Model Order Reduction Techniques I & II

RB: *A Posteriori* Error Estimation

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"Truth" Problem Statement

Given $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}} \subset \mathbb{R}^{P}$, evaluate

$(\cdot) = (\cdot)^{\mathcal{N}}$

$$s(\mu) = f(u(\mu); \mu)$$

where $u(x; \mu) \in X$ satisfies

$$a(u(\mu), v; \mu) = f(v; \mu), \quad \forall\, v \in X(\Omega).$$

Assumptions:

- linearity, coercivity, continuity;
- affine parameter dependence; and
- compliance: $\ell = f$, a symmetric.

Reduced Basis Sample and Space

Parameter samples:

$$S_N = \{\mu^1 \in \mathcal{D}, \ldots, \mu^N \in \mathcal{D}\}, \quad 1 \le N \le N_{\max},$$

with

$$S_1 \subset S_2 \subset \ldots \subset S_{N_{\max}-1} \subset S_{N_{\max}} \subset \mathcal{D}.$$

Lagrangian reduced basis spaces:

$$W_N = \operatorname{span}\{ \underbrace{u(\mu^n)}_{\text{"snapshots"}}, 1 \le n \le N\}, \quad 1 \le N \le N_{\max},$$

with

$$W_1 \subset W_2 \subset \ldots \subset W_{N_{\max}-1} \subset W_{N_{\max}} (\subset X).$$

Reduced Basis Approximation

Given $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}} \subset \mathbb{R}^{\boldsymbol{P}}$, evaluate

$$\boldsymbol{s_N(\mu) = f(u_N(\mu); \mu)}$$

where $\boldsymbol{u_N(x; \mu)} \in \boldsymbol{X_N} \subset \boldsymbol{X}$ satisfies

$$\boldsymbol{a(u_N(\mu), v; \mu) = f(v; \mu), \quad \forall\, v \in X_N.}$$

Recall:

- RB space: $\boldsymbol{X_N}$ = "Gram-Schmidt"$(\boldsymbol{W_N})$
- $\boldsymbol{u_N(\mu)}$ unique (from coercivity, continuity, linear independence)

Coercivity & Continuity

We assume that $a : X \times X \times \mathcal{D} \to \mathbb{R}$ is

- coercive

$$(0 <)\ \alpha(\mu) \equiv \inf_{w \in X} \frac{a(w, w; \mu)}{\|w\|_X^2};$$

- continuous

$$\gamma(\mu) \equiv \sup_{w \in X} \sup_{v \in X} \frac{a(w, v; \mu)}{\|w\|_X \|v\|_X}\ (< \infty).$$

Affine Parameter Dependence & Parametric Coercivity

We also assume that $a : X \times X \times \mathcal{D} \to \mathbb{R}$ is

- affine

$$a(w, v; \mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\, a^q(w, v), \qquad \forall w, v \in X$$

- and parametrically coercive

$$\Theta_a^q(\mu) > 0, \qquad \forall \mu \in \mathcal{D},\ 1 \le q \le Q_a,$$

and

$$a^q(v, v) \ge 0, \qquad \forall v \in X,\ 1 \le q \le Q_a.$$

Inner Products and Norms

- Energy inner product and induced norm (parameter-dependent)

$$
\begin{aligned}
(((w, v)))_{\mu} &= a(w, v; \mu), \qquad \forall w, v \in X \\
|||w|||_{\mu} &= \sqrt{a(w, w; \mu)}, \qquad \forall w \in X
\end{aligned}
$$

- X^{e}-inner product and induced norm (parameter-independent)

$$
\begin{aligned}
(w, v)_{X} &\equiv (((w, v)))_{\bar{\mu}} \, (= a(w, v; \bar{\mu})), \qquad \forall w, v \in X \\
\|w\|_{X} &\equiv |||w|||_{\bar{\mu}} \, (= \sqrt{a(w, w; \bar{\mu})}), \qquad \forall w \in X
\end{aligned}
$$

Motivation

How do we know that $u_N(\mu)$, $s_N(\mu)$ are accurate? ONLINE

$$\|u(\mu) - u_N(\mu)\|_X \leq \epsilon_{\text{tol,min}}, \quad \forall \mu \in \mathcal{D}$$

$$|s(\mu) - s_N(\mu)| \leq \epsilon^s_{\text{tol,min}}, \quad \forall \mu \in \mathcal{D}$$

How do we know what value of N to take? ONLINE/OFFLINE

N too large $\Rightarrow$ computational inefficiency
N too small $\Rightarrow$ unacceptable uncertainty

How do we choose the sample S_N optimally? OFFLINE

RB space has to approximate manifold $\mathcal{M}$ well, but
RB matrices need to be "well-conditioned."

