Model Order Reduction Techniques I & II

RB: Nonsymmetric Linear Parabolic Problems (Crank-Nicolson)

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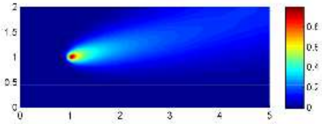
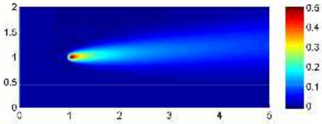
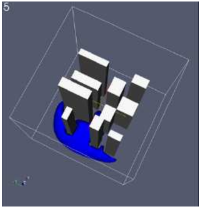
Contaminant Transport

- Application: control of emission [D]

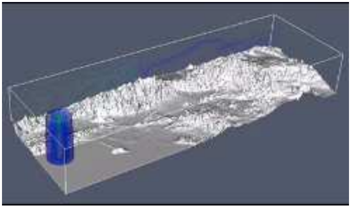




- Application: Identification of sources



Airborne contaminants
in urban canyon [B *et al.*]

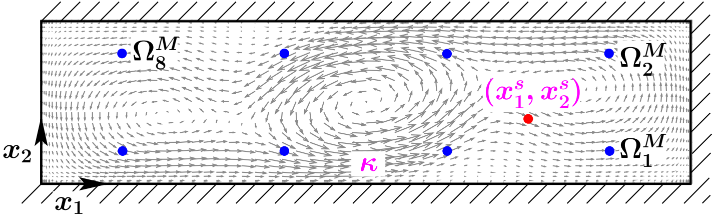


Airborne contaminants
in LA basis [H *et al.*]

Contaminant Transport

- Application: Identification of Sources

Dispersion of a pollutant $\qquad \Omega = [0,4] \times [0,1]$



Source: $g^{\mathrm{PS}}(x;\mu) = \frac{50}{\pi}\, e^{-50((x_1 - x_1^s)^2 + (x_2 - x_2^s)^2)}$

(say, $\mu \equiv (\kappa, x_1^s, x_2^s)$)

Contaminant Transport – Problem Statement

Scalar Convection-Diffusion $\quad y(x, t = 0; \mu) = 0$

$$\frac{\partial}{\partial t} y(t;\mu) + \mathrm{U} \cdot \nabla y(t;\mu) = \kappa \nabla^2 y(t;\mu) + g^{\mathrm{PS}}(x;\mu)\, u(t),$$

INPUTS: $\mu \equiv \kappa \in \mathcal{D} = [0.01, 1.0] \subset \mathbb{R}^{P=1}$;

$(x_1^s, x_2^s) = (3.0, 0.4)$;

$\mathrm{U}(\mathrm{Gr} = 10^5)$ from $\mathrm{Pr} = 0$

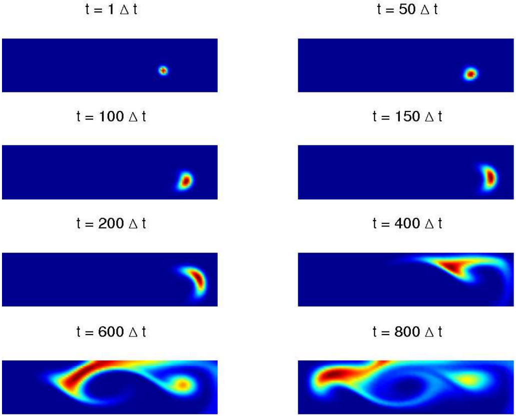
Natural Convection (Navier-Stokes);

$u(t)$ “control” input (source strength).

OUTPUTS: Measurements $s_q(t;\mu)$, $1 \le q \le 8$.

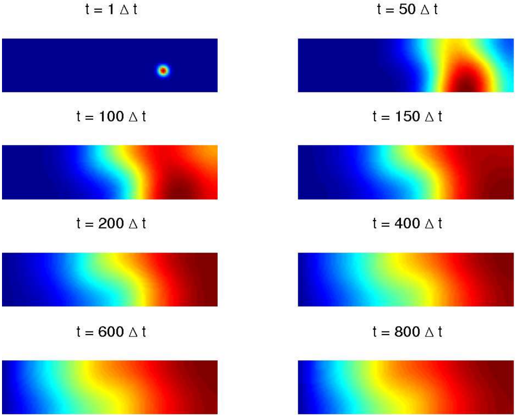
Contaminant Transport – Sample Solutions

Field variable: $\mu = 0.01$ $(\mathcal{N} = 3720)$



Contaminant Transport – Sample Solutions

Field variable: $\mu = 1.00$ $(\mathcal{N} = 3720)$



Application: Inverse Problem

Determine: $\mu^* \in \mathcal{D}$ (actual value)

