\|_{\boldsymbol{Y}}$: there exist positive constants $\boldsymbol{C_1},\ \boldsymbol{C_2}$ such that

$$\boldsymbol{C_1} \|\boldsymbol{v}\|_{\boldsymbol{Z}} \leq \|\boldsymbol{v}\|_{\boldsymbol{Y}} \leq \boldsymbol{C_2} \|\boldsymbol{v}\|_{\boldsymbol{Z}}.$$

Definitions

Definition 5 (Cartesian Product Spaces)**.** Given two inner product spaces $\boldsymbol{Z_1}$ and $\boldsymbol{Z_2}$, we define

$$\boldsymbol{Z = Z_1 \times Z_2 \equiv \{(w_1, w_2) | w_1 \in Z_1, w_2 \in Z_2\}}$$

and given $\boldsymbol{w = (w_1, w_2) \in Z}$, $\boldsymbol{v = (v_1, v_2) \in Z}$, we define

$$\boldsymbol{w + v \equiv (w_1 + v_1, w_2 + v_2)}.$$

We also equip $\boldsymbol{Z}$ with the inner product

$$\boldsymbol{(w, v)_Z = (w_1, v_1)_{Z_1} + (w_2, v_2)_{Z_2}}$$

and induced norm $$\boldsymbol{\|w\|_Z = \sqrt{(w, w)_Z}}.$$

1.3 Linear and Bilinear Forms

[Linear and Bilinear Forms]

Linear Forms

Definition 6 (Linear Functional)**.** A functional $\boldsymbol{g : Z \to \mathbb{R}}$ is a linear functional if, for any $\boldsymbol{\alpha} \in \mathbb{R}$, $\boldsymbol{w, v \in Z}$

$$\boldsymbol{g(\alpha w + v) = \alpha g(w) + g(v)}$$

A linear form is bounded, or continuous, over $\boldsymbol{Z}$ if

$$\boldsymbol{|g(v)| \le C \, \|v\|_Z, \qquad \forall v \in Z,}$$

for some finite real constant $\boldsymbol{C}$.

Dual Spaces

Definition 7 (Dual Space)**.** Given $\boldsymbol{Z}$, we define the dual space $\boldsymbol{Z'}$ as the space of all *bounded* linear functionals over $\boldsymbol{Z}$. We associate to $\boldsymbol{Z'}$ the *dual norm* [-1ex]

$$\boldsymbol{\|g\|_{Z'} = \sup_{v \in Z} \frac{g(v)}{\|v\|_Z}, \qquad \forall g \in Z'.}$$

Theorem 8 (Riesz representation)**.** *For any* $\boldsymbol{g \in Z'}$*, there exists a unique* $\boldsymbol{w_g \in Z}$ *such that*

$$\boldsymbol{(w_g, v)_Z = g(v), \qquad \forall v \in Z.}$$

It directly follows that $$\boxed{\boldsymbol{\|g\|_{Z'} = \|w_g\|_Z.}}$$

Bilinear Forms

Definition 9 (Bilinear Form)**.** A form $\boldsymbol{b} : \boldsymbol{Z_1} \times \boldsymbol{Z_2} \to \mathbb{R}$ is bilinear if, for any $\boldsymbol{\alpha} \in \mathbb{R}$,

$$\boldsymbol{b}(\boldsymbol{\alpha w} + \boldsymbol{v}, \boldsymbol{z}) = \boldsymbol{\alpha b}(\boldsymbol{w}, \boldsymbol{z}) + \boldsymbol{b}(\boldsymbol{v}, \boldsymbol{z}), \quad \forall \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{Z_1},\ \boldsymbol{z} \in \boldsymbol{Z_2}$$
$$\boldsymbol{b}(\boldsymbol{z}, \boldsymbol{\alpha w} + \boldsymbol{v}) = \boldsymbol{\alpha b}(\boldsymbol{z}, \boldsymbol{w}) + \boldsymbol{b}(\boldsymbol{z}, \boldsymbol{v}), \quad \forall \boldsymbol{z} \in \boldsymbol{Z_1},\ \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{Z_2}$$

The bilinear form $\boldsymbol{b} : \boldsymbol{Z} \times \boldsymbol{Z} \to \mathbb{R}$ is

- symmetric, if
$$\boldsymbol{b}(\boldsymbol{w}, \boldsymbol{v}) = \boldsymbol{b}(\boldsymbol{v}, \boldsymbol{w}), \qquad \forall \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{Z};$$
- skew-symmetric, if
$$\boldsymbol{b}(\boldsymbol{w}, \boldsymbol{v}) = -\boldsymbol{b}(\boldsymbol{v}, \boldsymbol{w}), \qquad \forall \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{Z};$$
- positive definite, if
$$\boldsymbol{b}(\boldsymbol{v}, \boldsymbol{v}) \geq \boldsymbol{0}, \qquad \forall \boldsymbol{v} \in \boldsymbol{Z},$$
  with equality only for $\boldsymbol{v} = \boldsymbol{0}$.

Bilinear Forms

The bilinear form $\boldsymbol{b} : \boldsymbol{Z} \times \boldsymbol{Z} \to \mathbb{R}$ is

- positive semidefinite, if
$$\boldsymbol{b}(\boldsymbol{v}, \boldsymbol{v}) \geq \boldsymbol{0}, \qquad \forall \boldsymbol{v} \in \boldsymbol{Z}.$$

We also define, for a general bilinear form $\boldsymbol{b} : \boldsymbol{Z} \times \boldsymbol{Z} \to \mathbb{R}$,

- the symmetric part as
$$\boldsymbol{b_S}(\boldsymbol{w}, \boldsymbol{v}) = \boldsymbol{1/2}\ (\boldsymbol{b}(\boldsymbol{w}, \boldsymbol{v}) + \boldsymbol{b}(\boldsymbol{v}, \boldsymbol{w}))\,, \quad \forall \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{Z};$$
- the skew-symmetric part as
$$\boldsymbol{b_{SS}}(\boldsymbol{w}, \boldsymbol{v}) = \boldsymbol{1/2}\ (\boldsymbol{b}(\boldsymbol{w}, \boldsymbol{v}) - \boldsymbol{b}(\boldsymbol{v}, \boldsymbol{w}))\,, \quad \forall \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{Z}.$$

Bilinear Forms

The bilinear form $\boldsymbol{b} : \boldsymbol{Z} \times \boldsymbol{Z} \to \mathbb{R}$ is

- coercive over $\boldsymbol{Z}$ if
$$\boldsymbol{\alpha} \equiv \inf_{\boldsymbol{w} \in \boldsymbol{Z}} \frac{\boldsymbol{b}(\boldsymbol{w}, \boldsymbol{w})}{\|\boldsymbol{w}\|_{\boldsymbol{Z}}^{\boldsymbol{2}}}$$
  is positive;
- continuous over $\boldsymbol{Z}$ if
$$\boldsymbol{\gamma} \equiv \sup_{\boldsymbol{w} \in \boldsymbol{Z}} \sup_{\boldsymbol{v} \in \boldsymbol{Z}} \frac{\boldsymbol{b}(\boldsymbol{w}, \boldsymbol{v})}{\|\boldsymbol{w}\|_{\boldsymbol{Z}} \|\boldsymbol{v}\|_{\boldsymbol{Z}}}$$
  is finite.

Parametric Linear and Bilinear Forms

We introduce

- $\mathcal{D} \in \mathbb{R}^P$: closed bounded parameter domain;
- $\mu = (\mu_1, \ldots, \mu_P) \in \mathcal{D}$: parameter vector.

We shall say that

- $g : Z \times \mathcal{D} \to \mathbb{R}$ is a parametric linear form if,

  for all $\mu \in \mathcal{D}$, $g(\cdot;\mu) : Z \to \mathbb{R}$ is a linear form;

- $b : Z \times Z \times \mathcal{D} \to \mathbb{R}$ is a parametric bilinear form if,

  for all $\mu \in \mathcal{D}$, $b(\cdot,\cdot;\mu) : Z \times Z \to \mathbb{R}$ is a bilinear form.