Requirements

Our *a posteriori* error bounds, $\boldsymbol{\Delta_N(\mu)}$ and $\boldsymbol{\Delta_N^s(\mu)}$, must be

- rigorous $\qquad \boldsymbol{1 \leq N \leq N_{\max}}$

$$\begin{aligned} \|u(\mu) - u_N(\mu)\|_X &\leq \Delta_N(\mu), \quad \forall \mu \in \mathcal{D}, \\ |s(\mu) - s_N(\mu)| &\leq \Delta_N^s(\mu), \quad \forall \mu \in \mathcal{D}. \end{aligned}$$

- reasonably sharp

$$\frac{\Delta_N(\mu)}{\|u(\mu) - u_N(\mu)\|_X} \leq C, \quad \frac{\Delta_N^s(\mu)}{|s(\mu) - s_N(\mu)|} \leq C,$$

where $C \approx 1$.

- efficient

$\Rightarrow$ Online cost depends on N and Q, but not on $\mathcal{N}$.

Coercivity Lower Bound

We require a positive lower bound for the coercivity constant

$$0 < \alpha_{\mathrm{LB}}(\mu) \leq \alpha(\mu), \quad \forall \mu \in \mathcal{D}.$$

This bound can be constructed and calculated explicitly using the

- "$\min \Theta$" Approach

  $\Rightarrow$ if a is parametrically coercive,

or (more generally) using the

- Successive Constraint Method

  $\Rightarrow$ also applicable if a is (only) "inf-sup" stable ($\beta_{\mathrm{LB}}(\mu)$).

The "$\min \Theta$" Approach

Lemma

For a parametrically coercive, we have

$$0 < \Theta_a^{\min,\bar{\mu}}(\mu) \leq \alpha(\mu), \quad \forall \mu \in \mathcal{D},$$

where $\Theta_a^{\min,\bar{\mu}} : \mathcal{D} \to \mathbb{R}_+$ is given by

$$\Theta_a^{\min,\bar{\mu}}(\mu) = \min_{q \in \{1,\dots,Q_a\}} \frac{\Theta_a^q(\mu)}{\Theta_a^q(\bar{\mu})}.$$

Recall X-inner product an norm

$$(w, v)_X \equiv a(w, v; \bar{\mu}), \quad \forall w, v \in X$$

$$||w||_X \equiv (w, w)_X^{1/2}, \quad \forall w \in X$$

The "$\min \Theta$" Approach

Similarly, we may develop an upper bound for the continuity constant:

$$\gamma(\mu) < \gamma_{\mathrm{UB}}(\mu) < \infty, \quad \forall \mu \in \mathcal{D}.$$

NOTE: we do not required $\gamma_{\mathrm{UB}}(\mu)$ for our *a posteriori* error estimation, but only for theoretical clarification.

Lemma

For a parametrically coercive and symmetric, we have

$$\gamma(\mu) \leq \Theta_a^{\max,\bar{\mu}}(\mu) < \infty, \quad \forall \mu \in \mathcal{D},$$

where $\Theta_a^{\max,\bar{\mu}} : \mathcal{D} \to \mathbb{R}_+$ is given by

$$\Theta_a^{\max,\bar{\mu}}(\mu) = \max_{q \in \{1,\ldots,Q_a\}} \frac{\Theta_a^q(\mu)}{\Theta_a^q(\bar{\mu})}.$$

The "$\min \Theta$" Approach

If a is parametrically coercive, we may choose

- coercivity constant lower bound
$$\alpha_{\mathrm{LB}}(\mu) \equiv \Theta_a^{\min,\bar{\mu}}(\mu),$$
- continuity constant upper bound (if a is also symmetric)
$$\gamma_{\mathrm{UB}}(\mu) \equiv \Theta_a^{\max,\bar{\mu}}(\mu).$$

Note:

- Online cost to evaluate $\Theta_a^{\min,\bar{\mu}}(\mu)$: $O(Q_a)$.
- Choice of inner product important: $(w, v)_X \equiv a(w, v; \bar{\mu})$.
- Lower bound directly extends to non-symmetric forms b by using the symmetric part
$$b_s(w, v; \mu) = \frac{1}{2}(b(w, v; \mu) + b(v, w; \mu)).$$

Prerequisites – Error Residual Equation

Recall:

- Truth problem statement: $\boldsymbol{u(x;\mu)} \in \boldsymbol{X}$ satisfies
$$\boldsymbol{a(u(\mu),v;\mu) = f(v;\mu), \quad \forall\, v \in X,}$$
- RB problem statement: $\boldsymbol{u_N(x;\mu)} \in \boldsymbol{X_N} \subset \boldsymbol{X}$ satisfies
$$\boldsymbol{a(u_N(\mu),v;\mu) = f(v;\mu), \quad \forall\, v \in X_N.}$$

Error Residual Equation

The error, $\boldsymbol{e(\mu) \equiv u(\mu) - u_N(\mu) \in X}$, satisfies
$$\boldsymbol{a(e(\mu),v;\mu) = r_N(v;\mu), \quad \forall\, v \in X,}$$
where the residual is defined as
$$\boldsymbol{r(v;\mu) \equiv f(v;\mu) - a(u_N(\mu),v;\mu), \quad \forall\, v \in X.}$$

Prerequisites – Dual Norm of Residual

Dual Norm of Residual

Given $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}}$, the dual norm of $\boldsymbol{r}(\boldsymbol{v};\boldsymbol{\mu})$ is defined as

$$\begin{aligned} \|\boldsymbol{r}(\cdot;\boldsymbol{\mu})\|_{\boldsymbol{X}'} &\equiv \sup_{\boldsymbol{v}\in\boldsymbol{X}} \frac{\boldsymbol{r}(\boldsymbol{v};\boldsymbol{\mu})}{\|\boldsymbol{v}\|_{\boldsymbol{X}}} \\ &= \|\hat{e}(\boldsymbol{\mu})\|_{\boldsymbol{X}}, \end{aligned}$$

where $\hat{e}(\boldsymbol{\mu}) \in \boldsymbol{X}$ satisfies

$$(\hat{e}(\boldsymbol{\mu}), \boldsymbol{v})_{\boldsymbol{X}} = \boldsymbol{r}(\boldsymbol{v};\boldsymbol{\mu}), \quad \forall \boldsymbol{v} \in \boldsymbol{X}.$$

It follows that the error residual equation can also be written as

$$\boldsymbol{a}(\boldsymbol{e}(\boldsymbol{\mu}), \boldsymbol{v};\boldsymbol{\mu}) = (\hat{e}(\boldsymbol{\mu}), \boldsymbol{v})_{\boldsymbol{X}}, \quad \forall \boldsymbol{v} \in \boldsymbol{X}.$$

Energy Error Bound

We define

- the error bound: $$\Delta_N^{\mathrm{en}}(\mu) \equiv \frac{\|\hat{e}(\mu)\|_X}{\sqrt{\alpha_{\mathrm{LB}}(\mu)}}$$
- and the effectivity: $$\eta_N^{\mathrm{en}}(\mu) \equiv \frac{\Delta_N^{\mathrm{en}}(\mu)}{|||e(\mu)|||_\mu}$$

Proposition (Energy Error Bound)

For any $N = 1, \dots, N_{\max}$, the effecitivity, $\eta_N^{\mathrm{en}}(\mu)$ satisfies

$$1 \leq \eta_N^{\mathrm{en}}(\mu) \leq \sqrt{\frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)}}, \qquad \forall \mu \in \mathcal{D}.$$

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$$1 \leq \eta_N^{\text{en}}(\mu) \leq \sqrt{\frac{\gamma_{\text{UB}}(\mu)}{\alpha_{\text{LB}}(\mu)}}, \qquad \forall \mu \in \mathcal{D}.$$

Energy Error Bound

Note that effectivity measures quality of error bound

- Lower bound implies rigor

$$\mathbf{1} \leq \boldsymbol{\eta}_{\boldsymbol{N}}^{\mathbf{en}}(\boldsymbol{\mu}) \quad \Rightarrow \quad |||e(\boldsymbol{\mu})|||_{\boldsymbol{\mu}} \leq \boldsymbol{\Delta}_{\boldsymbol{N}}^{\mathbf{en}}(\boldsymbol{\mu}).$$

- Upper bound measures sharpness

$$\eta_N^{\mathrm{en}}(\mu) \leq \sqrt{\frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)}} \Rightarrow \Delta_N^{\mathrm{en}}(\mu) \leq \sqrt{\frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)}} |||e(\mu)|||_{\mu}.$$

For a parametrically coercive and symmetric

$$\theta^{\bar{\mu}}(\mu) \equiv \frac{\Theta_a^{\max,\bar{\mu}}(\mu)}{\Theta_a^{\min,\bar{\mu}}(\mu)} = \frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)}.$$

