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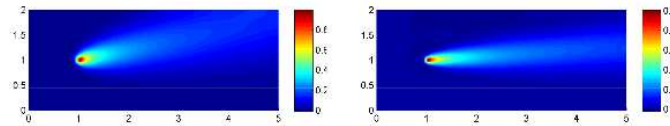
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1 Non-symmetric Parabolic Problems

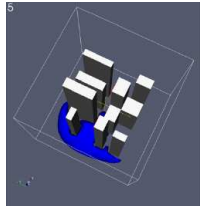
1.1 Motivation: Convection-Diffusion

Contaminant Transport

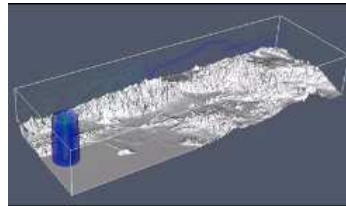
- Application: control of emission [D]



- Application: Identification of sources



Airborne contaminants
in urban canyon [B *et al.*]



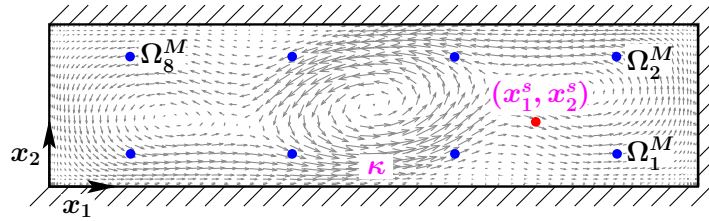
Airborne contaminants
in LA basin [H *et al.*]

Contaminant Transport

- Application: Identification of Sources

Dispersion of a pollutant

$$\Omega = [0, 4] \times [0, 1]$$



$$\text{Source: } g^{\text{PS}}(x; \mu) = \frac{50}{\pi} e^{-50((x_1 - x_1^s)^2 + (x_2 - x_2^s)^2)}$$

$$(\text{say, } \mu \equiv (\kappa, x_1^s, x_2^s))$$

Contaminant Transport – Problem Statement

Scalar Convection-Diffusion

$$y(x, t = 0; \mu) = 0$$

$$\frac{\partial}{\partial t} y(t; \mu) + \mathbf{U} \cdot \nabla y(t; \mu) = \kappa \nabla^2 y(t; \mu) + g^{\text{PS}}(x; \mu) u(t),$$

$$\text{INPUTS: } \mu \equiv \kappa \in \mathcal{D} = [0.01, 1.0] \subset \mathbb{R}^{P=1};$$

$$(x_1^s, x_2^s) = (3.0, 0.4);$$

$$\mathbf{U} (\text{Gr} = 10^5) \text{ from Pr} = 0$$

Natural Convection (Navier-Stokes);

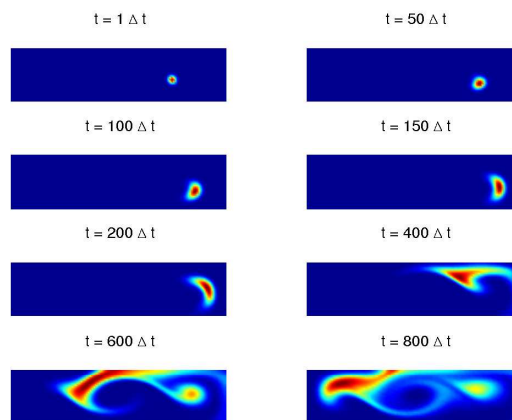
$u(t)$ “control” input (source strength).

OUTPUTS: Measurements $s_q(t; \mu)$, $1 \leq q \leq 8$.

Contaminant Transport – Sample Solutions

Field variable: $\mu = 0.01$

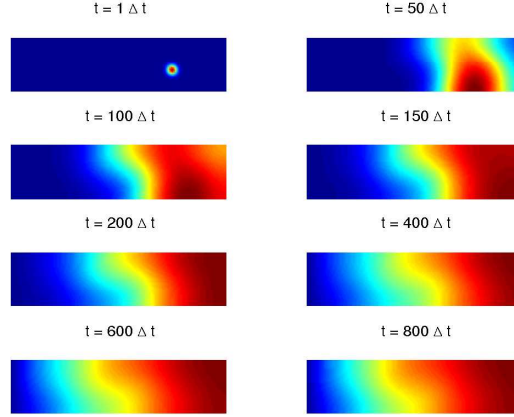
$$(\mathcal{N} = 3720)$$



Contaminant Transport – Sample Solutions

Field variable: $\mu = 1.00$

($\mathcal{N} = 3720$)