Given experimental data

measurements : $z(t^k) \in \mathcal{Z}^k_{\text{exp}}, \; \forall k \in \mathbb{K}_{\text{exp}}$, where

$$\mathcal{Z}^k_{\text{exp}} \equiv [s^{\mathcal{N}}(t^k;\mu^*) - \epsilon_{\text{exp}}, s^{\mathcal{N}}(t^k;\mu^*) + \epsilon_{\text{exp}}]$$

observations : $\mathbb{K}_{\text{exp}} \subset \mathbb{K}$

error : $\epsilon_{\text{exp}} \in \mathbb{R}$ (bounded, “white”)

input : $u(t^k) = \delta_{1k}, \; \forall k \in \mathbb{K}$

Application: Inverse Problem – (Regularized) Solution

Given noisy measurements, $z(t^k)$, $k \in \mathbb{K}_{\rm exp}$, solve

- Output least squares problem

$$\hat{\mu} = \arg\min_{\mu \in \mathcal{D}} \frac{1}{2} \sum_{k=1}^{\mathbb{K}_{\rm exp}} \|s^{\mathcal{N}}(t^k;\mu) - z(t^k)\|_W^2$$

s.t. $\mathrm{PDE}_{\mathcal{N}}(\mu)$ being satisfied; or

- Regularized problem

$$\hat{\mu} = \arg\min_{\mu \in \mathcal{D}} \frac{1}{2} \sum_{k=1}^{\mathbb{K}_{\rm exp}} \|s^{\mathcal{N}}(t^k;\mu) - z(t^k)\|_W^2 + \frac{1}{2}\delta_R R(\mu)$$

s.t. $\mathrm{PDE}_{\mathcal{N}}(\mu)$ being satisfied.

Application: Inverse Problem – (Regularized) Solution

Given noisy measurements, $z(t^k)$, $k \in \mathbb{K}_{\text{exp}}$, solve

- Output least squares problem

$$\hat{\mu} = \arg\min_{\mu \in \mathcal{D}} \frac{1}{2} \sum_{k=1}^{\mathbb{K}_{\text{exp}}} \| s^{\mathcal{N}}(t^k; \mu) - z(t^k) \|_W^2$$

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s.t. $\text{PDE}_{\mathcal{N}}(\mu)$ being satisfied.

$\Rightarrow$ Solution very expensive: $\mathcal{N}$-dependent cost

Application: Inverse Problem – (Regularized) Solution

Given noisy measurements, $z(t^k)$, $k \in \mathbb{K}_{\rm exp}$, solve

- Output least squares problem

$$\hat{\mu} = \arg\min_{\mu \in \mathcal{D}} \frac{1}{2} \sum_{k=1}^{\mathbb{K}_{\rm exp}} \| s_N(t^k;\mu) - z(t^k) \|_W^2$$

s.t. $\mathrm{PDE}_N(\mu)$ being satisfied; or

- Regularized problem

$$\hat{\mu} = \arg\min_{\mu \in \mathcal{D}} \frac{1}{2} \sum_{k=1}^{\mathbb{K}_{\rm exp}} \| s_N(t^k;\mu) - z(t^k) \|_W^2 + \frac{1}{2}\delta_R R(\mu)$$

s.t. $\mathrm{PDE}_N(\mu)$ being satisfied.

$\Rightarrow$ Surrogate model approach: N-dependent cost

Problem Statement

Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$, evaluate $\quad t \in (0, t_f]$

$$s^{\mathrm{e}}(t;\mu) = \ell(u^{\mathrm{e}}(x;t;\mu);\mu)$$

where $u^{\mathrm{e}}(x;t;\mu) \in L^2(0,T;X^{\mathrm{e}}(\Omega))$ satisfies $\quad u_0 = 0$

$$m\left(\frac{\partial u^{\mathrm{e}}}{\partial t}(x;t;\mu), v;\mu\right) + a^{\mathrm{CD}}(u^{\mathrm{e}}(x;t;\mu), v;\mu) = f(v;\mu)\,g(t), \quad \forall\, v \in X^{\mathrm{e}}.$$

Classical example: unsteady convection diffusion equation

Hypotheses

Linear forms and functions

$f(\cdot;\mu)$: linear, affine in μ,
X^{e}-bounded, $\forall \mu \in \mathcal{D}$

$g(\cdot)$: $L^2(0, t_f)$ "control" input

$\ell(\cdot;\mu)$: linear, affine in μ,
$L^2(\Omega)$-bounded, $\forall \mu \in \mathcal{D}$

Hypotheses

$a^{\mathrm{CD}}(\cdot,\cdot;\mu)$: bilinear, affine in μ,
non-symmetric,
X^{e}-continuous,
X^{e}-coercive form, $\forall \mu \in \mathcal{D}$;

$m(\cdot,\cdot;\mu)$: bilinear, affine in μ,
symmetric,
$L^2(\Omega)$-continuous,
$L^2(\Omega)$-coercive form, $\forall \mu \in \mathcal{D}$;

Recall that we can write, $\forall w, v \in X^{\mathrm{e}}$,

$$a^{\mathrm{CD}}(w,v;\mu) = \underbrace{a_S(w,v;\mu)}_{\text{symmetric}} + \underbrace{a_{SS}(w,v;\mu)}_{\text{skew-symmetric}}$$

Hypotheses

Require affine parameter dependence also $\ell(v;\mu)$, $f(v;\mu)$

$$a^{\mathrm{CD}}(w,v;\mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\, a^{\mathrm{CD},q}(w,v),$$

$$m(w,v;\mu) = \sum_{q=1}^{Q_m} \Theta_m^q(\mu)\, m^q(w,v);$$

where

$\Theta_{a,m}^q : \mathcal{D} \to \mathbb{R}$, μ-dependent functions;
representing coefficients, geometry, . . .