Concepts of symmetry, . . . directly extend to the parametric case.

Parametric Linear and Bilinear Forms

The parametric bilinear form $b : Z \times Z \times \mathcal{D} \to \mathbb{R}$ is

- coercive over Z if

$$\alpha(\mu) \equiv \inf_{w \in Z} \frac{b(w,w;\mu)}{\|w\|_Z^2}$$

  is positive for all $\mu \in \mathcal{D}$;

- continuous over Z if

$$\gamma(\mu) \equiv \sup_{w \in Z} \sup_{v \in Z} \frac{b(w,v;\mu)}{\|w\|_Z \|v\|_Z}$$

  is finite for all $\mu \in \mathcal{D}$.

We also define

$$\begin{aligned} (0 <)\ \alpha_0 &\equiv \min_{\mu \in \mathcal{D}} \alpha(\mu), \\ \gamma_0 &\equiv \max_{\mu \in \mathcal{D}} \gamma(\mu)\ (< \infty). \end{aligned}$$

Coercivity Eigenproblem

We have

$$\alpha(\mu) \equiv \inf_{w \in Z} \frac{b_S(w,w;\mu)}{\|w\|_Z^2};$$

Associated generalized eigenproblem:

Given $\mu \in \mathcal{D}$, find $(\chi^{co}, \nu^{co})_i(\mu) \in Z \times \mathbb{R},\ 1 \le i \le \dim(Z)$, such that

$$b_S(\chi_i^{co}(\mu), v; \mu) = \nu_i^{co}(\mu)(\chi_i^{co}(\mu), v)_Z, \qquad \forall v \in Z,$$

and

$$\|\chi_i^{co}(\mu)\|_Z = 1.$$

Let $\nu_1^{co}(\mu) \le \nu_2^{co}(\mu) \le \ldots \le \nu_{\dim(Z)}^{co}(\mu)$ and b coercive, then

$$\alpha(\mu) = \nu_1^{co}(\mu) > 0.$$

Affine Parameter Dependence

We assume [-3ex]

$$g(v;\mu) = \sum_{q=1}^{Q_g} \Theta_g^q(\mu)\; g^q(v), \quad \forall v \in Z,$$

where, for $1 \leq q \leq Q_g$ (finite),

- parameter-*dependent* functions $\Theta_g^q : \mathcal{D} \to \mathbb{R}$,
- parameter-*independent* forms $g^q : Z \to \mathbb{R}$;

and [-3ex]

$$b(w,v;\mu) = \sum_{q=1}^{Q_b} \Theta_b^q(\mu)\; b^q(w,v), \quad \forall w,v \in Z,$$

where, for $1 \leq q \leq Q_b$ (finite),

- parameter-*dependent* functions $\Theta_b^q : \mathcal{D} \to \mathbb{R}$,
- parameter-*independent* forms $b^q : Z \times Z \to \mathbb{R}$.

Parametric Coercivity

Definition 10 (Parametric Coercivity)**.** The coercive bilinear form $b : Z \times Z \times \mathcal{D} \to \mathbb{R}$

$$b(w,v;\mu) = \sum_{q=1}^{Q_b} \Theta_b^q(\mu)\, b^q(w,v), \qquad \forall w,v \in Z$$

is parametrically coercive if $c \equiv b_S$ is affine

$$c(w,v;\mu) = \sum_{q=1}^{Q_c} \Theta_c^q(\mu)\, c^q(w,v), \qquad \forall w,v \in Z$$

and satisfies

$$\Theta_c^q(\mu) > 0, \qquad \forall \mu \in \mathcal{D},\ 1 \leq q \leq Q_c,$$

and

$$c^q(v,v) \geq 0, \qquad \forall v \in Z,\ 1 \leq q \leq Q_c.$$

1.4 Classes of Functions

[Classes of Functions]

1.4.1 Scalar and Vector Fields

[Scalar and Vector Fields]

Scalar and Vector Field

We consider (real)

- scalar-valued field variables (e.g., temperature, pressure)
$$\boldsymbol{w} : \boldsymbol{\Omega} \to \mathbb{R}^{\boldsymbol{d_v = 1}}$$

- vector-valued field variables (e.g., displacement, velocity)
$$\boldsymbol{w} : \boldsymbol{\Omega} \to \mathbb{R}^{\boldsymbol{d_v}},$$

  where $\boldsymbol{w(x) = (w_1(x), \ldots, w_{d_v}(x))}$;

and

- $\boldsymbol{\Omega} \in \mathbb{R}^{\boldsymbol{d}},\ \boldsymbol{d = 1,\ 2,}$ or $\boldsymbol{3}$ is an open bounded; domain
- $\boldsymbol{x = (x_1, \ldots, x_d) \in \Omega}$;
- $\boldsymbol{\Omega}$ has Lipschitz continuous boundary $\boldsymbol{\partial\Omega}$; and
- we define the canonical basis vectors as $\boldsymbol{e_i,\ 1 \leq i \leq d}$.

Multi-index Derivatives

Given a scalar (or one component of a vector)

- field $\boldsymbol{w} : \boldsymbol{\Omega} \to \mathbb{R}$ SPATIAL DERIVATIVE
$$(\boldsymbol{D^{\sigma} w})(\boldsymbol{x}) = \frac{\boldsymbol{\partial^{\sigma} w}}{\boldsymbol{\partial x_1^{\sigma_1} \ldots \partial x_d^{\sigma_d}}}$$

- parametric field $\boldsymbol{w} : \boldsymbol{\Omega} \times \boldsymbol{\mathcal{D}} \to \mathbb{R}$ SENSITIVITY DERIVATIVE
$$(\boldsymbol{D_{\sigma} w})(\boldsymbol{x}) = \frac{\boldsymbol{\partial^{\sigma} w}}{\boldsymbol{\partial \mu_1^{\sigma_1} \ldots \partial \mu_d^{\sigma_d}}}$$

where

- $\boldsymbol{\sigma = (\sigma_1, \ldots, \sigma_d),\ \sigma_i,\ 1 \leq i \leq d,}$ non-negative integers;
- $|\boldsymbol{\sigma}| = \sum_{\boldsymbol{j=1}}^{\boldsymbol{d}} \boldsymbol{\sigma_j}$ is the order of the derivative; and
- $\boldsymbol{I^{d,n}}$ is set of all index vectors $\boldsymbol{\sigma} \in \mathbb{N}_{\boldsymbol{0}}^{\boldsymbol{d}}$ such that $|\boldsymbol{\sigma}| \leq \boldsymbol{n}$.

1.4.2 Function Spaces

[Function Spaces]

Function Spaces

Definition 11 (Spaces of Continuous Functions)**.** Let $m \in \mathbb{N}_0$, the space $C^m(\Omega)$ is defined as

$$C^m(\Omega) \equiv \{w \mid \; D^\sigma w \in C^0(\Omega), \; \forall \sigma \in I^{d,m}\},$$

and $C^0(\Omega)$ is the space of continuous functions over $\Omega \in \mathbb{R}^d$.

- We denote by $C^\infty(\Omega)$ the space of functions w for which D^σ exists and is continuous for any order $|\sigma|$.