Application: Inverse Problem

Determine: $\mu^* \in \mathcal{D}$

(actual value)

Given experimental data

measurements : $z(t^k) \in \mathcal{Z}_{\text{exp}}^k, \forall k \in \mathbf{IK}_{\text{exp}}$, where

$$\mathcal{Z}_{\text{exp}}^k \equiv [s^{\mathcal{N}}(t^k; \mu^*) - \epsilon_{\text{exp}}, s^{\mathcal{N}}(t^k; \mu^*) + \epsilon_{\text{exp}}]$$

observations : $\mathbf{IK}_{\text{exp}} \subset \mathbf{IK}$

error : $\epsilon_{\text{exp}} \in \mathbf{IR}$ (bounded, “white”)

input : $u(t^k) = \delta_{1k}, \forall k \in \mathbf{IK}$

Application: Inverse Problem – (Regularized) Solution

Given noisy measurements, $z(t^k), k \in \mathbf{IK}_{\text{exp}}$, solve

- Output least squares problem

$$\hat{\mu} = \arg \min_{\mu \in \mathcal{D}} \frac{1}{2} \sum_{k=1}^{\mathbf{IK}_{\text{exp}}} \|s^{\mathcal{N}}(t^k; \mu) - z(t^k)\|_W^2$$

s.t. $\text{PDE}_{\mathcal{N}}(\mu)$ being satisfied; or

- Regularized problem

$$\hat{\mu} = \arg \min_{\mu \in \mathcal{D}} \frac{1}{2} \sum_{k=1}^{\mathbf{IK}_{\text{exp}}} \|s^{\mathcal{N}}(t^k; \mu) - z(t^k)\|_W^2 + \frac{1}{2} \delta_R R(\mu)$$

s.t. $\text{PDE}_{\mathcal{N}}(\mu)$ being satisfied.

Application: Inverse Problem – (Regularized) Solution

Given noisy measurements, $z(t^k)$, $k \in \mathbf{K}_{\text{exp}}$, solve

- Output least squares problem

$$\hat{\mu} = \arg \min_{\mu \in \mathcal{D}} \frac{1}{2} \sum_{k=1}^{\mathbf{K}_{\text{exp}}} \|s^{\mathcal{N}}(t^k; \mu) - z(t^k)\|_W^2$$

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- Regularized problem

$$\hat{\mu} = \arg \min_{\mu \in \mathcal{D}} \frac{1}{2} \sum_{k=1}^{\mathbf{K}_{\text{exp}}} \|s^{\mathcal{N}}(t^k; \mu) - z(t^k)\|_W^2 + \frac{1}{2} \delta_R R(\mu)$$

s.t. $\text{PDE}_{\mathcal{N}}(\mu)$ being satisfied.

\Rightarrow Solution very expensive: \mathcal{N} -dependent cost

Application: Inverse Problem – (Regularized) Solution

Given noisy measurements, $z(t^k)$, $k \in \mathbf{K}_{\text{exp}}$, solve

- Output least squares problem

$$\hat{\mu} = \arg \min_{\mu \in \mathcal{D}} \frac{1}{2} \sum_{k=1}^{\mathbf{K}_{\text{exp}}} \|s_{\mathcal{N}}(t^k; \mu) - z(t^k)\|_W^2$$

s.t. $\text{PDE}_{\mathcal{N}}(\mu)$ being satisfied; or

- Regularized problem

$$\hat{\mu} = \arg \min_{\mu \in \mathcal{D}} \frac{1}{2} \sum_{k=1}^{\mathbf{K}_{\text{exp}}} \|s_{\mathcal{N}}(t^k; \mu) - z(t^k)\|_W^2 + \frac{1}{2} \delta_R R(\mu)$$

s.t. $\text{PDE}_{\mathcal{N}}(\mu)$ being satisfied.

\Rightarrow Surrogate model approach: \mathcal{N} -dependent cost

1.2 Problem Statement

Problem Statement

Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$, evaluate

$t \in (0, t_f]$

$$s^e(t; \mu) = \ell(u^e(x; t; \mu); \mu)$$

where $u^e(x; t; \mu) \in L^2(0, T; X^e(\Omega))$ satisfies

$u_0 = 0$

$$\begin{aligned} m \left(\frac{\partial u^e}{\partial t}(x; t; \mu), v; \mu \right) + a^{\text{CD}}(u^e(x; t; \mu), v; \mu) \\ = f(v; \mu) g(t), \quad \forall v \in X^e. \end{aligned}$$