$a^{\mathrm{CD},q}$ and m^q μ-independent forms.

Convection-Diffusion

Bilinear form a^{CD} can be written as

$$a^{\mathrm{CD}}(w,v;\mu) = \underbrace{a^{\mathrm{D}}(w,v;\mu)}_{\text{DIFFUSION}} + \underbrace{a^{\mathrm{C}}(w,v;\mu)}_{\text{CONVECTION}}$$

where

$$\begin{aligned} a^{\mathrm{D}}(w,v;\mu) &= \mu \textstyle\int_\Omega \nabla w \cdot \nabla v \\ a^{\mathrm{C}}(w,v;\mathrm{U}) &= \textstyle\int_\Omega v(\mathrm{U}\cdot\nabla w) + \frac{1}{2}\int_\Omega vw\,(\nabla\cdot\mathrm{U}) \end{aligned}$$

Conservation Form: Require $\nabla \cdot \mathrm{U}^{\mathrm{e}} = 0$, then a^{C} is skew-symmetric (although $\nabla \cdot \mathrm{U} \neq 0$ pointwise) for

(i) a contained flow, i.e., $\mathbf{n} \cdot \mathrm{U} = 0$ on Γ;

(i) a flow with Dirichlet boundary conditions.

Truth Approximation

- Spatial Discretization: Finite Element

  $$X^{\mathcal{N}} \subset X \text{ with } \dim(X^{\mathcal{N}}) = \mathcal{N}$$

  for given $\mathcal{N}$ ($\mathcal{N} \to \infty$).

  Also possible: Finite Volume [HO]

- Temporal Discretization: Finite Difference

  $$t^k = k\,\Delta t,\ \forall k \in \mathbb{K} \equiv \{(0), 1, 2, \ldots, K\}$$

  for given $\Delta t = t_f / K$ (fixed).

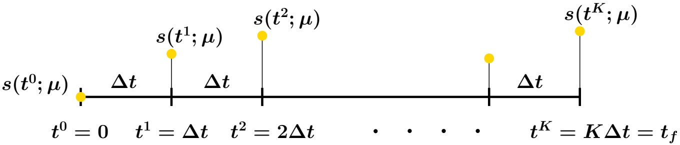
  Also possible: DG [RMM]

Truth Approximation

- Temporal Discretization: Finite Difference

$$\frac{\partial u}{\partial t}(t^k;\mu) \approx \frac{u(t^k;\mu) - u(t^{k-1};\mu)}{\Delta t}$$

  - Euler Backward
  - Crank-Nicolson (advection-dominated problems)



Problem Statement

Given $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}} \subset \mathbb{R}^{P}$, evaluate $\forall \boldsymbol{k} \in \mathbb{K}$

$$s^k(\mu) = \ell(u^k(\mu); \mu)$$

where $u^k(\mu) \in X$ satisfies $u_0 = 0$

$$m\left(\frac{u^k(\mu) - u^{k-1}(\mu)}{\Delta t}, v; \mu\right) + a^{\mathrm{CD}}\left(\frac{u^k(\mu) + u^{k-1}(\mu)}{2}, v; \mu\right) = f(v;\mu)\,\frac{1}{2}\,\left(g(t^k) + g(t^{k-1})\right), \quad \forall\, v \in X.$$

Reduced Basis Space

We define the Lagrangian RB space

$$X_N = \mathrm{span}\{\zeta^n,\ 1 \le n \le N\}, \quad 1 \le N \le N_{\max},$$

with mutually $(\cdot,\cdot)_X$-orthonormal basis functions

$$\zeta^n \in X, \quad 1 \le n \le N_{\max}.$$

We thus obtain

$$X_N \subset X, \quad \dim(X_N) = N, \quad 1 \le N \le N_{\max},$$

and hierarchical spaces

$$X_1 \subset X_2 \subset \ldots \subset X_{N_{\max}-1} \subset X_{N_{\max}} (\subset X).$$

The basis functions are constructed using the POD(t)-Greedy(μ) algorithm.

Galerkin Projection

Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$, evaluate $\quad \forall k \in \mathbb{K}$

$$s_N^k(\mu) = \ell(u_N^k(\mu); \mu)$$

where $u_N^k(\mu) \in X_N$ satisfies $\quad u_{N,0} = 0$

$$m\left(\frac{u_N^k(\mu) - u_N^{k-1}(\mu)}{\Delta t}, v; \mu\right) + a^{\mathrm{CD}}\left(\frac{u_N^k(\mu) + u_N^{k-1}(\mu)}{2}, v; \mu\right) = f(v; \mu)\,\frac{1}{2}\,\left(g(t^k) + g(t^{k-1})\right), \quad \forall\, v \in X_N.$$

- RB inherits fixed truth temporal discretization.
- Offline-online decomposition follows directly from symmetric case.