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- Lower bound implies rigor

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For a parametrically coercive and symmetric

$$\theta^{\bar{\mu}}(\mu) \equiv \frac{\Theta_a^{\max,\bar{\mu}}(\mu)}{\Theta_a^{\min,\bar{\mu}}(\mu)} = \frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)}.$$

Output Error Bound

We define

- the output error bound: $$\Delta_N^s(\mu) \equiv \frac{\|\hat{e}(\mu)\|_X^2}{\alpha_{\mathrm{LB}}(\mu)}$$
- and the output effectivity: $$\eta_N^s(\mu) \equiv \frac{\Delta_N^s(\mu)}{s(\mu) - s_N(\mu)}$$

Proposition (Output Error Bound)

For any $N = 1, \ldots, N_{\max}$, *the effecitivity,* $\eta_N^s(\mu)$ *satisfies*

$$1 \le \eta_N^s(\mu) \le \frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)}, \qquad \forall \mu \in \mathcal{D}.$$

Output Error Bound

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Proposition (Output Error Bound)

For any $N = 1, \ldots, N_{\max}$, the effecitivity, $\eta_N^s(\mu)$ satisfies

$$1 \leq \eta_N^s(\mu) \leq \frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)}, \qquad \forall \mu \in \mathcal{D}.$$

Relative Output Error Bound

We define

- the relative output error bound:

$$\Delta_N^{s,\text{rel}}(\mu) \equiv \frac{\|\hat{e}(\mu)\|_X^2}{\alpha_{\text{LB}}(\mu) s_N(\mu)} = \frac{\Delta_N^s(\mu)}{s_N(\mu)}$$

- and the relative output effectivity:

$$\eta_N^{s,\text{rel}}(\mu) \equiv \frac{\Delta_N^{s,\text{rel}}(\mu)}{(s(\mu) - s_N(\mu))/s_(\mu)}$$

Proposition (Relative Output Error Bound)

For any $N = 1, \ldots, N_{\max}$ and for $\Delta_N^{s,\text{rel}}(\mu) \leq 1$, the effectivity, $\eta_N^{s,\text{rel}}(\mu)$ satisfies

$$1 \leq \eta_N^{s,\text{rel}}(\mu) \leq 2\frac{\gamma_{\text{UB}}(\mu)}{\alpha_{\text{LB}}(\mu)}, \qquad \forall \mu \in \mathcal{D}.$$

Relative Output Error Bound

We define

- the relative output error bound:

$$\boldsymbol{\Delta}_N^{s,\mathrm{rel}}(\boldsymbol{\mu}) \equiv \frac{\|\hat{e}(\boldsymbol{\mu})\|_X^2}{\alpha_{\mathrm{LB}}(\boldsymbol{\mu}) s_N(\boldsymbol{\mu})} = \frac{\boldsymbol{\Delta}_N^s(\boldsymbol{\mu})}{s_N(\boldsymbol{\mu})}$$

- and the relative output effectivity:

$$\eta_N^{s,\mathrm{rel}}(\boldsymbol{\mu}) \equiv \frac{\boldsymbol{\Delta}_N^{s,\mathrm{rel}}(\boldsymbol{\mu})}{(s(\boldsymbol{\mu}) - s_N(\boldsymbol{\mu}))/s_(\boldsymbol{\mu})}$$

Proposition (Relative Output Error Bound)

For any $N = 1, \ldots, N_{\max}$ and for $\boldsymbol{\Delta}_N^{s,\mathrm{rel}}(\boldsymbol{\mu}) \leq 1$, the effecitivity, $\eta_N^{s,\mathrm{rel}}(\boldsymbol{\mu})$ satisfies

$$1 \leq \eta_N^{s,\mathrm{rel}}(\boldsymbol{\mu}) \leq 2\frac{\gamma_{\mathrm{UB}}(\boldsymbol{\mu})}{\alpha_{\mathrm{LB}}(\boldsymbol{\mu})}, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}.$$

X-Norm Error Bound

We define

- the error bound: $$\Delta_N(\mu) \equiv \frac{\|\hat{e}(\mu)\|_X}{\alpha_{\mathrm{LB}}(\mu)}$$
- and the effectivity: $$\eta_N(\mu) \equiv \frac{\Delta_N(\mu)}{\|e(\mu)\|_X}$$

Proposition (X-Norm Error Bound)

