Asymptotic behavior of complex fluid systems

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Scaled Navier-Stokes-Fourier system

Mass conservation:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,\ \ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \int_\Omega \varrho(t,\cdot)\ \mathrm{d}x = M_0$$

Balance of momentum:

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{f},\ \ \mathbf{u}|_{\partial\Omega} = 0$$

Entropy production:

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma \geq 0,\ \ \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Total energy balance:

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right)\ \mathrm{d}x = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u}\ \mathrm{d}x$$

Gibbs' relation:

$$\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta) D\left(\frac{1}{\varrho}\right)$$

Hypothesis of thermodynamic stability:

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0,\ \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0$$

Entropy production rate:

$$\sigma \geq \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right) \geq 0$$

Constitutive relations:

Newton's rheological law:

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

$$\mu = \mu(\varrho, \vartheta) > 0, \ \eta = \eta(\varrho, \vartheta) \geq 0$$

Fourier's law:

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

$$\kappa = \kappa(\varrho, \vartheta) > 0$$

Conservative driving force

- $\Omega \subset R^3$ a bounded (Lipschitz) domain
- $\mathbf{f} = \nabla_x F$, $F = F(x)$, $F \in W^{1,\infty}(\Omega)$
- $\partial_\varrho p(0, \vartheta) > 0$ for any $\vartheta > 0$

$$\varrho(t, \cdot) \to \tilde{\varrho} \text{ in } L^p(\Omega) \text{ as } t \to \infty$$

$$(\varrho \mathbf{u})(t, \cdot) \to 0 \text{ in } L^p(\Omega; R^3) \text{ as } t \to \infty$$

$$\vartheta(t, \cdot) \to \overline{\vartheta} > 0 \text{ in } L^p(\Omega) \text{ as } t \to \infty$$

Static problem:

$$\nabla_x p(\tilde{\varrho}, \overline{\vartheta}) = \tilde{\varrho} \nabla_x F, \ \int_\Omega \tilde{\varrho} \, \mathrm{d}x = M_0, \ \int_\Omega \Big(\tilde{\varrho} e(\tilde{\varrho}, \overline{\vartheta}) - \tilde{\varrho} F \Big) \, \mathrm{d}x = E_0$$

Uniform stabilization:

- $\int_\Omega \varrho \, \mathrm{d}x \geq M_0 > 0$
- $\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F \right) \, \mathrm{d}x \leq E_0$
- $\int_\Omega \varrho s(\varrho,\vartheta) \, \mathrm{d}x \geq S_0$

For any $\varepsilon > 0$ there is $T = T(\varepsilon)$ such that

$$\|\varrho(t,\cdot) - \tilde{\varrho}\|_{L^p(\Omega)} < \varepsilon$$

$$\|(\varrho\mathbf{u})(t,\cdot)\|_{L^p(\Omega;R^3)} < \varepsilon$$

$$\|\vartheta(t,\cdot) - \overline{\vartheta}\|_{L^p(\Omega)} < \varepsilon$$

for all $t > T(\varepsilon)$

Non-conservative stationary driving forces

- $\Omega \subset R^3$ a bounded (Lipschitz) domain
- $\mathbf{f} = \mathbf{f}(x)$, $\mathbf{f} \not\equiv \nabla_x F$

Total energy "blow up":

$$E(t) = \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \, \mathrm{d}x \to \infty$$

as $t \to \infty$

General bounded driving force

- $\Omega \subset R^3$ a bounded (Lipschitz) domain
- $\mathbf{f} \in L^\infty((0,\infty) \times \Omega; R^3)$

Either

$$E(t) = \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \ \mathrm{d}x \to \infty$$

as $t \to \infty$;

or

$$E(t) \leq E_\infty \text{ for a.a. } t > 0$$

In the later case, for any sequence $t_n \to \infty$, there is an $F = F(x)$ such that

$$\mathbf{f}(t_{n,k} + \cdot, \cdot) \to \nabla_x F \text{ weakly-(*) in } L^\infty((0,T) \times \Omega; R^3)$$

Highly oscillating driving forces

- $\Omega \subset R^3$ a bounded (Lipschitz) domain
- $$\mathbf{f} = \omega(t^\beta)\mathbf{w}(x),\ \beta > 2$$
$$\omega \in L^\infty(0,\infty),\ \omega \neq 0,\ \sup_{\tau>0} \left| \int_0^\tau \omega(t)\ \mathrm{d}t \right| < \infty$$

$$\varrho\mathbf{u}(t,\cdot) \to 0 \text{ in } L^p(\Omega;R^3)$$

$$\varrho(t,\cdot) \to \overline{\varrho} \text{ in } L^p(\Omega),\ M_0 = \overline{\varrho}|\Omega|$$

$$\vartheta(t,\cdot) \to \overline{\vartheta} \text{ in } L^p(\Omega)$$

as $t \to \infty$
