Numerical Approximation of a two-fluid two-pressure diphasic model

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We are interested in the computation of two-phase flows, **using a two-fluid approach.**

Generally speaking, **two-fluid models** have several drawbacks :

- **they are generally non conservative**
- they may not be hyperbolic in a large class of conditions
- they may not satify the maximum principle property on the void fraction α

Today, we will focus on the first point only.

The governing equations are the following :

$$\begin{cases} \partial_t \alpha_1 + u_I \partial_x \alpha_1 = 0 \\ \partial_t \alpha_1 \rho_1 + \partial_x \alpha_1 \rho_1 u_1 = 0 \\ \partial_t (\alpha_1 \rho_1 u_1) + \partial_x (\alpha_1 \rho_1 u_1^2 + \alpha_1 p_1(\rho_1)) - p_I \partial_x \alpha_1 = 0 \\ \partial_t \alpha_2 \rho_2 + \partial_x \alpha_2 \rho_2 u_2 = 0 \\ \partial_t (\alpha_2 \rho_2 u_2) + \partial_x (\alpha_2 \rho_2 u_2^2 + \alpha_2 p_2(\rho_2)) + p_I \partial_x \alpha_1 = 0 \end{cases}$$

Note first that this model is **barotropic** and focuses on **convective effects only.**

It is **unconditionally hyperbolic** (not strictly) with real eigenvalues given by

$$\lambda_0 = u_I,$$
$$\lambda_1 = u_1 - c_1, \ \lambda_2 = u_1 + c_1,$$
$$\lambda_3 = u_2 - c_2, \ \lambda_4 = u_2 + c_2$$

The last four characteristic fields are GNL.

The first characteristic field is LD under the condition

$$u_I = u_2, \quad p_I = p_1 \qquad (\text{or} \quad u_I = u_1, \quad p_I = p_2)$$

To summarize, the model is

- non conservative
- hyperbolic
- associated with a pure transport of the void fraction α_1

The literature is large on this subject : see for instance the works by Gallouët, Hérard and Seguin, Ransom and Hicks, Baer and Nunziato, Kapila *et al*, Gavrilyuk and Saurel, Saurel and Abgrall, ...

The Riemann problem is difficult to solve for this model because of

- the large number of waves
- the non linear pressure terms
- the non conservative products

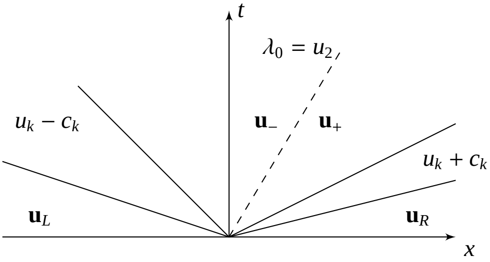


Fig.: General structure of a Riemann problem

Except of course if $\partial_x \alpha_1 = 0$ since we get two classical barotropic Euler systems in this case...

Outline of the talk

1. A relaxation approach (joint with A. Ambroso, T. Galié and F. Coquel)
2. A « sharp interface » approach with random sampling (still in progress)
3. Numerical illustrations

Introducing the condensed form

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) + \mathbf{c}(\mathbf{u})\partial_x \mathbf{u} = 0, \;\; t > 0, \;\; x \in \mathbb{R},$$

the general idea is to propose a larger but simpler relaxation model with source term

$$\partial_t \mathbf{v} + \partial_x \mathbf{g}(\mathbf{v}) + \mathbf{d}(\mathbf{v})\partial_x \mathbf{v} = \lambda \mathcal{R}(\mathbf{v}), \;\; t > 0, \;\; x \in \mathbb{R}.$$

This system is expected to be such that

$$\lim_{\lambda \to \infty} \mathbf{v}^\lambda = \mathbf{u}.$$

Here, simpler means that the Riemann problem of the convective part is easier to solve.