Lebesgue Spaces

Definition 12 (Lebesgue Spaces)**.** We define, for $1 \leq p < \infty$, the Lebesgue space $L^p(\Omega)$ as

$$L^p(\Omega) \equiv \left\{w \text{ measurable} \mid \|w\|_{L^p(\Omega)} < \infty\right\}$$

where

$$\|w\|_{L^p(\Omega)} \equiv \left(\int_\Omega |w|^p \, dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

$$\|w\|_{L^\infty(\Omega)} \equiv \operatorname*{ess\,sup}_{x \in \Omega} |w(x)|, \qquad p = \infty.$$

Hilbert Space

Definition 13 (Hilbert space)**.** Let $m \in \mathbb{N}_0$, the space $H^m(\Omega)$ is then defined as

$$H^m(\Omega) \equiv \{w \mid D^\sigma w \in L^2(\Omega), \; \forall \sigma \in I^{d,m}\},$$

with associated inner product

$$(w, v)_{H^m(\Omega)} \equiv \sum_{\sigma \in I^{d,m}} \int_\Omega D^\sigma w \, D^\sigma v \, dx,$$

and induced norm

$$\|w\|_{H^m(\Omega)} \equiv \sqrt{(w, w)_{H^m(\Omega)}}.$$

Special (Most Important) Cases

Since we only consider second-order PDEs, we require mostly

- $L^2(\Omega) = H^0(\Omega)$: Lebesgue Space $p = 2$

$$L^2(\Omega) \equiv \{w \text{ measurable} \mid \int_\Omega w^2 < \infty\}$$

with inner product and induced norm

$$
\begin{aligned}
(w, v)_{L^2(\Omega)} &\equiv \int_\Omega w\, v, && \forall w, v \in L^2(\Omega)\\
\|w\|_{L^2(\Omega)} &\equiv \sqrt{(w, w)_{L^2(\Omega)}}, && \forall w \in L^2(\Omega),
\end{aligned}
$$

$\Rightarrow$ Space of all functions $w : \Omega \to \mathbb{R}$ square-integrable over Ω.

Special (Most Important) Cases

Since we only consider second-order PDEs, we require mostly

- $H^1(\Omega)$

  $$H^1(\Omega) \equiv \{w \in L^2(\Omega) \mid \tfrac{\partial w}{\partial x_i} \in L^2(\Omega),\ 1 \le i \le d\}$$

  with inner product and induced norm

  $$
  \begin{aligned}
  (w, v)_{H^1(\Omega)} &\equiv \int_\Omega \nabla w \cdot \nabla v + wv, && \forall w, v \in H^1(\Omega),\\
  \|w\|_{H^1(\Omega)} &\equiv \sqrt{(w, w)_{H^1(\Omega)}}, && \forall w \in H^1(\Omega),
  \end{aligned}
  $$

  and seminorm

  $$|w|_{H^1(\Omega)} \equiv \int_\Omega \nabla w \cdot \nabla w, \qquad \forall w \in H^1(\Omega).$$

Special (Most Important) Cases

Since we only consider second-order PDEs, we require mostly

- and the space

  $$H_0^1(\Omega) \equiv \{v \in H^1(\Omega) \mid v|_{\partial\Omega} = 0\}$$

  where $v = 0$ on the boundary $\partial\Omega$.

Note that, for any $v \in H_0^1(\Omega)$, we have

$$C_{PF}\|v\|_{H^1(\Omega)} \le |v|_{H^1(\Omega)} \le \|v\|_{H^1(\Omega)},$$

and thus

$$|v|_{H^1(\Omega)} = 0 \text{ implies } v = 0.$$

$\Rightarrow$ $|v|_{H^1(\Omega)}$ constitutes a norm for $v \in H_0^1(\Omega)$.

Projection

Definition 14 (Projection)**.** Given Hilbert Spaces Y and $Z \subset Y$, the projection, $\Pi : Y \to Z$, of $y \in Y$ onto Z is defined as

$$(\Pi y, v)_Y = (y, v)_Y, \qquad \forall v \in Z$$

Properties

- Orthogonality: $\quad (y - \Pi y, v)_Y = 0, \qquad \forall v \in Z$

- Idempotence: $\Pi(\Pi y) = \Pi y$
- Best Approximation
$$\|y - \Pi y\|_Y^2 = \inf_{v \in Z} \|y - v\|_Y^2$$
- Given an orthonormal basis $\{\phi_i\}_{i=1}^N$, $N = \dim(Z)$, then
$$\Pi y = \sum_{i=1}^{N} (\phi_i, y)_Y \, \phi_i, \qquad \forall y \in Y$$

2 Linear Elliptic Problems I

Contents

2.1 Problem Statement

Problem Statement

Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$, evaluate — P-vector INPUT

$$s^e(\mu) = \ell(u^e(\mu); \mu) \qquad \text{OUTPUT}$$

where $u^e(x; \mu) \in X^e(\Omega)$ satisfies — FIELD $(x; \mu)$

$$a(u^e(\mu), v; \mu) = f(v; \mu), \quad \forall\, v \in X^e(\Omega). \qquad \mu PDE$$

- Here ^{e} refers to "exact," i.e., the exact (or analytic) solution to the problem for the prescribed mathematical model.
- To begin, we consider elliptic problems which are linear, coercive, affine, and compliant — explanations to follow …

Definitions

μ:	input parameter - $\mu = (\mu_1, \mu_2, \ldots, \mu_P)$;	P-tuple
$\mathcal{D}$:	parameter domain in $\mathbb{R}^P$;	
Ω:	spatial domain in $\mathbb{R}^d$;	
s^e:	output;	
ℓ:	output functional;	
u^e:	field variable;	
X^e:	function space $(H_0^1(\Omega))^\nu \subset X^e \subset (H^1(\Omega))^\nu$ †, with inner product $(w, v)_{X^e}$, $\forall w, v \in X^e$, and induced norm $\|w\|_{X^e} = \sqrt{(w, w)_{X^e}}$, $\forall w \in X^e$.	

† For simplicity we first assume $\nu = 1$.

Reference Geometry

Note $\boldsymbol{\Omega}$ is parameter-independent:

- the reduced basis requires a common spatial configuration, i.e., a reference domain $\boldsymbol{\Omega}_{\mathbf{ref}}$
- Introduce a piecewise affine mapping $\mathcal{T}(\cdot;\boldsymbol{\mu}) : \boldsymbol{\Omega} \to \boldsymbol{\Omega}_o(\boldsymbol{\mu})$

$$
\begin{array}{c}
a_o(w_o, v_o; \boldsymbol{\mu}) \text{ over } \boldsymbol{\Omega}_o(\boldsymbol{\mu}) \\
\Downarrow \\
\mathcal{T}(\cdot;\boldsymbol{\mu})^{-1} : \boldsymbol{\Omega}_o(\boldsymbol{\mu}) \to \boldsymbol{\Omega}_{\mathbf{ref}} \equiv \boldsymbol{\Omega} \qquad (\boldsymbol{\Omega}_{\mathbf{ref}} = \boldsymbol{\Omega}_o(\boldsymbol{\mu}_{\mathbf{ref}})) \\
\Downarrow \\
a(w, v; \boldsymbol{\mu}) \text{ over } \boldsymbol{\Omega}
\end{array}
$$

where $a(w, v; \boldsymbol{\mu}) = a_o(w_o \circ \mathcal{T}_\mu, v_o \circ \mathcal{T}_\mu; \boldsymbol{\mu})$

We will discuss this issue in detail later on.

Hypotheses – Continuity, Stability, Compliance

$$
\left.
\begin{array}{rl}
a(\cdot,\cdot;\boldsymbol{\mu}) : & \text{bilinear,} \\
& \text{symmetric,} \\
& X^{\mathrm{e}}\text{-continuous,} \\
& X^{\mathrm{e}}\text{-coercive form,} \quad \forall \boldsymbol{\mu} \in \mathcal{D}; \\
f(\cdot;\boldsymbol{\mu}),\ \ell(\cdot;\boldsymbol{\mu}) : & \text{linear,} \\
& X^{\mathrm{e}}\text{-bounded,} \quad \forall \boldsymbol{\mu} \in \mathcal{D}
\end{array}
\right\} \boldsymbol{\mu}\text{PDE}
$$

We first consider the compliant case:

- $\ell(\cdot;\boldsymbol{\mu}) = f(\cdot;\boldsymbol{\mu}),\ \forall \boldsymbol{\mu} \in \mathcal{D}$
- a symmetric

Hypotheses – Affine parameter dependence

Require also $f(v;\boldsymbol{\mu}),\ \ell(v;\boldsymbol{\mu})$

$$
a(w, v; \boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \Theta_a^q(\boldsymbol{\mu})\ a^q(w, v),
$$

where for $q = 1, \ldots, Q_a$

$\Theta_a^q : \mathcal{D} \to \mathbb{R}$, $\boldsymbol{\mu}$-dependent functions;
$a^q : X^{\mathrm{e}} \times X^{\mathrm{e}} \to \mathbb{R}$, $\boldsymbol{\mu}$-independent forms.