For any $N = 1, \ldots, N_{\max}$, the effecitivity, $\eta_N(\mu)$ satisfies

$$1 \leq \eta_N(\mu) \leq \frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)}, \qquad \forall \mu \in \mathcal{D}.$$

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Error Bounds – Remarks

Remarks:

- The error bounds are rigorous upper bounds for the reduced basis error for any $\boldsymbol{N = 1, \ldots, N_{\max}}$ and for all $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}}$.
- The upper bounds for the effectivities are
  - independent of $\boldsymbol{N}$, and
  - independent of $\boldsymbol{\mathcal{N}}$ if $\boldsymbol{\alpha_{\mathrm{LB}}(\mu)}$ only depends on $\boldsymbol{\mu}$,

  and are thus stable with respect to RB and FEM refinement.
- Results for energy norm (and $\boldsymbol{X}$-norm) bound directly extend to noncompliant (& nonsymmetric) problems
  - if we choose an appropriate definition for the energy (and $\boldsymbol{X}$) norm

Offline-Online Decomposition

Crucial ingredient: Dual norm of residual $\|\hat{e}(\mu)\|_X$

We expand $u_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\,\zeta^j$

and obtain from the definition of the residual and affine dependence

$$
\begin{aligned}
r(v;\mu) &= f(v) - a(u_N(\mu), v; \mu) \\
&= f(v) - a\Big(\sum_{n=1}^{N} u_{N\,n}(\mu)\zeta_n, v; \mu\Big) \\
&= f(v) - \sum_{n=1}^{N} u_{N\,n}(\mu) a(\zeta_n, v; \mu) \\
&= f(v) - \sum_{n=1}^{N} u_{N\,n}(\mu) \sum_{q=1}^{Q_a} \Theta_a^q(\mu) a^q(\zeta_n, v)
\end{aligned}
$$

For simplicity, we assume here that $f(v)$ does not depend on μ.

Offline-Online Decomposition

Riesz representation:

$$\begin{aligned}(\hat{e}(\mu), v)_X &= r(v;\mu) \\ &= f(v) - \sum_{q=1}^{Q_a}\sum_{n=1}^{N} \Theta_a^q(\mu) u_{N\,n}(\mu) a^q(\zeta_n, v),\end{aligned}$$

Linear Superposition:

$$\Rightarrow \quad \hat{e}(\mu) = \mathcal{C} + \sum_{q=1}^{Q_a}\sum_{n=1}^{N} \Theta_a^q(\mu) u_{N\,n}(\mu) \mathcal{A}_n^q$$

where

$$\begin{aligned}(\mathcal{C}, v)_X &= f(v), & \forall v \in X; \\ (\mathcal{A}_n^q, v)_X &= -a^q(\zeta_n, v), & \forall v \in X, \\ & & 1 \le n \le N, 1 \le q \le Q_a.\end{aligned}$$

Offline-Online Decomposition

Riesz representation:

$$\begin{aligned}(\hat{e}(\mu), v)_X &= r(v;\mu) \\ &= f(v) - \sum_{q=1}^{Q_a}\sum_{n=1}^{N} \Theta_a^q(\mu) u_{N\,n}(\mu) a^q(\zeta_n, v),\end{aligned}$$

Linear Superposition:

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Offline-Online Decomposition

Thus

$$\|\hat{e}(\mu)\|_X^2$$

$$= \left(\mathcal{C} + \sum_{q=1}^{Q_a} \sum_{n=1}^{N} \Theta_a^q(\mu) u_{N\,n}(\mu) \mathcal{A}_n^q \,,\, \cdot \right)_X$$

$$= (\mathcal{C},\mathcal{C})_X + \sum_{q=1}^{Q_a} \sum_{n=1}^{N} \Theta_a^q(\mu) u_{N\,n}(\mu) \Big\{$$

$$2(\mathcal{C},\mathcal{A}_n^q)_X + \sum_{q'=1}^{Q_a} \sum_{n'=1}^{N} \Theta_a^{q'}(\mu) u_{N\,n'}(\mu) (\mathcal{A}_n^q, \mathcal{A}_{n'}^{q'})_X \Big\}$$