Classical example: unsteady convection diffusion equation

Hypotheses

Linear forms and functions

$$f(\cdot; \mu) : \begin{array}{l} \text{linear, affine in } \mu, \\ X^e\text{-bounded,} \quad \forall \mu \in \mathcal{D} \end{array}$$

$$g(\cdot) : L^2(0, t_f) \quad \text{“control” input}$$

$$\ell(\cdot; \mu) : \begin{array}{l} \text{linear, affine in } \mu, \\ L^2(\Omega)\text{-bounded,} \quad \forall \mu \in \mathcal{D} \end{array}$$

Hypotheses

$$a^{\text{CD}}(\cdot, \cdot; \mu) : \begin{array}{l} \text{bilinear, affine in } \mu, \\ \text{non-symmetric,} \\ X^e\text{-continuous,} \\ X^e\text{-coercive form,} \quad \forall \mu \in \mathcal{D}; \end{array}$$

$$m(\cdot, \cdot; \mu) : \begin{array}{l} \text{bilinear, affine in } \mu, \\ \text{symmetric,} \\ L^2(\Omega)\text{-continuous,} \\ L^2(\Omega)\text{-coercive form,} \quad \forall \mu \in \mathcal{D}; \end{array}$$

Recall that we can write, $\forall w, v \in X^e$,

$$a^{\text{CD}}(w, v; \mu) = \underbrace{a_S(w, v; \mu)}_{\text{symmetric}} + \underbrace{a_{SS}(w, v; \mu)}_{\text{skew-symmetric}}$$

Hypotheses

Require affine parameter dependence also $\ell(v; \mu)$, $f(v; \mu)$

$$a^{\text{CD}}(w, v; \mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu) a^{\text{CD},q}(w, v),$$

$$m(w, v; \mu) = \sum_{q=1}^{Q_m} \Theta_m^q(\mu) m^q(w, v);$$

where

$$\Theta_{a,m}^q : \mathcal{D} \rightarrow \mathbb{R}, \quad \mu\text{-dependent functions;}$$

representing coefficients, geometry, ...

$$a^{\text{CD},q} \text{ and } m^q \quad \mu\text{-independent forms.}$$

Convection-Diffusion

Bilinear form a^{CD} can be written as

$$a^{\text{CD}}(w, v; \mu) = \underbrace{a^{\text{D}}(w, v; \mu)}_{\text{DIFFUSION}} + \underbrace{a^{\text{C}}(w, v; \mu)}_{\text{CONVECTION}}$$

where

$$\begin{aligned}
a^{\text{D}}(w, v; \mu) &= \mu \int_{\Omega} \nabla w \cdot \nabla v \\
a^{\text{C}}(w, v; \mathbf{U}) &= \int_{\Omega} v (\mathbf{U} \cdot \nabla w) + \frac{1}{2} \int_{\Omega} v w (\nabla \cdot \mathbf{U})
\end{aligned}$$

Conservation Form: Require $\nabla \cdot \mathbf{U}^e = 0$, then a^{C} is skew-symmetric (although $\nabla \cdot \mathbf{U} \neq 0$ pointwise) for

- (i) a contained flow, i.e., $\mathbf{n} \cdot \mathbf{U} = 0$ on Γ ;
- (i) a flow with Dirichlet boundary conditions.

1.3 Truth Approximation

Truth Approximation

- Spatial Discretization: Finite Element

$$X^{\mathcal{N}} \subset X \text{ with } \dim(X^{\mathcal{N}}) = \mathcal{N}$$

for given \mathcal{N} ($\mathcal{N} \rightarrow \infty$).

Also possible: Finite Volume [HO]

- Temporal Discretization: Finite Difference

$$t^k = k \Delta t, \forall k \in \mathbb{K} \equiv \{(0), 1, 2, \dots, K\}$$

for given $\Delta t = t_f / K$ (fixed).

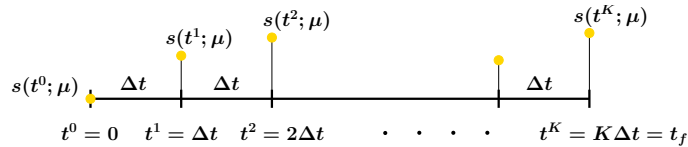
Also possible: DG [RMM]

Truth Approximation

- Temporal Discretization: Finite Difference

$$\frac{\partial u}{\partial t}(t^k; \mu) \approx \frac{u(t^k; \mu) - u(t^{k-1}; \mu)}{\Delta t}$$

- Euler Backward
- Crank-Nicolson (advection-dominated problems)