This assumption

- is broadly applicable to many instances of property and geometry parametric variations, but
- can also be relaxed (see lectures on "nonaffine problems").

Inner Products and Norms

- Energy inner product and induced norm (parameter-dependent)

$$
\begin{aligned}
(((w, v)))_{\mu} &= a(w, v; \mu), \qquad \forall w, v \in X^{e} \\
|||w|||_{\mu} &= \sqrt{a(w, w; \mu)}, \qquad \forall w \in X^{e}
\end{aligned}
$$

- X^{e}-inner product and induced norm (parameter-independent)

$$
\begin{aligned}
(w, v)_{X^{e}} &\equiv (((w, v)))_{\bar{\mu}} \, (= a(w, v; \bar{\mu})), \qquad \forall w, v \in X^{e} \\
\|w\|_{X^{e}} &\equiv |||w|||_{\bar{\mu}} \, (= \sqrt{a(w, w; \bar{\mu})}), \qquad \forall w \in X^{e}
\end{aligned}
$$

Coercivity and Continuity constants

Recall

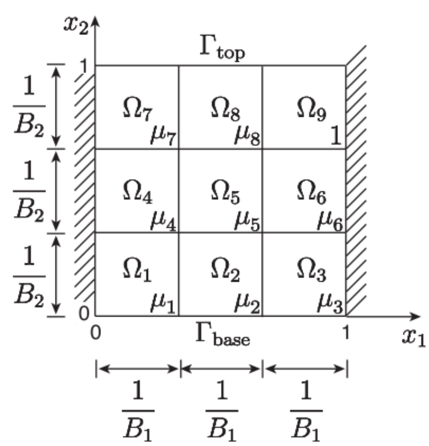
- Coercivity constant

$$
(0 <) \; \alpha^{e}(\mu) \equiv \inf_{w \in X^{e}} \frac{a(w, w; \mu)}{\|w\|^2_{X^{e}}};
$$

- Continuity constant

$$
\gamma^{e}(\mu) \equiv \sup_{w \in X^{e}} \sup_{v \in X^{e}} \frac{a(w, v; \mu)}{\|w\|_{X^{e}} \|v\|_{X^{e}}} \; (< \infty).
$$

Example: ThermalBlock



$\overline{\Omega} = \cup_{i=1}^{B_1 B_2} \overline{\Omega}_i$

Example: ThermalBlock
Problem Statement

Given $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_P) \subset \mathcal{D} \equiv [\mu^{\min}, \mu^{\max}]^P$, evaluate

$$s^{\mathrm{e}}(\mu) = f(u^{\mathrm{e}}(\mu)) \qquad\qquad \ell = f$$

where $u^{\mathrm{e}}(\mu) \in X^{\mathrm{e}} \equiv \{v \in H^1(\Omega) \mid v|_{\Gamma_{\mathrm{top}}} = 0\}$ satisfies

$$a(u^{\mathrm{e}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathrm{e}}.$$

Here, $P = B_1 B_2 - 1$, and we assume that

$$\mu^{\min} = 1/\sqrt{\mu_r},\ \mu^{\max} = \sqrt{\mu_r}, \text{ for } 1 < \mu_r < \infty,$$

such that $\mu^{\max}/\mu^{\min} = \mu_r$.

Example: ThermalBlock

Here, $\forall v \in X^{\mathrm{e}}$, compliant $\ell = f$

$$f(v) \equiv \int_{\Gamma_{\mathrm{base}}} v,$$

and, $\forall w, v \in X^{\mathrm{e}}$,

$$a(w, v; \mu) = \sum_{i=1}^{P} \mu_i \int_{\Omega_i} \nabla w \cdot \nabla v + \int_{\Omega_{P+1}} \nabla w \cdot \nabla v,$$

where $\overline{\Omega} = \cup_{i=1}^{P+1} \overline{\Omega}_i$.

Example: ThermalBlock

Inner product, $w, v \in X^{\mathrm{e}}$,

$$(w, v)_{X^{\mathrm{e}}} = \sum_{i=1}^{P} \overline{\mu}_i \int_{\Omega_i} \nabla w \cdot \nabla v + \int_{\Omega_{P+1}} \nabla w \cdot \nabla v,$$

where $\overline{\mu}_i$ is the reference parameter.

The bilinear form a is

- symmetric,
- (parametrically) coercive,
  $$0 < \frac{1}{\sqrt{\mu_r}} \leq \mathrm{Min}(\mu_1/\overline{\mu}_1, \ldots, \mu_P/\overline{\mu}_P, 1) \leq \alpha^{\mathrm{e}}(\mu),$$
- and continuous,
  $$\gamma^{\mathrm{e}}(\mu) < \mathrm{Max}(\mu_1/\overline{\mu}_1, \ldots, \mu_P/\overline{\mu}_P, 1) \leq \sqrt{\mu_r} < \infty.$$

The linear form f is bounded.

Example: ThermalBlock

We obtain

$$a(w,v;\mu) = \sum_{q=1}^{Q_a=P+1} \Theta^q(\mu)\, a^q(w,v):$$

for

$$\begin{aligned}
\Theta^1(\mu) &= \mu_1, & a^1(w,v) &= \int_{\Omega_1} \nabla w \cdot \nabla v\ ; \\
& & \vdots & \\
\Theta^P(\mu) &= \mu_P, & a^P(w,v) &= \int_{\Omega_P} \nabla w \cdot \nabla v\ ; \\
\Theta^{P+1}(\mu) &= 1, & a^{P+1}(w,v) &= \int_{\Omega_{P+1}} \nabla w \cdot \nabla v\ .
\end{aligned}$$

$\Rightarrow$ affine assumption is honored

2.2 Truth Approximation

Approximation Space

We replace the "exact" statement by a "truth" approximation, i.e., we introduce

- an approximation space $X^{\mathcal{N}} \subset X^{\mathrm{e}}$ of dimension $\dim(X^{\mathcal{N}}) = \mathcal{N}$, and
- an associated set of basis functions $\varphi_k^{\mathcal{N}} \in X^{\mathcal{N}},\ 1 \le k \le \mathcal{N}$.

We also equip $X^{\mathcal{N}}$ with inner product and induced norm

$$\begin{aligned}
(w,v)_{X^{\mathcal{N}}} &\equiv (w,v)_{X^{\mathrm{e}}} \equiv a(w,v;\bar{\mu}), \qquad \forall w,v \in X^{\mathcal{N}} \\
\|w\|_{X^{\mathcal{N}}} &\equiv \|w\|_{X^{\mathrm{e}}} \equiv \sqrt{a(w,w;\bar{\mu})}, \qquad \forall w \in X^{\mathcal{N}}
\end{aligned}$$

Approximation Space

- Coercivity constant of a over $X^{\mathcal{N}}$

$$\alpha^{\mathcal{N}}(\mu) \equiv \inf_{w \in X^{\mathcal{N}}} \frac{a(w,w;\mu)}{\|w\|^2_{X^{\mathcal{N}}}};$$

- Continuity constant of a over $X^{\mathcal{N}}$

$$\gamma^{\mathcal{N}}(\mu) \equiv \sup_{w \in X^{\mathcal{N}}} \sup_{v \in X^{\mathcal{N}}} \frac{a(w,v;\mu)}{\|w\|_{X^{\mathcal{N}}} \|v\|_{X^{\mathcal{N}}}}.$$

Note: since $X^{\mathcal{N}} \subset X^{\mathrm{e}}$,

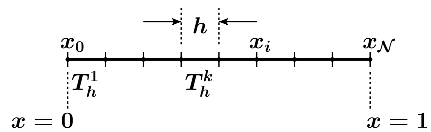
$$\begin{aligned}
\alpha^{\mathcal{N}}(\mu) &\ge \alpha^{\mathrm{e}}(\mu)\ (>0), \qquad \forall \mu \in \mathcal{D}, \\
\gamma^{\mathcal{N}}(\mu) &\le \gamma^{\mathrm{e}}(\mu)\ (<\infty), \qquad \forall \mu \in \mathcal{D}.
\end{aligned}$$