Offline-Online Decomposition

Thus

$$\|\hat{e}(\mu)\|_X^2$$

$$= \left(\mathcal{C} + \sum_{q=1}^{Q_a}\sum_{n=1}^{N} \Theta_a^q(\mu) u_{N\,n}(\mu) \mathcal{A}_n^q\,,\ \cdot\ \right)_X$$

$$= (\mathcal{C},\mathcal{C})_X + \sum_{q=1}^{Q_a}\sum_{n=1}^{N} \Theta_a^q(\mu) u_{N\,n}(\mu) \Bigg\{$$

$$2(\mathcal{C},\mathcal{A}_n^q)_X + \sum_{q'=1}^{Q_a}\sum_{n'=1}^{N} \Theta_a^{q'}(\mu) u_{N\,n'}(\mu) (\mathcal{A}_n^q, \mathcal{A}_{n'}^{q'})_X \Bigg\}$$

Offline-Online Decomposition

Offline: *once, parameter independent*

- Compute $\mathcal{C}, \mathcal{A}_n^q,\ 1 \le n \le N_{\max},\ 1 \le q \le Q_a$, from

$$\begin{aligned} (\mathcal{C}, v)_X &= f(v), && \forall v \in X; \\ (\mathcal{A}_n^q, v)_X &= -a^q(\zeta_n, v), && \forall v \in X, \\ & && 1 \le n \le N, 1 \le q \le Q_a. \end{aligned}$$

- Form/Store $(\mathcal{C}, \mathcal{C})_X,\ (\mathcal{C}, \mathcal{A}_n^q)_X,\ (\mathcal{A}_n^q, \mathcal{A}_{n'}^{q'})_X,$

$$1 \le n, n' \le N_{\max}, 1 \le q, q' \le Q_a.$$

Complexity depends on N, Q_a, and $\mathcal{N}$.

Offline-Online Decomposition

Offline: *once, parameter independent*

- Compute $\mathcal{C}, \mathcal{A}_n^q,\ 1 \le n \le N_{\max},\ 1 \le q \le Q_a$, from

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- Form/Store $(\mathcal{C}, \mathcal{C})_X,\ (\mathcal{C}, \mathcal{A}_n^q)_X,\ (\mathcal{A}_n^q, \mathcal{A}_{n'}^{q'})_X,$

$$1 \le n, n' \le N_{\max}, 1 \le q, q' \le Q_a.$$

Complexity depends on N, Q_a, and $\mathcal{N}$.

Offline-Online Decomposition

Online: *many times, for each new* μ

(and associated solution $u_N(\mu)$)

- Evaluate

$$\|\hat{e}(\mu)\|_X^2$$

$$= (\mathcal{C},\mathcal{C})_X + \sum_{q=1}^{Q_a} \sum_{n=1}^{N} \Theta_a^q(\mu) u_{N\,n}(\mu) \Big\{ 2(\mathcal{C},\mathcal{A}_n^q)_X + \sum_{q'=1}^{Q_a} \sum_{n'=1}^{N} \Theta_a^{q'}(\mu) u_{N\,n'}(\mu) (\mathcal{A}_n^q, \mathcal{A}_{n'}^{q'})_X \Big\}$$

– $O(Q_a^2 N^2)$

Complexity depends on N, Q_a, but not $\mathcal{N}$.

Offline-Online Decomposition

Summary of computational cost:

OFFLINE —

$$\underbrace{O(Q_a N_{\max} \mathcal{N}^{\bullet})}_{\text{solve Poisson problems}} + \underbrace{O(Q_a^2 N_{\max}^2 \mathcal{N})}_{\text{form } \mu\text{-independent inner products}} ;$$

ONLINE —

$$\underbrace{O(Q_a^2 N^2)}_{\text{evaluate } \|\hat{e}(\mu)\|_X\text{-sum}} ;$$

Online cost is independent of $\mathcal{N}$.

Algebraic Equations

Offline: *once, parameter independent*

- Compute $\underline{\mathcal{C}} \in \mathbb{R}^{\mathcal{N}}$ and $\underline{\mathcal{A}}^q \in \mathbb{R}^{\mathcal{N}\times N}$, $1 \leq q \leq Q_a$, from

$$\begin{aligned} \underline{\mathbb{X}}^{\mathcal{N}}\underline{\mathcal{C}} &= \underline{F}^{\mathcal{N}}; \\ \underline{\mathbb{X}}^{\mathcal{N}}\underline{\mathcal{A}}^q &= -\underline{\mathbb{A}}^{\mathcal{N}q}\underline{\mathbb{Z}}_N, \quad 1 \leq q \leq Q_a. \end{aligned}$$