Energy Norm

Crank-Nicolson

- "Spatio-temporal" energy norm (parameter-dependent)

$$
\begin{aligned}
(((w^k, v^k)))_{\mathrm{CN}} &= m(w^k, v^k; \mu) \\
&\quad + \sum_{k'=1}^{k} \Delta t \, a_S\left(\frac{w^{k'}+w^{k'-1}}{2}, \frac{v^{k'}+v^{k'-1}}{2}; \mu\right),
\end{aligned}
$$

$$
\begin{aligned}
|||w^k|||_{\mathrm{CN}} &= \Bigg(m(w^k, w^k; \mu) \\
&\quad + \sum_{k'=1}^{k} \Delta t \, a_S\left(\frac{w^{k'}+w^{k'-1}}{2}, \frac{w^{k'}+w^{k'-1}}{2}; \mu\right)\Bigg)^{1/2}, \\
&\qquad 1 \le k \le K.
\end{aligned}
$$

Coercivity and Coercivity Lower Bound

We also define

- Coercivity constant

$$\alpha(\mu) \equiv \inf_{w \in X} \frac{a_S(w, w; \mu)}{\|w\|_X^2};$$

- Positive lower bound, $\alpha_{\mathrm{LB}} : \mathcal{D} \to \mathbb{R}$, such that

$$0 < \alpha_{\mathrm{LB}}(\mu) \leq \alpha(\mu), \quad \forall \mu \in \mathcal{D}.$$

This bound can be calculated using the

- "$\min \Theta$" Approach (if a is parametrically coercive), or
- Successive Constraint Method

exactly as in elliptic case.

Dual Norm of Residual

We define the residual, $\forall k \in \mathbb{K}$,

$$r^k(v;\mu) \equiv f(v;\mu)\frac{1}{2}(g(t^k)+g(t^{k-1})) - m\left(\frac{u_N^k(\mu)-u_N^{k-1}(\mu)}{\Delta t}, v;\mu\right) - a^{\mathrm{CD}}\left(\frac{u_N^k(\mu)+u_N^{k-1}(\mu)}{2}, v;\mu\right), \quad \forall\, v \in X$$

Dual Norm of Residual

Given $\mu \in \mathcal{D}$, the dual norm of $r^k(v;\mu)$ is defined as

$$\|r^k(\cdot;\mu)\|_{X'} \equiv \sup_{v\in X} \frac{r^k(v;\mu)}{\|v\|_X} = \|\hat{e}^k(\mu)\|_X,$$

where $\hat{e}^k(\mu) \in X$ satisfies

$$(\hat{e}^k(\mu), v)_X = r^k(v;\mu), \quad \forall v \in X.$$

Energy Error Bound

We define the error bound, $\boldsymbol{\Delta}_N^k(\boldsymbol{\mu}) = \boldsymbol{\Delta}_N(\boldsymbol{t}^k; \boldsymbol{\mu})$, $\mathbf{1} \leq \boldsymbol{k} \leq \boldsymbol{K}$, as

$$\boldsymbol{\Delta}_N^k(\boldsymbol{\mu}) = \alpha_{\mathrm{LB}}^{-1/2}(\boldsymbol{\mu}) \left(\sum_{k'=1}^{k} \boldsymbol{\Delta t} \, \| \hat{e}^{k'}(\boldsymbol{\mu}) \|_X^2 \right)^{1/2} .$$

We can then prove

Proposition (Energy Error Bound)

For any $N = 1, \ldots, N_{\max}$, the error in the field variable, $e^k(\mu) = u^k(\mu) - u_N^k(\mu)$, is bounded by

$$|||e^k(\mu)|||_{\mathrm{CN}} \leq \Delta_N^k(\mu), \quad \forall\, \mu \in \mathcal{D}, \; \forall\, k \in \mathbb{K}.$$

Energy Error Bound

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We can then prove

Proposition (Energy Error Bound)

For any $N = 1, \ldots, N_{\max}$, *the error in the field variable,* $e^k(\boldsymbol{\mu}) = u^k(\boldsymbol{\mu}) - u_N^k(\boldsymbol{\mu})$, *is bounded by*

$$|||e^k(\boldsymbol{\mu})|||_{\mathrm{CN}} \leq \boldsymbol{\Delta}_N^k(\boldsymbol{\mu}), \quad \forall\, \boldsymbol{\mu} \in \mathcal{D}, \; \forall\, k \in \mathbb{K}.$$

Simple Output Error Bound

We define the output error bound, $\Delta_N^{s\,k}(\mu) = \Delta_N^s(t^k;\mu)$, $1 \leq k \leq K$, as

.

$$\Delta_N^{s\,k}(\mu) \equiv \sigma_{\mathrm{LB}}^{-1}(\mu) \left(\sup_{v \in X} \frac{\ell(v;\mu)}{\|v\|} \right) \Delta_N^k(\mu)$$

Proposition (Simple Output Error Bound)

For any $N = 1, \ldots, N_{\max}$, the error in the output is bounded by

$$|s^k(\mu) - s_N^k(\mu)| \leq \Delta_N^{s\,k}(\mu), \quad \forall \mu \in \mathcal{D},\ \forall k \in \mathbb{K}.$$