Problem Statement

Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$, evaluate

$\forall k \in \mathbb{K}$

$$s^k(\mu) = \ell(u^k(\mu); \mu)$$

where $u^k(\mu) \in X$ satisfies

$$u_0 = \mathbf{0}$$

$$\begin{aligned} m \left(\frac{u^k(\mu) - u^{k-1}(\mu)}{\Delta t}, v; \mu \right) + a^{\text{CD}} \left(\frac{u^k(\mu) + u^{k-1}(\mu)}{2}, v; \mu \right) \\ = f(v; \mu) \frac{1}{2} (g(t^k) + g(t^{k-1})), \quad \forall v \in X. \end{aligned}$$

2 Reduced Basis Approximation

2.1 RB Spaces & Bases

Reduced Basis Space

We define the Lagrangian RB space

$$X_N = \text{span}\{\zeta^n, 1 \leq n \leq N\}, \quad 1 \leq N \leq N_{\max},$$

with mutually $(\cdot, \cdot)_X$ -orthonormal basis functions

$$\zeta^n \in X, \quad 1 \leq n \leq N_{\max}.$$

We thus obtain

$$X_N \subset X, \quad \dim(X_N) = N, \quad 1 \leq N \leq N_{\max},$$

and

hierarchical spaces

$$X_1 \subset X_2 \subset \dots \subset X_{N_{\max}-1} \subset X_{N_{\max}} (\subset X).$$

The basis functions are constructed using the POD(t)-Greedy(μ) algorithm.

2.2 Approximation

Galerkin Projection

Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$, evaluate

$\forall k \in \mathbb{K}$

$$s_N^k(\mu) = \ell(u_N^k(\mu); \mu)$$

where $u_N^k(\mu) \in X_N$ satisfies

$$u_{N,0} = \mathbf{0}$$

$$\begin{aligned} m \left(\frac{u_N^k(\mu) - u_N^{k-1}(\mu)}{\Delta t}, v; \mu \right) + a^{\text{CD}} \left(\frac{u_N^k(\mu) + u_N^{k-1}(\mu)}{2}, v; \mu \right) \\ = f(v; \mu) \frac{1}{2} (g(t^k) + g(t^{k-1})), \quad \forall v \in X_N. \end{aligned}$$

- RB inherits fixed truth temporal discretization.
- Offline-online decomposition follows directly from symmetric case.

3 A Posteriori Error Estimation

3.1 Preliminaries

Energy Norm

Crank-Nicolson

- “Spatio-temporal” energy norm (parameter-dependent)

$$\begin{aligned} ((w^k, v^k))_{\text{CN}} &= m(w^k, v^k; \mu) \\ &\quad + \sum_{k'=1}^k \Delta t a_S \left(\frac{w^{k'} + w^{k'-1}}{2}, \frac{v^{k'} + v^{k'-1}}{2}; \mu \right), \end{aligned}$$

$$\begin{aligned} |||w^k|||_{\text{CN}} &= \left(m(w^k, w^k; \mu) \right. \\ &\quad \left. + \sum_{k'=1}^k \Delta t a_S \left(\frac{w^{k'} + w^{k'-1}}{2}, \frac{w^{k'} + w^{k'-1}}{2}; \mu \right) \right)^{1/2}, \\ &\qquad\qquad\qquad 1 \leq k \leq K. \end{aligned}$$

Coercivity and Coercivity Lower Bound

We also define

- Coercivity constant

$$\alpha(\mu) \equiv \inf_{w \in X} \frac{a_S(w, w; \mu)}{\|w\|_X^2};$$

- Positive lower bound, $\alpha_{\text{LB}} : \mathcal{D} \rightarrow \mathbb{R}$, such that

$$0 < \alpha_{\text{LB}}(\mu) \leq \alpha(\mu), \quad \forall \mu \in \mathcal{D}.$$

This bound can be calculated using the

- “min Θ ” Approach (if a is parametrically coercive), or
- Successive Constraint Method

exactly as in elliptic case.

Dual Norm of Residual

We define the residual, $\forall k \in \mathbb{K}$,

$$\begin{aligned} r^k(v; \mu) &\equiv f(v; \mu) \frac{1}{2} (g(t^k) + g(t^{k-1})) - m \left(\frac{u_N^k(\mu) - u_N^{k-1}(\mu)}{\Delta t}, v; \mu \right) \\ &\quad - a^{\text{CD}} \left(\frac{u_N^k(\mu) + u_N^{k-1}(\mu)}{2}, v; \mu \right), \quad \forall v \in X \end{aligned}$$