Approximation Space – Finite Element

We consider linear ($\mathbb{P}_1$) or quadratic ($\mathbb{P}_2$) Finite Element Spaces

Example: Consider the domain $\boldsymbol{\Omega} \equiv]\boldsymbol{0}, \boldsymbol{1}[\subset \mathbb{R}^{d=1}$

- Triangulation $\mathcal{T}_h$ with segments T_h^j, $1 \le j \le \mathcal{N}$

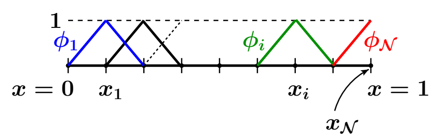


  Properties

  - T_h^j are open (closure $\overline{T}_h^j$)
  - $T_h^j \cap T_h^i = \emptyset$ for $j \neq i$
  - $\overline{\Omega} = \bigcup_{T_h \in \mathcal{T}_h} \overline{T}_h$

Approximation Space – Finite Element

- Nodal basis functions $\varphi_i(x)$, $1 \le i \le \mathcal{N}$



  Properties

  - Compact support in Ω (i.e., φ_i nonzero only on $\overline{T}_h^i \bigcup \overline{T}_h^{i+1}$)
  - For all $v \in X^{\mathcal{N}}$, $\exists$ unique $v_n \in \mathbb{R}$, such that $v(x) = \sum_{n=1}^{\mathcal{N}} v_n \varphi_n(x)$

- "Truth" finite element approximation space

  $X^{\mathcal{N}} = \{v \in X^{\mathrm{e}} \mid v|_{T_h^j} \in \mathbb{P}_1(T_h^j),\ 1 \le j \le \mathcal{N}\};$

  (Space of functions which are linear over each element)

"Truth" Problem Statement

Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$, evaluate — Galerkin projection

$$s^{\mathcal{N}}(\mu) = f(u^{\mathcal{N}}(\mu); \mu)$$

where $u^{\mathcal{N}}(x; \mu) \in X^{\mathcal{N}}(\Omega)$ satisfies

$$a(u^{\mathcal{N}}(\mu), v; \mu) = f(v; \mu), \quad \forall\, v \in X^{\mathcal{N}}(\Omega).$$

Assumption:

No variational crimes in Galerkin approximation, i.e.,

- truth approximation represents exact geometry,
- all quadratures are exact.

"Truth" Problem Statement

Galerkin projection convergent:

- $\boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu}) \to \boldsymbol{u}^{\mathrm{e}}(\boldsymbol{\mu})$ as $\boldsymbol{\mathcal{N}} \to \infty$; and ($\boldsymbol{f}$ is bounded)
- $\boldsymbol{s}^{\mathcal{N}}(\boldsymbol{\mu}) \to \boldsymbol{s}^{\mathrm{e}}(\boldsymbol{\mu})$ as $\boldsymbol{\mathcal{N}} \to \infty$.

We choose $\boldsymbol{\mathcal{N}}$ such that $\boldsymbol{\mathcal{N}}$ *conservative*

$$\varepsilon^{\mathcal{N}} = \max_{\boldsymbol{\mu}\in\boldsymbol{\mathcal{D}}} \|\boldsymbol{u}^{\mathrm{e}}(\boldsymbol{\mu}) - \boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu})\|_{\boldsymbol{X}^{\mathrm{e}}} \ll \epsilon_{\textbf{tol, min}},$$

and hence we require that our RB formulation be

(a) numerically stable as $\boldsymbol{\mathcal{N}} \to \infty$; and[1ex]

(b) computationally efficient as $\boldsymbol{\mathcal{N}} \to \infty$.

Truth Approximation – Role

We shall

(i) *build* our reduced basis approximation upon "truth" solutions $\boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu}) \in \boldsymbol{X}^{\mathcal{N}}$;

(ii) *measure* the error in the reduced basis approximation relative to the "truth" solution $\boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu}) \in \boldsymbol{X}^{\mathcal{N}}$ (and $\boldsymbol{s}^{\mathcal{N}}(\boldsymbol{\mu})$);

$\Rightarrow \boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu})$ is a *calculable surrogate* for $\boldsymbol{u}^{\mathrm{e}}(\boldsymbol{\mu})$.

$$\|\boldsymbol{u}^{\mathrm{e}}(\boldsymbol{\mu}) - \boldsymbol{u}_N(\boldsymbol{\mu})\|_{\boldsymbol{X}^{\mathrm{e}}} \le \underbrace{\|\boldsymbol{u}^{\mathrm{e}}(\boldsymbol{\mu}) - \boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu})\|_{\boldsymbol{X}^{\mathrm{e}}}}_{\le \varepsilon_{\textbf{exact}}} + \underbrace{\|\boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu}) - \boldsymbol{u}_N(\boldsymbol{\mu})\|_{\boldsymbol{X}^{\mathrm{e}}}}_{\le \varepsilon_{\textbf{tol,min}}}$$

with $\varepsilon_{\textbf{exact}} \ll \varepsilon_{\textbf{tol,min}}$.

Algebraic Equations

Expand $\quad \boldsymbol{u}^{\mathcal{N}}(\boldsymbol{x};\boldsymbol{\mu}) = \sum_{j=1}^{\mathcal{N}} \boldsymbol{u}_j^{\mathcal{N}}(\boldsymbol{\mu})\varphi_j^{\mathcal{N}}(\boldsymbol{x})$

Given $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}}$, evaluate

$$\boldsymbol{s}^{\mathcal{N}}(\boldsymbol{\mu}) = (\underline{\boldsymbol{F}}^{\mathcal{N}}(\boldsymbol{\mu}))^{\boldsymbol{T}}\, \underline{\boldsymbol{u}}^{\mathcal{N}}(\boldsymbol{\mu})$$

where $\underline{\boldsymbol{u}}^{\mathcal{N}}(\boldsymbol{\mu}) \equiv [\boldsymbol{u}_1^{\mathcal{N}}(\boldsymbol{\mu})\ \boldsymbol{u}_2^{\mathcal{N}}(\boldsymbol{\mu}) \ldots \boldsymbol{u}_{\mathcal{N}}^{\mathcal{N}}(\boldsymbol{\mu})]^{\boldsymbol{T}} \in \mathbb{R}^{\mathcal{N}}$ satisfies

$$\underline{\boldsymbol{A}}^{\mathcal{N}}(\boldsymbol{\mu})\, \underline{\boldsymbol{u}}^{\mathcal{N}}(\boldsymbol{\mu}) = \underline{\boldsymbol{F}}^{\mathcal{N}}(\boldsymbol{\mu}).$$

- Stiffness matrix $\underline{\boldsymbol{A}}^{\mathcal{N}}(\boldsymbol{\mu}) \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$
$$A_{ij}^{\mathcal{N}}(\mu) = a(\varphi_j^{\mathcal{N}}, \varphi_i^{\mathcal{N}}; \mu), \qquad 1 \le i, j \le \mathcal{N}$$

- Load/source vector $\underline{\boldsymbol{F}}^{\mathcal{N}}(\boldsymbol{\mu}) \in \mathbb{R}^{\mathcal{N}}$
$$F_i^{\mathcal{N}}(\mu) = f(\varphi_i^{\mathcal{N}}; \mu), \qquad 1 \le i \le \mathcal{N}$$

Algebraic Equations

Affine Assumption

- Stiffness matrix: $\underline{\boldsymbol{A}}^{\mathcal{N}}(\boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\underline{\mathbb{A}}^{\mathcal{N}\,q}$, where $\underline{\mathbb{A}}^{\mathcal{N}\,q} \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$, $1 \le q \le Q_a$, and

  $\mathbb{A}_{ij}^{\mathcal{N}\,q} = a^q(\varphi_j^{\mathcal{N}}, \varphi_i^{\mathcal{N}}), \quad 1 \le i, j \le \mathcal{N},\ 1 \le q \le Q_a.$

- Load/source vector: $\underline{\boldsymbol{F}}^{\mathcal{N}}(\boldsymbol{\mu}) = \sum_{q=1}^{Q_f} \Theta_f^q(\mu)\underline{\mathbb{F}}^{\mathcal{N}\,q}$, where

  $\underline{\mathbb{F}}^{\mathcal{N}\,q} \in \mathbb{R}^{\mathcal{N}}$, $1 \le q \le Q_f$, and

  $\mathbb{F}_i^{\mathcal{N}\,q} = f^q(\varphi_i^{\mathcal{N}}), \quad 1 \le i \le \mathcal{N},\ 1 \le q \le Q_f.$

Note: the $\underline{\mathbb{A}}^{\mathcal{N}\,q}$ and $\underline{\mathbb{F}}^{\mathcal{N}\,q}$ are parameter-independent.