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Proposition (Simple Output Error Bound)

For any $N = 1, \ldots, N_{\max}$*, the error in the output is bounded by*

$$|s^k(\mu) - s_N^k(\mu)| \le \Delta_N^{s\,k}(\mu), \quad \forall \mu \in \mathcal{D},\ \forall k \in \mathbb{K}.$$

Dual Problem – Truth Approximation

Given $\mu \in \mathcal{D}$, the dual variable $\Psi^k(\mu) \in X$, $1 \le k \le K$, satisfies

$$m\left(v, \frac{\Psi^k(\mu) - \Psi^{k+1}(\mu)}{\Delta t}; \mu\right) - a\left(v, \frac{\Psi^k(\mu) + \Psi^{k+1}(\mu)}{2}; \mu\right) = 0, \quad \forall v \in X,$$

with final condition

$$m(v, \Psi(t^{K+1}; \mu); \mu) \equiv l(v; \mu), \quad \forall v \in X.$$

- Dual problem inherits spatial and temporal discretization from primal problem.

Dual Problem – Galerkin Projection

Given $\mu \in \mathcal{D}$, the dual variable $\Psi_N^k(\mu) \in X_{N_{\mathrm{du}}}^{\mathrm{du}}$, $1 \leq k \leq K$, satisfies

$$m\left(v, \frac{\Psi_N^k(\mu) - \Psi_N^{k+1}(\mu)}{\Delta t}; \mu\right) - a\left(v, \frac{\Psi_N^k(\mu) + \Psi_N^{k+1}(\mu)}{2}; \mu\right) = 0, \quad \forall v \in X_{N_{\mathrm{du}}}^{\mathrm{du}},$$

with final condition

$$m(v, \Psi_N(t^{K+1}; \mu); \mu) \equiv l(v; \mu), \quad \forall v \in X_{N_{\mathrm{du}}}^{\mathrm{du}}.$$

- $X_{N_{\mathrm{du}}}^{\mathrm{du}}$ constructed using POD(t)-Greedy(μ) sampling procedure.
- Offline-online decomposition follows from symmetric case.

Primal Problem – Galerkin Projection

Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$ and $\Psi_N^k(\mu)$, evaluate $\qquad \forall k \in \mathbb{K}$

$$s_N^k(\mu) = \ell(u_N^k(\mu); \mu)$$
$$+ \sum_{k'=1}^{k} r^{k'} \left(\frac{1}{2} \left(\Psi_N^{K-k+k'}(\mu) + \Psi_N^{K-k+k'+1}(\mu) \right); \mu \right) \Delta t$$

where $u_N^k(\mu) \in X_N^{(\mathrm{pr})}$ satisfies $\qquad u_{N,0} = 0$

$$m\left(\frac{u_N^k(\mu) - u_N^{k-1}(\mu)}{\Delta t}, v; \mu \right) + a^{\mathrm{CD}}\left(\frac{u_N^k(\mu) + u_N^{k-1}(\mu)}{2}, v; \mu \right)$$
$$= f(v;\mu)\, \frac{1}{2}\, \left(g(t^k) + g(t^{k-1}) \right), \quad \forall\, v \in X_N.$$

$\Rightarrow X_N = X_N^{\mathrm{pr}}$ and $r^k(v;\mu)$ is the primal residual.

Inner Products and Norms

Crank-Nicolson

- "Spatio-temporal" energy norm (parameter-dependent)

$$(((w^k, v^k)))^{\mathrm{du}}_{\mathrm{CN}} = m(w^k, v^k; \mu) + \sum_{k'=k}^{K} \Delta t \, a_S\left(\frac{w^{k'} + w^{k'+1}}{2}, \frac{v^{k'} + v^{k'+1}}{2}; \mu\right),$$

$$|||w^k|||^{\mathrm{du}}_{\mathrm{CN}} = \left(m(w^k, w^k; \mu) + \sum_{k'=k}^{K} \Delta t \, a_S\left(\frac{w^{k'} + w^{k'+1}}{2}, \frac{w^{k'} + w^{k'+1}}{2}; \mu\right)\right)^{1/2}, \quad 1 \le k \le K.$$

Dual Final Condition

If m or ℓ are parameter-dependent, we define the residual

$$r^{\Psi_f}(v;\mu) \equiv \ell(v;\mu) - m(v, \Psi_N^{K+1};\mu), \quad \forall\, v \in X.$$

Lemma (Dual Error Bound – Final Condition)

Given $\mu \in \mathcal{D}$, the error $e^{\mathrm{du}}(t^{K+1};\mu) = \Psi^{K+1}(\mu) - \Psi_N^{K+1}(\mu)$ is bounded by

$$\|e^{\mathrm{du}}(t^{K+1};\mu)\| \le \Delta_N^{\Psi_f}(\mu) \equiv \frac{\varepsilon_N^{\Psi_f}(\mu)}{\sigma_{\mathrm{LB}}(\mu)}$$

where

$$\varepsilon_N^{\Psi_f}(\mu) \equiv \sup_{v\in X} \frac{r^{\Psi_f}(v;\mu)}{\|v\|}$$