Dual Norm of Residual

Given $\mu \in \mathcal{D}$, the dual norm of $r^k(v; \mu)$ is defined as

$$\begin{aligned}\|r^k(\cdot; \mu)\|_{X'} &\equiv \sup_{v \in X} \frac{r^k(v; \mu)}{\|v\|_X} \\ &= \|\hat{e}^k(\mu)\|_X,\end{aligned}$$

where $\hat{e}^k(\mu) \in X$ satisfies

$$(\hat{e}^k(\mu), v)_X = r^k(v; \mu), \quad \forall v \in X.$$

3.2 Primal-only Formulation

Energy Error Bound

We define the error bound, $\Delta_N^k(\mu) = \Delta_N(t^k; \mu)$, $1 \leq k \leq K$, as

$$\Delta_N^k(\mu) = \alpha_{\text{LB}}^{-1/2}(\mu) \left(\sum_{k'=1}^k \Delta t \|\hat{e}^{k'}(\mu)\|_X^2 \right)^{1/2}.$$

We can then prove

Proposition 1 (Energy Error Bound). *For any $N = 1, \dots, N_{\max}$, the error in the field variable, $e^k(\mu) = \mathbf{u}^k(\mu) - \mathbf{u}_N^k(\mu)$, is bounded by*

$$\|e^k(\mu)\|_{\text{CN}} \leq \Delta_N^k(\mu), \quad \forall \mu \in \mathcal{D}, \forall k \in \mathbb{K}.$$

Simple Output Error Bound

We define the **output error bound**, $\Delta_N^{sk}(\mu) = \Delta_N^s(t^k; \mu)$, $1 \leq k \leq K$, as

$$\Delta_N^{sk}(\mu) \equiv \sigma_{\text{LB}}^{-1}(\mu) \left(\sup_{v \in X} \frac{\ell(v; \mu)}{\|v\|} \right) \Delta_N^k(\mu)$$

Proposition 2 (Simple Output Error Bound). *For any $N = 1, \dots, N_{\max}$, the error in the output is bounded by*

$$|s^k(\mu) - s_N^k(\mu)| \leq \Delta_N^{sk}(\mu), \quad \forall \mu \in \mathcal{D}, \forall k \in \mathbb{K}.$$

3.3 Primal-Dual Formulation

Dual Problem – Truth Approximation

Given $\mu \in \mathcal{D}$, the dual variable $\Psi^k(\mu) \in X$, $1 \leq k \leq K$, satisfies

$$m\left(v, \frac{\Psi^k(\mu) - \Psi^{k+1}(\mu)}{\Delta t}; \mu\right) - a\left(v, \frac{\Psi^k(\mu) + \Psi^{k+1}(\mu)}{2}; \mu\right) = 0, \quad \forall v \in X,$$

with final condition

$$m(v, \Psi(t^{K+1}; \mu); \mu) \equiv l(v; \mu), \quad \forall v \in X.$$

- Dual problem inherits spatial and temporal discretization from primal problem.

Dual Problem – Galerkin Projection

Given $\mu \in \mathcal{D}$, the dual variable $\Psi_N^k(\mu) \in X_{N_{\text{du}}}^{\text{du}}$, $1 \leq k \leq K$, satisfies

$$m\left(v, \frac{\Psi_N^k(\mu) - \Psi_N^{k+1}(\mu)}{\Delta t}; \mu\right) - a\left(v, \frac{\Psi_N^k(\mu) + \Psi_N^{k+1}(\mu)}{2}; \mu\right) = 0,$$

$$\forall v \in X_{N_{\text{du}}}^{\text{du}},$$

with final condition

$$m(v, \Psi_N(t^{K+1}; \mu); \mu) \equiv l(v; \mu), \quad \forall v \in X_{N_{\text{du}}}^{\text{du}}.$$

- $X_{N_{\text{du}}}^{\text{du}}$ constructed using POD(t)-Greedy(μ) sampling procedure.
- Offline-online decomposition follows from symmetric case.

Primal Problem – Galerkin Projection

Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$ and $\Psi_N^k(\mu)$, evaluate

$$\forall k \in \mathbb{K}$$

$$s_N^k(\mu) = \ell(u_N^k(\mu); \mu)$$

$$+ \sum_{k'=1}^k r^{k'} \left(\frac{1}{2} \left(\Psi_N^{K-k+k'}(\mu) + \Psi_N^{K-k+k'+1}(\mu) \right); \mu \right) \Delta t$$

where $u_N^k(\mu) \in X_N^{(\text{pr})}$ satisfies

$$u_{N,0} = 0$$

$$m\left(\frac{u_N^k(\mu) - u_N^{k-1}(\mu)}{\Delta t}, v; \mu\right) + a^{\text{CD}}\left(\frac{u_N^k(\mu) + u_N^{k-1}(\mu)}{2}, v; \mu\right)$$

$$= f(v; \mu) \frac{1}{2} (g(t^k) + g(t^{k-1})), \quad \forall v \in X_N.$$

$\Rightarrow X_N = X_N^{\text{pr}}$ and $r^k(v; \mu)$ is the primal residual.