As a consequence, the numerical strategy for solving the equilibrium system

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) + \mathbf{c}(\mathbf{u})\partial_x \mathbf{u} = 0, \quad t > 0, \quad x \in \mathbb{R},$$

is based on a splitting strategy on the relaxation model :

- **First step :** solve by a Godunov scheme the convective part

$$\partial_t \mathbf{v} + \partial_x \mathbf{g}(\mathbf{v}) + \mathbf{d}(\mathbf{v})\partial_x \mathbf{v} = 0$$

- **Second step :** solve in the regime $\lambda \to \infty$

$$\partial_t \mathbf{v} = \lambda \mathcal{R}(\mathbf{v})$$

  that is, impose on each cell the relation

$$\mathcal{R}(\mathbf{v}) = 0$$

How to designe the relaxation model

$$\partial_t \mathbf{v} + \partial_x \mathbf{g}(\mathbf{v}) + \mathbf{d}(\mathbf{v})\partial_x \mathbf{v} = \lambda \mathcal{R}(\mathbf{v}), \ \ t > 0, \ \ x \in \mathbb{R}.$$

such that

- $\lim_{\lambda\to\infty} \mathbf{v}^\lambda = \mathbf{u}$
- the model is larger but simpler to solve

Recall that **the Riemann problem is difficult to solve** for this model because of

- the large number of waves
- the non linear pressure terms
- the non conservative products

Here, our relaxation process will concern the last two points only.

We first focus on the non linear pressure terms.

We propose the following relaxation system :

$$\begin{cases} \partial_t \alpha_1 + u_2 \partial_x \alpha_1 = 0 \\ \partial_t \alpha_1 \rho_1 + \partial_x \alpha_1 \rho_1 u_1 = 0 \\ \partial_t(\alpha_1 \rho_1 u_1) + \partial_x(\alpha_1 \rho_1 u_1^2 + \alpha_1 \Pi_1) - \Pi_1 \partial_x \alpha_1 = 0 \\ \partial_t \alpha_2 \rho_2 + \partial_x \alpha_2 \rho_2 u_2 = 0 \\ \partial_t(\alpha_2 \rho_2 u_2) + \partial_x(\alpha_2 \rho_2 u_2^2 + \alpha_2 \Pi_2) + \Pi_1 \partial_x \alpha_1 = 0 \\ \partial_t \alpha_1 \rho_1 \Pi_1 + \partial_x \alpha_1 \rho_1 \Pi_1 u_1 + \alpha_1 a_1^2 \partial_x u_1 - a_1^2 (u_1 - u_I) \partial_x \alpha_1 = \lambda \alpha_1 \rho_1 (p_1 - \Pi_1), \\ \partial_t \alpha_2 \rho_2 \Pi_2 + \partial_x \alpha_2 \rho_2 \Pi_2 u_2 + \alpha_2 a_2^2 \partial_x u_2 - a_2^2 (u_2 - u_I) \partial_x \alpha_2 = \lambda \alpha_2 \rho_2 (p_2 - \Pi_2), \end{cases}$$

avec $a_k > \rho_k c_k,\ k = 1, 2.$ (the so-called Whitham conditions).

This is **unconditionally hyperbolic** (not strictly) with real eigenvalues given by

$$\begin{gathered} \lambda_0^r = u_2, \\ \lambda_1^r = u_1 - a_1 \tau_1,\ \lambda_2^r = u_1 + a_1 \tau_1, \\ \lambda_3^r = u_2 - a_2 \tau_2,\ \lambda_4^r = u_2 + a_2 \tau_2. \end{gathered}$$

All the fields are LD.

We then focus on the non conservative products.

We propose the following modified relaxation system :