Algebraic Equations

We also introduce $\underline{\mathbb{X}}^{\mathcal{N}} \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$, such that
$$\mathbb{X}_{ij}^{\mathcal{N}} = (\varphi_j^{\mathcal{N}}, \varphi_i^{\mathcal{N}})_{X^{\mathcal{N}}}, \quad 1 \le i, j \le \mathcal{N}$$

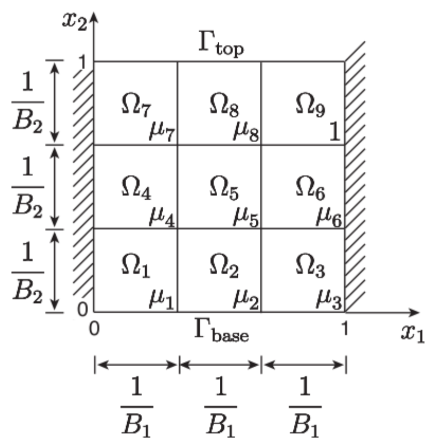
Then, given $w, v \in X^{\mathcal{N}}$, we expand
$$w = \sum_{j=1}^{\mathcal{N}} w_j \varphi_j^{\mathcal{N}} \text{ and } v = \sum_{j=1}^{\mathcal{N}} v_j \varphi_j^{\mathcal{N}},$$
and thus
$$(w, v)_{X^{\mathcal{N}}} = \sum_{j=1}^{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} w_j v_i (\varphi_j^{\mathcal{N}}, \varphi_i^{\mathcal{N}})_{X^{\mathcal{N}}} = \underline{w}^T \underline{\mathbb{X}}^{\mathcal{N}} \underline{v},$$
where
$$\underline{w} \equiv [w_1\ w_2 \ldots w_{\mathcal{N}}]^T \in \mathbb{R}^{\mathcal{N}} \text{ and } \underline{v} \equiv [v_1\ v_2 \ldots v_{\mathcal{N}}]^T \in \mathbb{R}^{\mathcal{N}}.$$

Nomenclature

For the rest of this course we will use the following notation

$()^{\mathrm{e}}$	exact problem statement X^{e}, u^{e}, s^{e}, ...
$() = ()^{\mathcal{N}}$	truth approximation e.g. $X = X^{\mathcal{N}}$, $u = u^{\mathcal{N}}$, $s = s^{\mathcal{N}}$, ...
$()_N = ()_N^{\mathcal{N}}$	reduced basis approximation e.g. $X_N = X_N^{\mathcal{N}}$, $u_N = u_N^{\mathcal{N}}$, $s_N = s_N^{\mathcal{N}}$, ...

Example: thermal block



$\overline{\Omega} = \cup_{i=1}^{B_1 B_2} \overline{\Omega}_i$

Example: thermal block

"Truth" Problem Statement

Given $\mu = (\mu_1, \ldots, \mu_P) \subset \mathcal{D} \equiv [\mu^{\min}, \mu^{\max}]^P$, evaluate

$$s^{\mathcal{N}}(\mu) = f(u^{\mathcal{N}}(\mu)) \qquad\qquad \ell = f$$

where $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}}$ satisfies

$$a(u^{\mathcal{N}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathcal{N}}.$$

Here, $P = B_1 B_2 - 1$ and we require $0 < \mu^{\min} < \mu^{\max} < \infty$.

Example: thermal block

"Truth" Problem Statement

Given $\mu = (\mu_1, \ldots, \mu_P) \subset \mathcal{D} \equiv [\mu^{\min}, \mu^{\max}]^P$, evaluate

$$s(\mu) = f(u(\mu)) \qquad \ell = f$$

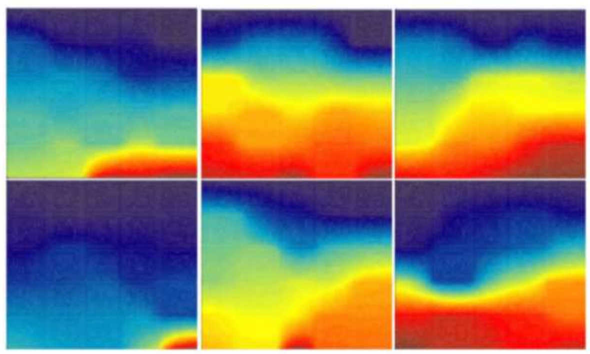
where $u(\mu) \in X$ satisfies

$$a(u(\mu), v; \mu) = f(v), \quad \forall\, v \in X.$$

Here, $P = B_1 B_2 - 1$ and we require $0 < \mu^{\min} < \mu^{\max} < \infty$.

Example: thermal block

Representative Solution ($P = 24$)



Source: S. Sen, NHT Part B

3 Reduced Basis Approximation

Contents

3.1 Objective

Reduced Basis (RB) Objective

For any $\varepsilon_{\text{des}} > 0$, evaluate — ACCURACY

$$\mu \in \mathcal{D} \to s_N(\mu) \ (\approx s^{\mathcal{N}}(\mu)) \text{ and } \Delta_N^s(\mu)$$

that provably achieves desired accuracy — RELIABILITY

$$|s^{\mathcal{N}}(\mu) - s_N(\mu)| \le \Delta_N^s(\mu) \le \varepsilon_{\text{des}}$$

but at (very low) marginal cost ∂t_{comp} — EFFICIENCY

independent of $\mathcal{N}$ as $\mathcal{N} \to \infty$.

Here, ∂t_{comp} is the time to perform

one additional certified evaluation $\mu \to (s_N(\mu), \Delta_N^s(\mu))$.

RB Objective – Rapid Convergence

Find a reduced basis approximation

$$\boldsymbol{s_N(\mu) \in \mathbb{R}} \text{ and } \boldsymbol{u_N(\mu) \in X_N \subset X^{\mathcal{N}}}$$

for all $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}}$, such that

$$\boldsymbol{s_N(\mu) \to s^{\mathcal{N}}(\mu)} \text{ and } \boldsymbol{u_N(\mu) \to u^{\mathcal{N}}(\mu)}$$

rapidly as $\boldsymbol{N = \mathbf{dim}(X_N) \to \infty (= 10 - 200)}$.

Note that the convergence rate should be independent of $\boldsymbol{\mathcal{N}}$.

RB Objective – Rigor & Certainty

A posteriori error bounds $\boldsymbol{\Delta_N(\mu)}$ and $\boldsymbol{\Delta_N^s(\mu)}$:

$$\mathbf{1} \text{ (rigor)} \leq \frac{\boldsymbol{\Delta_N(\mu)}}{\boldsymbol{\|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_X}} \leq \boldsymbol{C} \text{ (sharpness)}$$

and

$$\mathbf{1} \text{ (rigor)} \leq \frac{\boldsymbol{\Delta_N^s(\mu)}}{\boldsymbol{|s^{\mathcal{N}}(\mu) - s_N(\mu)|}} \leq \boldsymbol{C} \text{ (sharpness)}$$

for all $\boldsymbol{N \in \mathbb{N} \equiv \{1, \ldots, N_{\mathbf{max}}\}}$ and all $\boldsymbol{\mu \in \mathcal{D}}$.