Inner Products and Norms

Crank-Nicolson

- “Spatio-temporal” energy norm (parameter-dependent)

$$\begin{aligned} (((w^k, v^k)))_{\text{CN}}^{\text{du}} &= m(w^k, v^k; \mu) \\ &+ \sum_{k'=k}^K \Delta t a_S \left(\frac{w^{k'} + w^{k'+1}}{2}, \frac{v^{k'} + v^{k'+1}}{2}; \mu \right), \end{aligned}$$

$$\begin{aligned} |||w^k|||_{\text{CN}}^{\text{du}} &= \left(m(w^k, w^k; \mu) \right. \\ &\left. + \sum_{k'=k}^K \Delta t a_S \left(\frac{w^{k'} + w^{k'+1}}{2}, \frac{w^{k'} + w^{k'+1}}{2}; \mu \right) \right)^{1/2}, \\ &1 \leq k \leq K. \end{aligned}$$

Dual Final Condition

If m or ℓ are parameter-dependent, we define the residual

$$r^{\Psi_f}(v; \mu) \equiv \ell(v; \mu) - m(v, \Psi_N^{K+1}; \mu), \quad \forall v \in X.$$

Lemma 3 (Dual Error Bound – Final Condition). *Given $\mu \in \mathcal{D}$, the error $e^{\text{du}}(t^{K+1}; \mu) = \Psi^{K+1}(\mu) - \Psi_N^{K+1}(\mu)$ is bounded by*

$$\|e^{\text{du}}(t^{K+1}; \mu)\| \leq \Delta_N^{\Psi_f}(\mu) \equiv \frac{\varepsilon_N^{\Psi_f}(\mu)}{\sigma_{\text{LB}}(\mu)}$$

where

$$\varepsilon_N^{\Psi_f}(\mu) \equiv \sup_{v \in X} \frac{r^{\Psi_f}(v; \mu)}{\|v\|}$$

Dual Norm of Dual Residual

We define the residual, $\forall k \in \mathbb{K}$,

$$r^{\text{du},k}(v; \mu) \equiv -m\left(v, \frac{\Psi_N^k(\mu) - \Psi_N^{k+1}(\mu)}{\Delta t}; \mu\right) - a\left(v, \frac{\Psi_N^k(\mu) + \Psi_N^{k+1}(\mu)}{2}; \mu\right), \quad \forall v \in X$$

Dual Norm of Residual

Given $\mu \in \mathcal{D}$, the dual norm of $r^{\text{du},k}(v; \mu)$ is defined as

$$\begin{aligned} \|r^{\text{du},k}(\cdot; \mu)\|_{X'} &\equiv \sup_{v \in X} \frac{r^{\text{du},k}(v; \mu)}{\|v\|_X} \\ &= \|\hat{e}^{\text{du},k}(\mu)\|_X, \end{aligned}$$

where $\hat{e}^{\text{du},k}(\mu) \in X$ satisfies

$$(\hat{e}^{\text{du},k}(\mu), v)_X = r^{\text{du},k}(v; \mu), \quad \forall v \in X.$$

Energy Error Bound – Dual

We define the error bound, $\Delta_N^{\text{du},k}(\mu)$, $1 \leq k \leq K$, as

$$\Delta_N^{\text{du},k}(\mu) = \left(\frac{\Delta t}{\alpha_{\text{LB}}(\mu)} \sum_{k'=1}^k \|\hat{e}^{\text{du},k'}(\mu)\|_X^2 + \sigma_{\text{LB}}(\mu) \Delta_N^{\Psi_f}(\mu)^2 \right)^{1/2}.$$

We can then prove

Proposition 4 (Energy Error Bound). *For any $N = 1, \dots, N_{\text{du},\text{max}}$, the error in the dual variable, $e^{\text{du},k}(\mu) = \Psi^k(\mu) - \Psi_N^k(\mu)$, is bounded by*

$$\|e^{\text{du},k}(\mu)\|_{\text{CN}}^{\text{du}} \leq \Delta_N^{\text{du},k}(\mu), \quad \forall \mu \in \mathcal{D}, \forall k \in \mathbb{K}.$$

Output Error Bound – Primal-Dual Formulation

We (re-)define the **output error bound**, $\Delta_N^{s^k}(\mu)$, $1 \leq k \leq K$, as

$$\Delta_N^{s^k}(\mu) \equiv \Delta_{N_{\text{pr}}}^{\text{pr},k}(\mu) \Delta_{N_{\text{du}}}^{\text{du},K-k+1}(\mu)$$