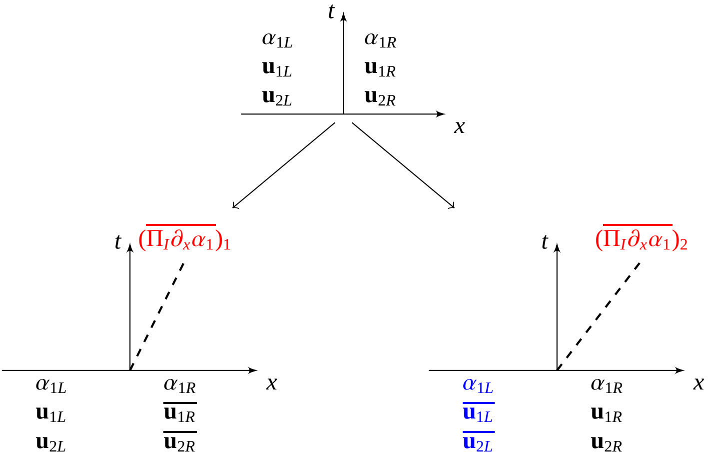
$$\begin{cases} \partial_t\alpha_1 + u_2\partial_x\alpha_1 = 0 \\ \partial_t\alpha_1\rho_1 + \partial_x\alpha_1\rho_1 u_1 = 0 \\ \partial_t(\alpha_1\rho_1 u_1) + \partial_x(\alpha_1\rho_1 u_1^2 + \alpha_1\Pi_1) = \overline{\Pi_1\partial_x\alpha_1}\delta_{x-u_2^* t} \\ \partial_t\alpha_2\rho_2 + \partial_x\alpha_2\rho_2 u_2 = 0 \\ \partial_t(\alpha_2\rho_2 u_2) + \partial_x(\alpha_2\rho_2 u_2^2 + \alpha_2\Pi_2) = -\overline{\Pi_1\partial_x\alpha_1}\delta_{x-u_2^* t} \\ \partial_t\alpha_1\rho_1\Pi_1 + \partial_x\alpha_1\rho_1\Pi_1 u_1 + \alpha_1 a_1^2\partial_x u_1 - a_1^2(u_1 - u_I)\partial_x\alpha_1 = \lambda\alpha_1\rho_1(p_1 - \Pi_1), \\ \partial_t\alpha_2\rho_2\Pi_2 + \partial_x\alpha_2\rho_2\Pi_2 u_2 + \alpha_2 a_2^2\partial_x u_2 - a_2^2(u_2 - u_I)\partial_x\alpha_2 = \lambda\alpha_2\rho_2(p_2 - \Pi_2), \end{cases}$$

avec $a_k > \rho_k c_k,\ \ k = 1, 2$. (the so-called Whitham conditions).

The Riemann problem is now explicitly solved.

How do we guess $\overline{\Pi_I \partial_x \alpha_1}$?

We chose to be exact on contact discontinuity solutions for the equilibrium system



We need to chose one of the two estimates. *Do we ?*

This way, we did construct a Relaxation system such that

- the Riemann solution is explicitly known
- the Riemann solution is exact for isolated contact discontinuities
- the corresponding numerical method is stable and conservative for the mass of each fluid and for the total momentum

Outline of the talk

Here, the key point is to remind that the model consists in two classical Euler systems for both phases when $\partial_x \alpha_1 = 0$.

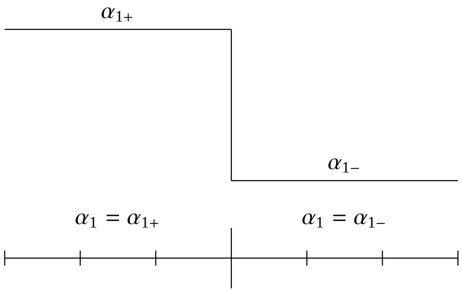
$$\begin{cases} \partial_t \alpha_1 + u_I \partial_x \alpha_1 = 0 \\ \partial_t \alpha_1 \rho_1 + \partial_x \alpha_1 \rho_1 u_1 = 0 \\ \partial_t (\alpha_1 \rho_1 u_1) + \partial_x (\alpha_1 \rho_1 u_1^2 + \alpha_1 p_1(\rho_1)) - p_I \partial_x \alpha_1 = 0 \\ \partial_t \alpha_2 \rho_2 + \partial_x \alpha_2 \rho_2 u_2 = 0 \\ \partial_t (\alpha_2 \rho_2 u_2) + \partial_x (\alpha_2 \rho_2 u_2^2 + \alpha_2 p_2(\rho_2)) + p_I \partial_x \alpha_1 = 0 \end{cases}$$

Our objective is then to design a numerical scheme which

- is based on our « favorite » scheme for the usual Euler equations
- provides in addition **sharp contact discontinuities**
- obeys stability properties

Our strategy is based on a **Ghost-Fluid** approach coupled with a **Glimm random sampling** strategy.