- Form/Store

$$\begin{aligned} \underline{\mathcal{C}}^T\underline{\mathbb{X}}^{\mathcal{N}}\underline{\mathcal{C}} &\in \mathbb{R}; \\ (\underline{\mathcal{A}}^q)^T\underline{\mathbb{X}}^{\mathcal{N}}\underline{\mathcal{C}} &\in \mathbb{R}^N, \qquad 1 \leq q \leq Q_a; \\ (\underline{\mathcal{A}}^{q'})^T\underline{\mathbb{X}}^{\mathcal{N}}\underline{\mathcal{A}}^q &\in \mathbb{R}^{N\times N}, \quad 1 \leq q, q' \leq Q_a. \end{aligned}$$

Algebraic Equations

Online: *many times, for each new* $\boldsymbol{\mu}$

(and associated solution $\underline{u}_N(\mu) \in \mathbb{R}^N$)

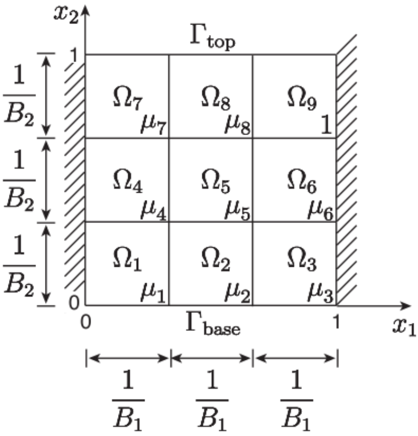
- Evaluate

$$\|\hat{e}(\mu)\|_X^2$$

$$= \underline{\mathcal{C}}^T \underline{\mathbb{X}}^{\mathcal{N}} \underline{\mathcal{C}} + \sum_{q=1}^{Q_a} \Theta_a^q(\mu)(\underline{u}_N(\mu))^T \Big\{ 2(\underline{\mathcal{A}}^q)^T \underline{\mathbb{X}}^{\mathcal{N}} \underline{\mathcal{C}} + \sum_{q'=1}^{Q_a} \Theta_a^{q'}(\mu)\, (\underline{\mathcal{A}}^{q'})^T \underline{\mathbb{X}}^{\mathcal{N}} \underline{\mathcal{A}}^q\, \underline{u}_N(\mu) \Big\}$$

– $O(Q_a^2 N^2)$

Example: ThermalBlock



$$\overline{\boldsymbol{\Omega}} = \cup_{i=1}^{B_1 B_2} \overline{\boldsymbol{\Omega}}_i$$

Example: ThermalBlock

Problem Statement

Given $\mu = (\mu_1, \ldots, \mu_P) \subset \mathcal{D} \equiv [\mu^{\min}, \mu^{\max}]^P$, evaluate

$$s(\mu) = f(u(\mu)) \qquad\qquad \ell = f$$

where $u(\mu) \in X$ satisfies

$$a(u(\mu), v; \mu) = f(v), \quad \forall\, v \in X.$$

Here, $P = B_1 B_2 - 1$, and we assume that

$$\mu^{\min} = 1/\sqrt{\mu_r},\ \mu^{\max} = \sqrt{\mu_r},\ \text{for } 1 < \mu_r < \infty,$$

such that $\mu^{\max}/\mu^{\min} = \mu_r$.

Example: ThermalBlock

Here, $\forall v \in X$, compliant $\ell = f$

$$f(v) \equiv \int_{\Gamma_{\text{base}}} v,$$

and, $\forall w, v \in X$,

$$a(w, v; \mu) = \sum_{i=1}^{P} \mu_i \int_{\Omega_i} \nabla w \cdot \nabla v + \int_{\Omega_{P+1}} \nabla w \cdot \nabla v,$$

where $\overline{\Omega} = \cup_{i=1}^{P+1} \overline{\Omega}_i$.

Example: ThermalBlock

Inner product, $w, v \in X$,

$$(w, v)_X = \sum_{i=1}^{P} \overline{\mu}_i \int_{\Omega_i} \nabla w \cdot \nabla v + \int_{\Omega_{P+1}} \nabla w \cdot \nabla v,$$

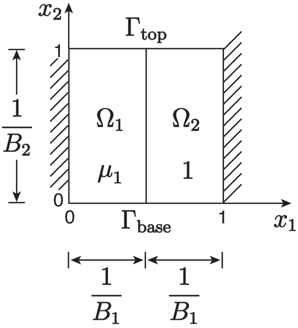
where $\overline{\mu}_i$ is the reference parameter.