Proposition 5 (Simple Output Error Bound). *For any $N = 1, \dots, N_{\text{max}}$, the error in the output is bounded by*

$$|s^k(\mu) - s_N^k(\mu)| \leq \Delta_N^{s^k}(\mu), \quad \forall \mu \in \mathcal{D}, \forall k \in \mathbb{K}.$$

- Offline-online decomposition same as for Euler Backward
- Computational cost same as for Euler Backward

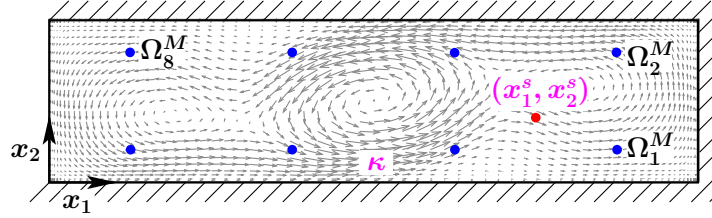
4 Numerical Results

4.1 Model Problem

Contaminant Transport – Problem Statement

Dispersion of a pollutant

$$\Omega = [0, 4] \times [0, 1]$$



$$\text{Source: } g^{\text{PS}}(x; \mu) = \frac{50}{\pi} e^{-50((x_1 - x_1^s)^2 + (x_2 - x_2^s)^2)}$$

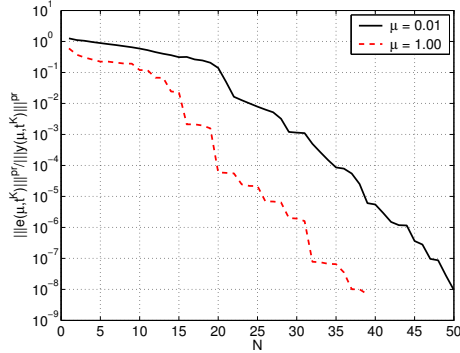
$$(\text{say, } \mu \equiv (\kappa, x_1^s, x_2^s))$$

Truth: $\mathcal{N} = 3720$, $K = 800$

4.2 Convergence Results

Dependence on conductivity

Convergence Rate for diffusivities $\mu = 0.01$ and $\mu = 1.00$



Primal: Energy Norm

Primal problem: convergence energy norm error and bound

N_{pr}	$\epsilon_{\text{max,rel}}^{\text{pr}}$	$\Delta_{\text{max,rel}}^{\text{pr}}$	$\bar{\eta}^{\text{pr}}$
20	3.96 E-01	2.45 E+00	5.20
40	1.39 E-01	5.04 E-01	2.77
60	4.60 E-02	1.08 E-01	2.17
80	1.95 E-02	3.60 E-02	2.03
100	1.17 E-02	1.46 E-02	1.85
120	3.03 E-03	3.91 E-03	1.74
140	7.61 E-04	1.37 E-03	1.66
160	6.01 E-04	7.17 E-04	1.62
180	1.74 E-04	2.88 E-04	1.58
200	9.59 E-05	1.33 E-04	1.60

Dual: Energy Norm

Dual problem: convergence energy norm error & bound output 1

N_{du}	$\epsilon_{\text{max,rel}}^{\text{du}}$	$\Delta_{\text{max,rel}}^{\text{du}}$	$\bar{\eta}^{\text{du}}$
20	4.23 E-01	1.85 E+00	4.58
40	1.75 E-01	7.51 E-01	3.93
60	7.66 E-02	1.55 E-01	2.43
80	3.43 E-02	4.38 E-02	2.06
100	1.22 E-02	1.54 E-02	1.90
120	4.54 E-03	6.35 E-03	1.82
140	1.53 E-03	2.04 E-03	1.73
160	1.10 E-03	1.31 E-03	1.69
180	3.77 E-04	5.39 E-04	1.62
200	1.97 E-04	2.25 E-04	1.62