Dual Norm of Dual Residual

We define the residual, $\forall k \in \mathbb{K}$,

$$r^{\mathrm{du},k}(v;\mu) \equiv -m\left(v, \frac{\Psi_N^k(\mu)-\Psi_N^{k+1}(\mu)}{\Delta t}; \mu\right) - a\left(v, \frac{\Psi_N^k(\mu)+\Psi_N^{k+1}(\mu)}{2}; \mu\right), \ \forall\, v \in X$$

Dual Norm of Residual

Given $\mu \in \mathcal{D}$, the dual norm of $r^{\mathrm{du},k}(v;\mu)$ is defined as

$$\|r^{\mathrm{du},k}(\cdot;\mu)\|_{X'} \equiv \sup_{v\in X} \frac{r^{\mathrm{du},k}(v;\mu)}{\|v\|_X} = \|\hat{e}^{\mathrm{du},k}(\mu)\|_X,$$

where $\hat{e}^{\mathrm{du},k}(\mu) \in X$ satisfies

$$(\hat{e}^{\mathrm{du},k}(\mu), v)_X = r^{\mathrm{du},k}(v;\mu), \quad \forall v \in X.$$

Energy Error Bound – Dual

We define the error bound, $\Delta_N^{\mathrm{du},k}(\mu)$, $1 \le k \le K$, as

$$\Delta_N^{\mathrm{du},k}(\mu) = \left(\frac{\Delta t}{\alpha_{\mathrm{LB}}(\mu)} \sum_{k'=1}^{k} \|\hat{e}^{\mathrm{du},k'}(\mu)\|_X^2 + \sigma_{\mathrm{LB}}(\mu) \Delta_N^{\Psi_f}(\mu)^2 \right)^{1/2}.$$

We can then prove

Proposition (Energy Error Bound)

For any $N = 1, \ldots, N_{\mathrm{du,max}}$, the error in the dual variable, $e^{\mathrm{du},k}(\mu) = \Psi^k(\mu) - \Psi_N^k(\mu)$, is bounded by

$$|||e^{\mathrm{du},k}(\mu)|||_{\mathrm{CN}}^{\mathrm{du}} \le \Delta_N^{\mathrm{du},k}(\mu), \quad \forall\, \mu \in \mathcal{D},\ \forall\, k \in \mathbb{K}.$$

Energy Error Bound – Dual

We define the error bound, $\Delta_N^{\mathrm{du},k}(\mu)$, $1 \le k \le K$, as

$$\Delta_N^{\mathrm{du},k}(\mu) = \left(\frac{\Delta t}{\alpha_{\mathrm{LB}}(\mu)} \sum_{k'=1}^{k} \|\hat{e}^{\mathrm{du},k'}(\mu)\|_X^2 + \sigma_{\mathrm{LB}}(\mu) \Delta_N^{\Psi_f}(\mu)^2 \right)^{1/2} .$$

We can then prove

Proposition (Energy Error Bound)

For any $N = 1, \ldots, N_{\mathrm{du,max}}$*, the error in the dual variable,* $e^{\mathrm{du},k}(\mu) = \Psi^k(\mu) - \Psi_N^k(\mu)$*, is bounded by*

$$|||e^{\mathrm{du},k}(\mu)|||_{\mathrm{CN}}^{\mathrm{du}} \le \Delta_N^{\mathrm{du},k}(\mu), \quad \forall\, \mu \in \mathcal{D},\ \forall\, k \in \mathbb{K}.$$

Output Error Bound – Primal-Dual Formulation

We (re-)define the output error bound, $\Delta_N^{s\,k}(\mu)$, $1 \le k \le K$, as

.

$$\Delta_N^{s\,k}(\mu) \equiv \Delta_{N_{\mathrm{pr}}}^{\mathrm{pr},k}(\mu)\;\Delta_{N_{\mathrm{du}}}^{\mathrm{du},K-k+1}(\mu)$$

Proposition (Simple Output Error Bound)

For any $N = 1, \ldots, N_{\max}$, *the error in the output is bounded by*

$$|s^k(\mu) - s_N^k(\mu)| \le \Delta_N^{s\,k}(\mu), \quad \forall \mu \in \mathcal{D},\ \forall k \in \mathbb{K}.$$

- Offline-online decomposition same as for Euler Backward
- Computational cost same as for Euler Backward

Output Error Bound – Primal-Dual Formulation

We (re-)define the output error bound, $\Delta_N^{s\,k}(\mu)$, $1 \leq k \leq K$, as

.

$$\Delta_N^{s\,k}(\mu) \equiv \Delta_{N_{\mathrm{pr}}}^{\mathrm{pr},k}(\mu)\; \Delta_{N_{\mathrm{du}}}^{\mathrm{du},K-k+1}(\mu)$$

Proposition (Simple Output Error Bound)

For any $N = 1, \ldots, N_{\max}$, *the error in the output is bounded by*

$$|s^k(\mu) - s_N^k(\mu)| \leq \Delta_N^{s\,k}(\mu), \quad \forall \mu \in \mathcal{D},\ \forall k \in \mathbb{K}.$$