RB Objective – Computational Efficiency

Offline-Online computational strategies:

Offline: expensive preprocessing

$\boldsymbol{t_{\mathbf{comp}}^{\mathbf{Offline}} \gg \mathbf{cost}\{\mu \to s^{\mathcal{N}}(\mu)\}}$

BUT

Online: very rapid certified RB input-output evaluation

$\boldsymbol{\partial t_{\mathbf{comp}} \equiv \mathbf{marginal\ cost}\{\mu \to s_N(\mu), \Delta_N^s(\mu)\}}$

depends only on $\boldsymbol{Q}$ and $\boldsymbol{N}$ – not on $\boldsymbol{\mathcal{N}}$.

⇒ we may thus choose $\boldsymbol{\mathcal{N}}$ very conservatively.

RB Objective – Relevance

Real-Time Context (parameter estimation, . . .):

$$\begin{array}{ccc} \boldsymbol{\mu} & \rightarrow & \boldsymbol{s_N(\mu),\ \Delta^s_N(\mu)}. \\ \boldsymbol{t_0} \text{ (“need”)} & & \boldsymbol{t_0 + \partial t_{\mathrm{comp}}} \text{ (“response”)} \end{array}$$

Many-Query Context (design, . . .):

$$\begin{array}{ccc} \boldsymbol{\mu_j} & \rightarrow & \boldsymbol{(s_N(\mu_j),\ \Delta^s_N(\mu_j)),\ j = 1, \ldots, J}. \\ \boldsymbol{t_0} & & \boldsymbol{t_0 + \partial t_{\mathrm{comp}}\, J} \text{ as } \boldsymbol{J \rightarrow \infty} \end{array}$$

⇒ Low marginal (real-time) and/or low average (many-query) cost.

Parametric Manifold $\boldsymbol{\mathcal{M}^{\mathcal{N}}}$

We assume

- the form $\boldsymbol{a}$ is continuous and coercive (or inf-sup stable); and
- affine $\boldsymbol{\mu}$-dependence; and
- the $\boldsymbol{\Theta^q(\mu)}$, $\boldsymbol{1 \le q \le Q}$, are smooth (i.e., $\boldsymbol{\Theta^q \in C^\infty(\mathcal{D})}$);

then

$$\boldsymbol{\mathcal{M}^{\mathcal{N}} \equiv \{u^{\mathcal{N}}(\mu) \,|\, \forall \mu \in \mathcal{D}\}}$$

is a smooth $\boldsymbol{P}$-dimensional manifold in $\boldsymbol{X^{\mathcal{N}}}$, since

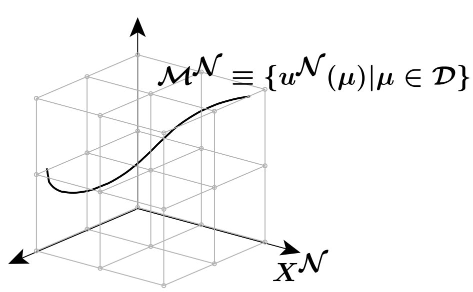
$$\boldsymbol{\|D_\sigma y^{\mathcal{N}}(\mu)\|_X \le C_\sigma,\ \forall \mu \in \mathcal{D}}, \text{ for any order } \boldsymbol{|\sigma| \in \mathbb{N}_{+0}}.$$

3.2 RB Approximation Space

Parametric Manifold $\boldsymbol{\mathcal{M}^{\mathcal{N}}}$

To approximate $\boldsymbol{u^{\mathcal{N}}(\mu)}$, and hence $\boldsymbol{s^{\mathcal{N}}(\mu)}$,

we need not represent every possible function in $\boldsymbol{X^{\mathcal{N}}}$.

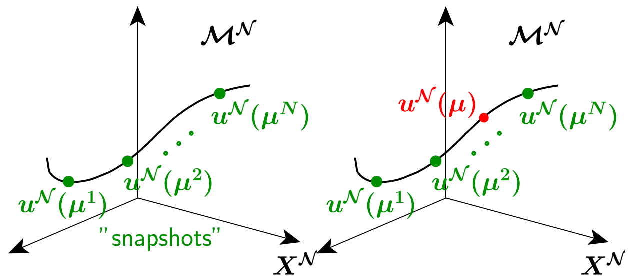


Early work: [ASB], [NP], [FR], [Po], [Pe], [G], [IR], . . .

Parametric Manifold $\mathcal{M}^{\mathcal{N}}$

To approximate $u^{\mathcal{N}}(\mu)$, and hence $s^{\mathcal{N}}(\mu)$,

we need not represent every possible function in $X^{\mathcal{N}}$.



LOCALIZATION SMOOTHNESS

Spaces & Bases

We define the RB approximation space

$$X_N = \text{span}\{\xi^n,\ 1 \le n \le N\}, \quad 1 \le N \le N_{\max},$$

with linearly independent basis functions

$$\xi^n \in X, \quad 1 \le n \le N_{\max}.$$

We thus obtain

$$X_N \subset X, \quad \dim(X_N) = N, \quad 1 \le N \le N_{\max},$$

and "nested" (hierarchical) spaces

$$X_1 \subset X_2 \subset \ldots \subset X_{N_{\max}-1} \subset X_{N_{\max}} (\subset X).$$

We denote non-hierarchical RB spaces as X_N^{nh}, $1 \le N \le N_{\max}$,

$$X_N^{\text{nh}} \subset X, \quad \dim(X_N^{\text{nh}}) = N, \quad 1 \le N \le N_{\max}.$$

Spaces & Bases – Lagrangian

Parameter samples:

$$S_N = \{\mu^1 \in \mathcal{D}, \ldots, \mu^N \in \mathcal{D}\}, \quad 1 \le N \le N_{\max},$$

with nested

$$S_1 \subset S_2 \subset \ldots \subset S_{N_{\max}-1} \subset S_{N_{\max}} \subset \mathcal{D}.$$

Lagrangian reduced basis spaces: hierarchical

$$W_N = \text{span}\{\underbrace{u(\mu^n)}_{\text{"snapshots"}},\ 1 \le n \le N\}, \quad 1 \le N \le N_{\max},$$

with

$$W_1 \subset W_2 \subset \ldots \subset W_{N_{\max}-1} \subset W_{N_{\max}} (\subset X).$$

Spaces & Bases – Taylor & Hermite

Taylor reduced basis spaces: hierarchical

$$W_N^{\text{Taylor}} = \text{span}\{D_\sigma u(\mu),\ \forall \sigma \in I^{P,N-1}\}, \quad 1 \le N \le N_{\max},$$

field variable and sensitivity derivatives at one point in $\mathcal{D}$.

Hermite reduced basis spaces: hierarchical

$$W_N^{\text{Hermite}} \text{ “}=\text{” } W_N^{\text{Lagrangian}} \cup W_N^{\text{Taylor}}$$

field variable and sensitivity derivatives at several points in $\mathcal{D}$

Note: We will exclusively use Lagrangian RB spaces in this course.

Spaces & Bases – Orthogonal Basis

Given $\xi^n = u(\mu^n),\ 1 \le n \le N_{\max}$ (Lagrange case) we construct the basis set $\{\zeta^n\},\ 1 \le n \le N_{\max}$, from

Gram-Schmidt Orthogonalization

$\zeta^1 = \xi^1 / \|\xi^1\|_X;$

`for` $n = 2 : N_{\max}$

$$z^n = \xi^n - \sum_{m=1}^{n-1} (\xi^n, \zeta^m)_X\, \zeta^m;$$

$$\zeta^n = z^n / \|z^n\|_X;$$

`end.`

Note: $(\zeta^n, \zeta^m)_X = \delta_{nm}, \quad 1 \le n, m \le N_{\max}$

Spaces & Bases – Orthogonal Basis

Given reduced basis space

$$X_N = \text{span}\{\zeta^n,\ 1 \le n \le N\}, \quad 1 \le N \le N_{\max}.$$

we can express any $w_N \in X_N$ as

$$w_N = \sum_{n=1}^{N} w_{N\,n} \zeta^n$$

for unique $w_{N\,n} \in \mathbb{R},\ 1 \le n \le N.$

Reduced basis “matrices” $\mathbb{Z}_N \in \mathbb{R}^{\mathcal{N} \times N},\ 1 \le N \le N_{\max}$:

$$\mathbb{Z}_N = [\zeta^1\ \zeta^2 \ldots \zeta^N], \quad 1 \le N \le N_{\max},$$

where, from orthogonality, $\mathbb{Z}_{N_{\max}}^T \mathbb{X} \mathbb{Z}_{N_{\max}}^T = \mathbb{I}_{N_{\max}}$, and $\mathbb{I}_M$ is the Identity matrix in $\mathbb{R}^{M \times M}$.