Let us assume without restriction that α_1 is a step function :



According to the Ghost-Fluid approach, we first split the actual mesh in two α_1-constant meshes :

Mesh $\pm$ s.t. $\alpha_1 = \alpha_{1\pm}$

$\alpha_1 = \alpha_{1+}$ | $\alpha_1 = \alpha_{1-}$

$\mathbf{u}_{j-3}$ $\mathbf{u}_{j-2}$ $\mathbf{u}_{j-1}$ | $\mathbf{u}_{j}$ $\mathbf{u}_{j+1}$ $\mathbf{u}_{j+2}$

$\Downarrow$

Mesh $+$ s.t. $\alpha_1 = \alpha_{1+}$

$\mathbf{u}_{j-3}$ $\mathbf{u}_{j-2}$ $\mathbf{u}_{j-1}$ | $\tilde{\mathbf{u}}_{j}$ $\tilde{\mathbf{u}}_{j+1}$ $\tilde{\mathbf{u}}_{j+2}$

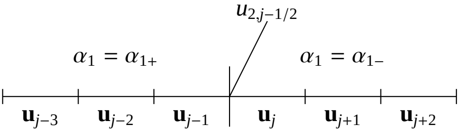
Mesh $-$ s.t. $\alpha_1 = \alpha_{1-}$

$\tilde{\mathbf{u}}_{j-3}$ $\tilde{\mathbf{u}}_{j-2}$ $\tilde{\mathbf{u}}_{j-1}$ | $\mathbf{u}_{j}$ $\mathbf{u}_{j+1}$ $\mathbf{u}_{j+2}$

We define the reconstructed states such that $\mathbf{u}_j$ and $\tilde{\mathbf{u}}_j$ are joined by an admissible contact discontinuity.

Using a Glimm scheme, we then solve numerically $\partial_t \alpha_1 + u_{2,j+1/2} \partial_x \alpha_1 = 0$

Mesh $\pm$

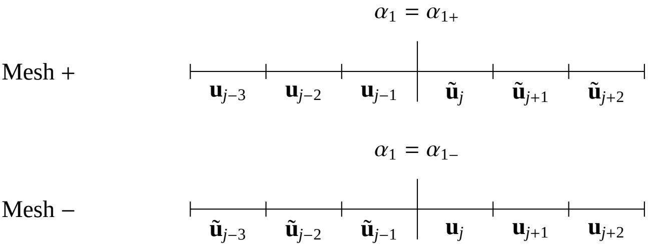


Being given an equidistributed random sequence $a_n \in (0,1)$, it amounts to set :

$$\alpha_{1j}^{n+1} = \begin{cases} \alpha_{1+} & \text{if} \quad a_n \leq u_{2,j-1/2} \frac{\Delta t}{\Delta x} \\ \alpha_{1-} & \text{if} \quad a_n > u_{2,j-1/2} \frac{\Delta t}{\Delta x} \end{cases}$$

Note that α_1 remains sharp.

With our « favorite » scheme, we numerically solve two classical Euler equations



and we choose on each cell j between the two meshes according to the value $(\alpha_1)_j^{n+1}$.

We immediately get the positivity properties of our « favorite » scheme.
Note that the global scheme is not strictly conservative.

For a similar approach in the context of van der Waals fluids, we refer to a work by C. Rohde and C. Merkle which inspired this work (note the presence of a level-set function instead of a random sampling).

Outline of the talk

Moving contact discontinuity

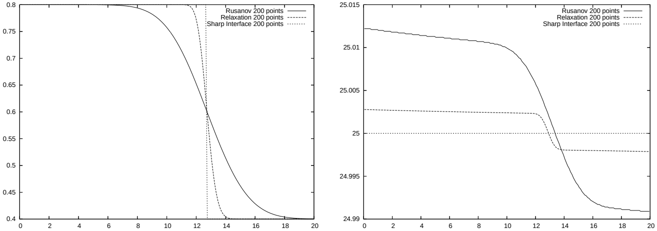


Fig.: α_1 and u_2

Shock tube problem (1D)