- Offline-online decomposition same as for Euler Backward
- Computational cost same as for Euler Backward

Output Error Bound – Primal-Dual Formulation

We (re-)define the output error bound, $\boldsymbol{\Delta}_N^{s\,k}(\boldsymbol{\mu})$, $\mathbf{1} \leq \boldsymbol{k} \leq \boldsymbol{K}$, as

.

$$\boldsymbol{\Delta}_N^{s\,k}(\boldsymbol{\mu}) \equiv \boldsymbol{\Delta}_{N_{\mathrm{pr}}}^{\mathrm{pr},k}(\boldsymbol{\mu})\ \boldsymbol{\Delta}_{N_{\mathrm{du}}}^{\mathrm{du},K-k+1}(\boldsymbol{\mu})$$

Proposition (Simple Output Error Bound)

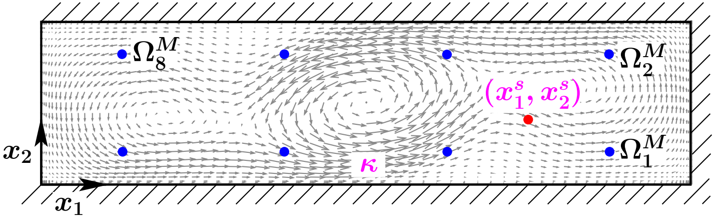
For any $\boldsymbol{N} = \mathbf{1}, \ldots, \boldsymbol{N}_{\mathbf{max}}$, *the error in the output is bounded by*

$$|\boldsymbol{s}^k(\boldsymbol{\mu}) - \boldsymbol{s}_N^k(\boldsymbol{\mu})| \leq \boldsymbol{\Delta}_N^{s\,k}(\boldsymbol{\mu}), \quad \forall \boldsymbol{\mu} \in \boldsymbol{\mathcal{D}},\ \forall \boldsymbol{k} \in \mathbb{K}.$$

- Offline-online decomposition same as for Euler Backward
- Computational cost same as for Euler Backward

Contaminant Transport – Problem Statement

Dispersion of a pollutant $\qquad \Omega = [0,4] \times [0,1]$



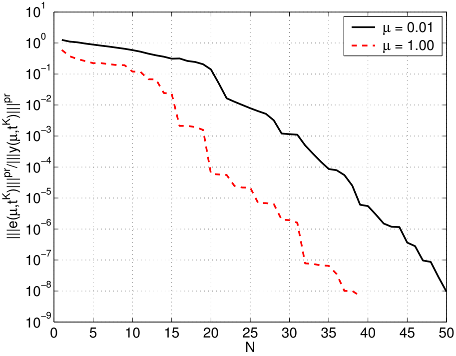
Source: $g^{\mathrm{PS}}(x;\mu) = \frac{50}{\pi}\, e^{-50((x_1 - x_1^s)^2 + (x_2 - x_2^s)^2)}$

(say, $\mu \equiv (\kappa, x_1^s, x_2^s)$)

Truth: $\mathcal{N} = 3720$, $K = 800$

Dependence on conductivity

Convergence Rate for diffusivities $\mu = 0.01$ and $\mu = 1.00$



Primal: Energy Norm

Primal problem: convergence energy norm error and bound

$N_{\rm pr}$	$\epsilon^{\rm pr}_{\rm max,rel}$	$\Delta^{\rm pr}_{\rm max,rel}$	$\overline{\eta}^{\rm pr}$
20	3.96 E−01	2.45 E+00	5.20
40	1.39 E−01	5.04 E−01	2.77
60	4.60 E−02	1.08 E−01	2.17
80	1.95 E−02	3.60 E−02	2.03
100	1.17 E−02	1.46 E−02	1.85
120	3.03 E−03	3.91 E−03	1.74
140	7.61 E−04	1.37 E−03	1.66
160	6.01 E−04	7.17 E−04	1.62
180	1.74 E−04	2.88 E−04	1.58
200	9.59 E−05	1.33 E−04	1.60

Dual: Energy Norm

Dual problem: convergence energy norm error & bound output 1

N_{du}	$\epsilon^{\text{du}}_{\text{max,rel}}$	$\Delta^{\text{du}}_{\text{max,rel}}$	$\overline{\eta}^{\text{du}}$
20	4.23 E−01	1.85 E+00	4.58
40	1.75 E−01	7.51 E−01	3.93
60	7.66 E−02	1.55 E−01	2.43
80	3.43 E−02	4.38 E−02	2.06
100	1.22 E−02	1.54 E−02	1.90
120	4.54 E−03	6.35 E−03	1.82
140	1.53 E−03	2.04 E−03	1.73
160	1.10 E−03	1.31 E−03	1.69
180	3.77 E−04	5.39 E−04	1.62
200	1.97 E−04	2.25 E−04	1.62