The bilinear form a is

- symmetric,
- (parametrically) coercive,
  $0 < \frac{1}{\sqrt{\mu_r}} \leq \mathrm{Min}(\mu_1/\overline{\mu}_1, \ldots, \mu_P/\overline{\mu}_P, 1) \leq \alpha^{\mathrm{e}}(\mu)$,
- and continuous,
  $\gamma^{\mathrm{e}}(\mu) < \mathrm{Max}(\mu_1/\overline{\mu}_1, \ldots, \mu_P/\overline{\mu}_P, 1) \leq \sqrt{\mu_r} < \infty$.

The linear form f is bounded.

Example: ThermalBlock $P = 1$



We assume

$$1/\mu_1^{\min} = \mu_1^{\max} = \sqrt{\mu_r} = 10,$$

and choose $\mathcal{N} = 256$.

We set $\overline{\mu} = 1$ and have

$$\Theta_a^1(\mu) = \mu_1, \ \Theta_a^2(\mu) = 1.$$

Thus,

$$\Theta_a^{\min,\overline{\mu}}(\mu_1) = \operatorname{Min}(\mu_1, 1),$$
$$\Theta_a^{\max,\overline{\mu}}(\mu_1) = \operatorname{Max}(\mu_1, 1),$$

and therefore

$$\theta^{\overline{\mu}}(\mu_1) = \operatorname{Max}\left(\frac{1}{\mu_1}, \mu_1\right),$$

and

$$\theta^{\overline{\mu}}(\mu_1) \le \sqrt{\mu_r}, \quad \forall \mu_1 \in \mathcal{D}.$$

Example: ThermalBlock $P = 1$

Output error bound and effectivities

N	$\Delta^s_{N,\max}$	$\eta^s_{N,ave}$	$\eta^s_{N,\max}$
1	7.2084 E+00	2.3417	3.3305
2	4.5371 E−01	2.4858	3.6850
3	6.9652 E−04	6.2195	9.8551
4	1.3744 E−07	3.3219	7.2632
5	3.1140 E−11	6.0789	7.0453

[RHP2008]

Note that: $$\eta^s_{N,\max}(\mu_1) \le \eta^s_{\max,\mathrm{UB}} \equiv \sqrt{\mu_r} = 10.$$

Here

- Maximum output error bound: $\Delta^s_{N,\max} = \max_{\mu \in \Xi_{\mathrm{train}}} \Delta^s_N(\mu)$
- Average output effectivity: $\eta^s_{N,ave} = \frac{1}{|\Xi_{\mathrm{train}}|} \sum_{\mu \in \Xi_{\mathrm{train}}} \eta^s_N(\mu)$
- Maximum output effectivity: $\eta^s_{N,\max} = \max_{\mu \in \Xi_{\mathrm{train}}} \eta^s_N(\mu)$

Example: ThermalBlock $P = 8$

We assume $B_1 = B_2 = 3$, $\mathcal{N} = 661$

$$1/\mu_i^{\min} = \mu_i^{\max} = \sqrt{\mu_r} = 10, \quad 1 \le i \le 8,$$

and set $\overline{\mu}_i = 1, \ 1 \le i \le 8$.
We have

$$\Theta_a^i(\mu) = \mu_i, \ 1 \le i \le 8; \text{ and } \Theta_a^9(\mu) = 1.$$

Thus,

$$\Theta_a^{\min,\overline{\mu}}(\mu) = \text{Min}\,(1, \mu_1, \ldots, \mu_8)\,,$$
$$\Theta_a^{\max,\overline{\mu}}(\mu) = \text{Max}\,(1, \mu_1, \ldots, \mu_8)\,,$$

and therefore

$$\theta^{\overline{\mu}}(\mu_1) = \text{Max}\left[1, \frac{1}{\mu_1}, \ldots, \frac{1}{\mu_8}\right] \times \text{Max}\,[1, \mu_1, \ldots, \mu_8]\,,$$

and

$$\theta^{\overline{\mu}}(\mu_1) \le \mu_r, \quad \forall \mu \in \mathcal{D}.$$

Example: ThermalBlock $P = 8$

Output error bound and effectivities

N	$\Delta^s_{N,\max}$	$\eta^s_{N,ave}$	$\eta^s_{N,\max}$
10	2.2036 E+00	6.7067	31.2850
20	2.0020 E−01	7.5587	37.3024
30	1.5100 E−02	12.1138	62.2537
40	1.2000 E−03	14.4598	73.1151
50	1.0000 E−04	10.2566	57.5113
60	1.0000 E−05	8.0103	43.3108
70	2.0000 E−06	8.4598	36.5435

[RHP2008]

Note that: $\eta^s_{N,\max}(\mu_1) \le \eta^s_{\max,UB} \equiv \mu_r = 100.$