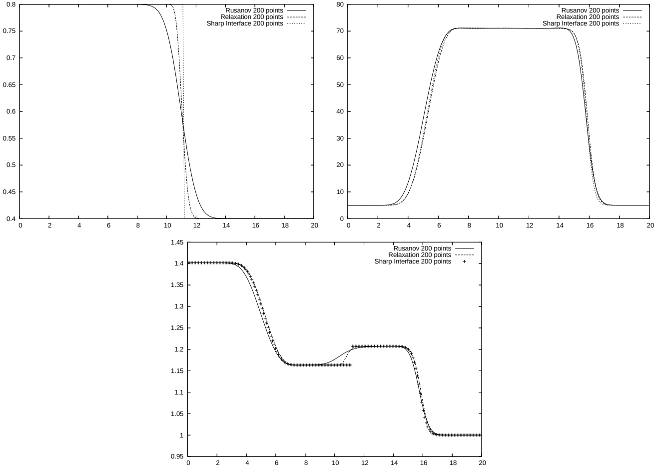


Fig.: α_1, u_2 and ρ_2

Shock tube problem (2D) - 1a

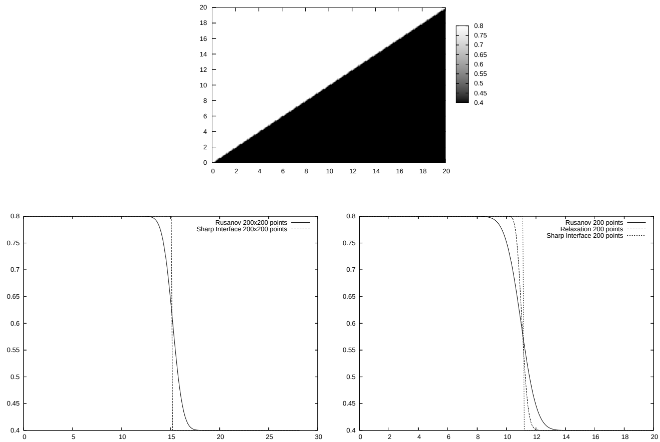


Fig.: Initial data and α_1

Shock tube problem (2D) - 1b

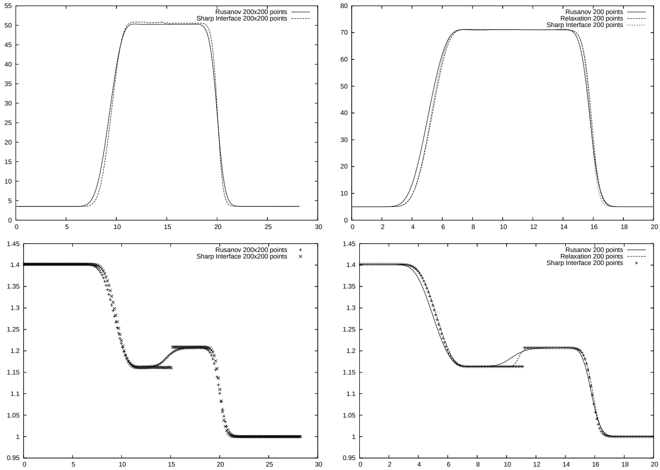
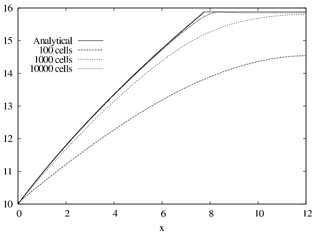


Fig.: u_2 and ρ_2

Ransom faucet test case



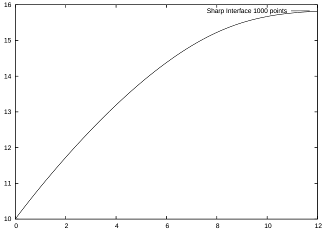


Fig.: u_2

This work was motivated by a Joint Research Group LJLL-CEA Saclay on multiphase flows and coupling of multiscale models, to which the following persons take part :

LJLL : C. Chalons, F. Coquel, E. Godlewski, F. Lagoutière, N. Seguin, P.-A. Raviart,
CEA Saclay : A. Ambroso (now Areva), B. Boutin, T. Galié (now in a service company),
+ some-time participation from EDF : J.-M. Hérard, O. Hurisse.

Advertising :

NTMC'09 (New Trends in Model Coupling)

workshop on coupling problems of multiscale phenomena that are of increasing interest in Applied Mathematics...