Primal-only: Output

Primal-only formulation: convergence output error and bound

$N_{\rm pr}$	$\epsilon^{\hat{s}}_{\rm max,rel}$	$\Delta^{\hat{s}}_{\rm max,rel}$	$\overline{\eta}^{\hat{s}}$
20	2.84 E−01	9.86 E+00	41.8
40	4.62 E−02	2.03 E+00	41.1
60	1.13 E−02	4.35 E−01	55.2
80	2.46 E−03	1.45 E−01	48.9
100	1.69 E−03	5.86 E−02	42.3
120	2.52 E−04	1.57 E−02	78.5
140	1.40 E−04	5.49 E−03	107
160	3.43 E−05	2.88 E−03	146
180	1.50 E−05	1.16 E−03	113
200	1.11 E−05	5.37 E−04	148

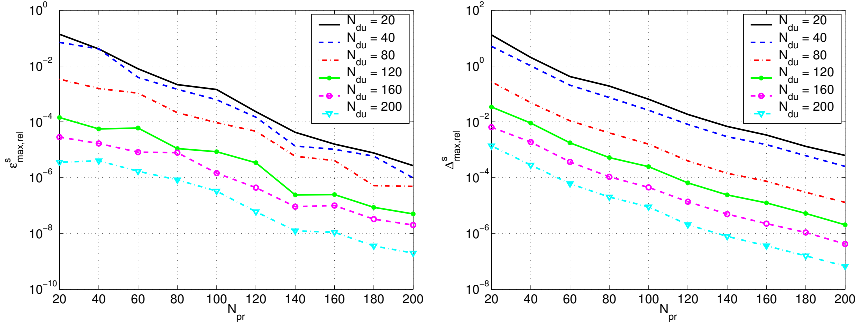
Primal-Dual: Output

Primal-dual formulation: convergence output bound ($N_{\mathbf{pr}} = N_{\mathbf{du}}$)

$N_{\mathbf{pr}}$	$N_{\mathbf{du}}$	$\epsilon^s_{\mathbf{max,rel}}$	$\Delta^s_{\mathbf{max,rel}}$	$\overline{\eta}^s$
20	20	1.37 E−01	1.30 E+01	1908
40	40	4.09 E−02	1.02 E+00	150
60	60	3.07 E−03	4.62 E−02	182
80	80	2.12 E−04	4.01 E−03	185
100	100	3.87 E−05	6.37 E−04	229
120	120	3.40 E−06	6.49 E−05	119
140	140	1.22 E−07	7.51 E−06	150
160	160	9.99 E−08	2.24 E−06	175
180	180	4.55 E−09	3.71 E−07	140
200	200	1.98 E−09	6.72 E−08	110

Primal-Dual: Output

Primal-dual formulation: convergence output error and bound



Primal-Dual: Online RB vs. Truth

Primal-dual formulation: online compuational times

$N_{\text{pr}} = N_{\text{du}}$	$s_N(\mu, t^k)$	$\Delta^s_N(\mu, t^k)$	$s(\mu, t^k)$
20	4.33 E–02	2.10 E–03	1
40	5.22 E–02	2.95 E–03	1
60	5.70 E–02	4.00 E–03	1
80	6.27 E–02	7.86 E–03	1
100	6.78 E–02	1.63 E–02	1
120	7.78 E–02	2.63 E–02	1
140	8.73 E–02	3.53 E–02	1
160	9.44 E–02	4.56 E–02	1

Output & Bound for $1 \leq k \leq K$

Savings with respect to truth: ≈ 16

Primal-Dual: Online RB vs. Truth

Primal-dual formulation: online compuational times

$N_{\mathrm{pr}} = N_{\mathrm{du}}$	$s_N(\mu, t^k)$	$\Delta_N^s(\mu, t^k)$	$s(\mu, t^k)$
20	5.34 E–03	2.10 E–03	1
40	7.00 E–03	2.95 E–03	1
60	8.53 E–03	4.00 E–03	1
80	1.03 E–02	7.86 E–03	1
100	1.35 E–02	1.63 E–02	1
120	1.96 E–02	2.63 E–02	1
140	2.73 E–02	3.53 E–02	1
160	3.36 E–02	4.56 E–02	1

Output & Bound for every tenth timestep $k = [10, 20, \ldots, K]$

Savings with respect to truth: ≈ 80

Primal-only: Online RB vs. Truth

Primal-only formulation: online compuational times

$N_{\rm pr}$	$\hat{s}_N(\mu, t^k)$	$\hat{\Delta}_N^s(\mu, t^k)$	$s(\mu, t^k)$
20	8.63E−04	9.30E−04	1
40	1.72E−03	1.13E−03	1
60	2.82E−03	1.28E−03	1
80	4.13E−03	2.10E−03	1
100	7.02E−03	5.38E−03	1
120	1.23E−02	7.91E−03	1
140	1.69E−02	1.00E−02	1
160	2.10E−02	1.30E−02	1

Output & Bound for every timestep $1 \leq k \leq K$

Savings with respect to truth: ≈ 50

Outlook

- Reduced basis methods have been extended to other classes of time-dependent problems
  - Hyperbolic problems,
  - Nonlinear Reaction-Diffusion Equations,
  - Navier-Stokes [KNP] (and viscous Burgers Equation [NRP]),
  - Fokker-Planck Equation [KP].
- The methodology directly extends to (non-parametric) dynamic systems.