3.3 Galerkin Projection

Galerkin Projection

Given $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}} \subset \mathbb{R}^{\boldsymbol{P}}$, evaluate optimality

$$s_N(\mu) = f(u_N(\mu); \mu)$$

where $u_N(x; \mu) \in X_N \subset X^{\mathcal{N}}$ satisfies

$$a(u_N(\mu), v; \mu) = f(v; \mu), \quad \forall\, v \in X_N.$$

Note:

- Lagrangian reduced basis space: $X_N = W_N$
- Solution $u_N(\mu)$ unique (from coercivity, continuity, linear independence)

Galerkin Optimality

Proposition

For any $\mu \in \mathcal{D}$, we have

$$\begin{aligned} |||u(\mu) - u_N(\mu)|||_\mu &= \inf_{w_N \in X_N} |||u(\mu) - w_N(\mu)|||_\mu, \\ \|u(\mu) - u_N(\mu)\|_X &\le \sqrt{\frac{\gamma^{\mathrm{e}}(\mu)}{\alpha^{\mathrm{e}}(\mu)}} \inf_{w_N \in X_N} \|u(\mu) - w_N(\mu)\|_X, \end{aligned}$$

and furthermore

$$\begin{aligned} s(\mu) - s_N(\mu) &= |||u(\mu) - u_N(\mu)|||_\mu^2 \\ &= \inf_{w_N \in X_N} |||u(\mu) - w_N(\mu)|||_\mu^2, \end{aligned}$$

as well as

$$0 \le s(\mu) - s_N(\mu) \le \gamma^{\mathrm{e}}(\mu) \inf_{w_N \in X_N} ||u(\mu) - w_N(\mu)||_X^2.$$

3.4 Offline-Online Computational Procedure

Offline-Online Decomposition

We expand $\quad u_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\zeta^j$

and obtain $\qquad v = \zeta^i, 1 \le i \le N$

$$a(u_N(\mu), v; \mu) = f(v; \mu)$$

$$\sum_{j=1}^{N} u_{N\,j}(\mu) a(\zeta^j, \zeta^i; \mu) = f(\zeta^i; \mu)$$

$$\underbrace{\sum_{j=1}^{N} u_{N\,j}(\mu) \underbrace{\sum_{q=1}^{Q_a} \Theta_a^q(\mu) \underbrace{a^q(\zeta^j, \zeta^i)}_{\text{OFFLINE: } O(\mathcal{N})}}_{\text{ONLINE: } O(Q_a N^2)} = \underbrace{\sum_{q=1}^{Q_f} \Theta_f^q(\mu) \underbrace{f^q(\zeta^i)}_{\text{OFFLINE: } O(\mathcal{N})}}_{\text{ONLINE: } O(Q_f N)}}_{\text{ONLINE: } O(N^3)}$$

Offline-Online Decomposition

Given $u_{N\,j}(\mu), 1 \leq j \leq N$, we evaluate the output from

$$\begin{aligned} s_N(\mu) = f(u_N(\mu);\mu) &= \sum_{j=1}^{N} u_{N\,j}(\mu) f(\zeta^j;\mu) \\ &= \underbrace{\sum_{j=1}^{N} u_{N\,j}(\mu) \underbrace{\sum_{q=1}^{Q_f} \Theta_f^q(\mu) \underbrace{f^q(\zeta^j)}_{\text{OFFLINE: } O(\mathcal{N})}}_{\text{ONLINE: } O(Q_f N)}}_{\text{ONLINE: } O(N)} \end{aligned}$$

Offline-Online Decomposition

Summary computational cost: $(Q = Q_a + Q_f)$

OFFLINE — once, parameter *independent*

$$\underbrace{O(N_{\max}\mathcal{N}^{\bullet})}_{\text{solve for } \zeta_n} + \underbrace{O(Q N_{\max}^2 \mathcal{N})}_{\text{form } \mu\text{-independent quantities}};$$

ONLINE — many times, parameter *dependent* μ^{new}

$$\underbrace{O(QN^2)}_{\text{form RB matrices}} + \underbrace{O(N^3)}_{\text{solve for } u_{Nj}(\mu)} + \underbrace{O(N)}_{\text{evaluate output}};$$

Online cost is independent of $\mathcal{N}$.

Algebraic Equations

Evaluation of RB Stiffness Matrix $\underline{A}_N \in \mathbb{R}^{N\times N}$:

Parameter-independent matrices $\mathbb{A}_N^q \in \mathbb{R}^{N\times N}, \; 1 \leq q \leq Q_a$:

$$\begin{aligned} \mathbb{A}^q_{N\,n\,m} &= a^q(\zeta^m, \zeta^n) \\ &= \sum_{i=1}^{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} \zeta_i^m \, a^q(\varphi_i^{\mathcal{N}}, \varphi_j^{\mathcal{N}}) \, \zeta_j^n, \quad 1 \leq n, m \leq N, \end{aligned}$$

thus

$$\mathbb{A}_N^q = \mathbb{Z}_N^T \, \mathbb{A}^{\mathcal{N} q} \, \mathbb{Z}_N.$$

We finally assemble

$$\underline{A}_N = \sum_{q=1}^{Q_a} \Theta_a^q(\mu) \, \mathbb{A}_N^q.$$

Algebraic Equations

Evaluation of RB Load/Source/Output Vector $\underline{F}_N \in \mathbb{R}^N$:

Parameter-independent vectors $\mathbb{F}_N^q \in \mathbb{R}^N, \; 1 \leq q \leq Q_f$:

$$\begin{aligned} \mathbb{F}^q_{N\,n} &= f^q(\zeta^n) \\ &= \sum_{i=1}^{\mathcal{N}} \zeta_i^m \, f^q(\varphi_i^{\mathcal{N}}), \quad 1 \leq n \leq N, \end{aligned}$$

thus

$$\underline{\mathbb{F}}_N^q = \mathbb{Z}_N^T \, \underline{\mathbb{F}}^{\mathcal{N} q}.$$

We finally assemble

$$\underline{F}_N = \sum_{q=1}^{Q_f} \Theta_f^q(\mu) \, \underline{\mathbb{F}}_N^q.$$

Algebraic Equations

We expand $\quad u_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu) \zeta^j$

and define the vector of RB coefficients

$$\underline{u}_N(\mu) = [u_{N\,1}(\mu) \; u_{N\,2}(\mu) \ldots u_{N\,N}(\mu)] \in \mathbb{R}^N.$$

Then, given $\mu \in \mathcal{D} \subset \mathbb{R}^P$, we evaluate

$$s_N(\mu) = \underline{F}_N^T(\mu) \, \underline{u}_N(\mu)$$

where $\underline{u}_N(\mu) \in \mathbb{R}^N$ satisfies

$$\underline{A}_N(\mu) \, \underline{u}_N(\mu) = \underline{F}_N(\mu).$$

Algebraic Equations

Proposition

The condition number of $\underline{A}_N(\mu)$ is bounded from above by $\gamma^{\mathrm{e}}(\mu)/\alpha^{\mathrm{e}}(\mu)$, the ratio of the continuity and coercivity constants for the continuous problem.

Sketch of proof:

- Rayleigh Quotient

$$\frac{\underline{w}_N^T \underline{A}_N(\mu) \underline{w}_N}{\underline{w}_N^T \underline{w}_N}, \quad \forall \underline{w}_N \in \boldsymbol{R}^N$$

- Express

$$\underline{w}_N = \sum_{m=1}^{N} w_{N\,m} \zeta^m$$

- Invoke coercivity (resp. continuity) and orthogonality.