Primal-only: Output

Primal-only formulation: convergence output error and bound

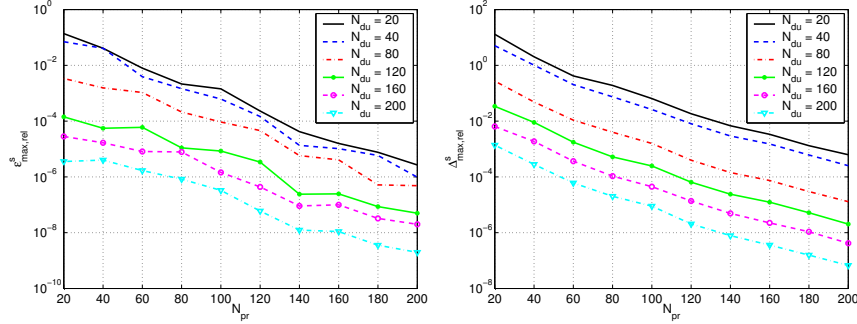
N_{pr}	$\epsilon_{\text{max,rel}}^{\hat{s}}$	$\Delta_{\text{max,rel}}^{\hat{s}}$	$\bar{\eta}^{\hat{s}}$
20	2.84 E-01	9.86 E+00	41.8
40	4.62 E-02	2.03 E+00	41.1
60	1.13 E-02	4.35 E-01	55.2
80	2.46 E-03	1.45 E-01	48.9
100	1.69 E-03	5.86 E-02	42.3
120	2.52 E-04	1.57 E-02	78.5
140	1.40 E-04	5.49 E-03	107
160	3.43 E-05	2.88 E-03	146
180	1.50 E-05	1.16 E-03	113
200	1.11 E-05	5.37 E-04	148

Primal-Dual: OutputPrimal-dual formulation: convergence output bound ($N_{\text{pr}} = N_{\text{du}}$)

N_{pr}	N_{du}	$\epsilon_{\text{max,rel}}^s$	$\Delta_{\text{max,rel}}^s$	$\bar{\eta}^s$
20	20	1.37 E-01	1.30 E+01	1908
40	40	4.09 E-02	1.02 E+00	150
60	60	3.07 E-03	4.62 E-02	182
80	80	2.12 E-04	4.01 E-03	185
100	100	3.87 E-05	6.37 E-04	229
120	120	3.40 E-06	6.49 E-05	119
140	140	1.22 E-07	7.51 E-06	150
160	160	9.99 E-08	2.24 E-06	175
180	180	4.55 E-09	3.71 E-07	140
200	200	1.98 E-09	6.72 E-08	110

Primal-Dual: Output

Primal-dual formulation: convergence output error and bound



4.3 Computational Times

Primal-Dual: Online RB vs. Truth

Primal-dual formulation: online computational times

$N_{\text{pr}} = N_{\text{du}}$	$s_N(\mu, t^k)$	$\Delta_N^s(\mu, t^k)$	$s(\mu, t^k)$
20	4.33 E-02	2.10 E-03	1
40	5.22 E-02	2.95 E-03	1
60	5.70 E-02	4.00 E-03	1
80	6.27 E-02	7.86 E-03	1
100	6.78 E-02	1.63 E-02	1
120	7.78 E-02	2.63 E-02	1
140	8.73 E-02	3.53 E-02	1
160	9.44 E-02	4.56 E-02	1

Output & Bound for $1 \leq k \leq K$

Savings with respect to truth: ≈ 16

Primal-Dual: Online RB vs. Truth

Primal-dual formulation: online computational times

$N_{\text{pr}} = N_{\text{du}}$	$s_N(\mu, t^k)$	$\Delta_N^s(\mu, t^k)$	$s(\mu, t^k)$
20	5.34 E-03	2.10 E-03	1
40	7.00 E-03	2.95 E-03	1
60	8.53 E-03	4.00 E-03	1
80	1.03 E-02	7.86 E-03	1
100	1.35 E-02	1.63 E-02	1
120	1.96 E-02	2.63 E-02	1
140	2.73 E-02	3.53 E-02	1
160	3.36 E-02	4.56 E-02	1

Output & Bound for every tenth timestep $k = [10, 20, \dots, K]$

Savings with respect to truth: ≈ 80

Primal-only: Online RB vs. Truth

Primal-only formulation: online computational times

N_{pr}	$\hat{s}_N(\mu, t^k)$	$\hat{\Delta}_N^s(\mu, t^k)$	$s(\mu, t^k)$
20	8.63 E-04	9.30 E-04	1
40	1.72 E-03	1.13 E-03	1
60	2.82 E-03	1.28 E-03	1
80	4.13 E-03	2.10 E-03	1
100	7.02 E-03	5.38 E-03	1
120	1.23 E-02	7.91 E-03	1
140	1.69 E-02	1.00 E-02	1
160	2.10 E-02	1.30 E-02	1

Output & Bound for every timestep $1 \leq k \leq K$

Savings with respect to truth: ≈ 50

Outlook

- Reduced basis methods have been extended to other classes of time-dependent problems
 - Hyperbolic problems,
 - Nonlinear Reaction-Diffusion Equations,
 - Navier-Stokes [KNP] (and viscous Burgers Equation [NRP]),
 - Fokker-Planck Equation [KP].
- The methodology directly extends to (non-parametric) dynamic systems.