Theory and Applications of Local Discontinuous-Galerkin Schemes in Phase Transition Theory

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A1: Numerical Solution of the NSK-System

Model problems in phase transition theory

$$\boxed{\begin{array}{rcrcl} \rho_t & + & \nabla\cdot(\rho\mathbf{v}) & = & 0 \\ (\rho\mathbf{v})_t & + & \nabla\cdot(\rho\mathbf{v}\mathbf{v}^T) + \nabla p(\rho) & = & \varepsilon\nabla\cdot\boldsymbol{\tau} + \lambda\rho\nabla(D^\varepsilon[\rho]) \end{array}} \quad \text{in } \mathbb{R}^d\times(0,T)$$

with either the local version $\qquad D^\varepsilon_{local}[\rho] = \varepsilon^2\Delta\rho$

or the non-local version $\qquad D^\varepsilon_{global}[\rho] = \gamma(\Phi_\varepsilon * \rho - \rho)$

Convolution $[\Phi_\varepsilon * \rho](x) = \int_{\mathbb{R}^d} \Phi_\varepsilon(x-y)\rho(y)\,dy$
with a symmetric, non-negative kernel function satisfying
$\Phi_\varepsilon(x) = \frac{1}{\varepsilon^d}\Phi(\frac{x}{\varepsilon})$ and $\int_{\mathbb{R}^d}\Phi(x)\,dx = 1$.

$$\begin{aligned} D^\varepsilon_{global}[\rho](x) &= \gamma\int_{\mathbb{R}} \Phi_\varepsilon(x-y)[\rho(y)-\rho(x)]\,dy \\ &\approx \gamma\int_{\mathbb{R}} \frac{1}{\varepsilon}\Phi\left(\frac{x-y}{\varepsilon}\right)\left[\rho_x(x)(y-x) + \rho_{xx}(x)\frac{1}{2}(y-x)^2\right]dy \\ &= \varepsilon^2\rho_{xx}(x)\,\frac{\gamma}{2}\int_{\mathbb{R}}\Phi(z)z^2\,dz \qquad \left(\text{set } \gamma := \frac{2}{\int_{\mathbb{R}}\Phi(z)z^2\,dz}\right) \\ &= D^\varepsilon_{local}[\rho](x) \end{aligned}$$

Outline

1. Scalar model problem
2. Elasticity system
3. Navier-Stokes-Korteweg system

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1. Scalar model problem
2. Elasticity system
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2. Scalar model problem

$$\boxed{u_t + f(u)_x = \varepsilon u_{xx} + \lambda D^{\varepsilon}_{local/global}[u]_x} \quad \text{in } \mathbb{R} \times (0, T)$$

Unknown: $u = u(x, t) \in \mathbb{R}$
Given parameters: $\lambda > 0$, parameter $\varepsilon > 0$ scales between diffusion and dispersion

Local version: $D^{\varepsilon}_{local}[u] = \varepsilon^2 u_{xx}$
(Jacobs&McKinney&Shearer '95, Hayes&LeFloch '97, . . .)

Non-local version: $D^{\varepsilon}_{global}[u] = \gamma(\Phi_{\varepsilon} * u - u)$
(Rohde '05)

2. Scalar model problem

$$\boxed{u_t + f(u)_x = \varepsilon u_{xx} + \lambda D^{\varepsilon}_{local/global}[u]_x} \quad \text{in } \mathbb{R} \times (0, T)$$

Unknown: $u = u(x, t) \in \mathbb{R}$
Given parameters: $\lambda > 0$, parameter $\varepsilon > 0$ scales between diffusion and dispersion

Local version: $$D^{\varepsilon}_{local}[u] = \varepsilon^2 u_{xx}$$

$$\begin{aligned} u_t + (f(u) - \varepsilon q - \lambda \varepsilon^2 p)_x &= 0 \\ q - u_x &= 0 \\ p - q_x &= 0 \end{aligned}$$

2. Scalar model problem

$$\boxed{u_t + f(u)_x = \varepsilon u_{xx} + \lambda D^{\varepsilon}_{local/global}[u]_x} \quad \text{in } \mathbb{R} \times (0, T)$$

Unknown: $u = u(x,t) \in \mathbb{R}$
Given parameters: $\lambda > 0$, parameter $\varepsilon > 0$ scales between diffusion and dispersion

Local version: $$D^{\varepsilon}_{local}[u] = \varepsilon^2 u_{xx}$$

$$\begin{aligned}
0 = \int_{I_j} u_{h,t}\phi \, dx - \int_{I_j} (f(u_h) - \varepsilon q_h - \lambda\varepsilon^2 p_h)\phi_x \, dx \\
+ (\tilde{f}_{j+1/2} - \varepsilon \tilde{q}_{j+1/2} - \lambda\varepsilon^2 \tilde{p}_{j+1/2})\phi(x_{j+1/2}) \\
- (\tilde{f}_{j-1/2} - \varepsilon \tilde{q}_{j-1/2} - \lambda\varepsilon^2 \tilde{p}_{j-1/2})\phi(x_{j-1/2})
\end{aligned}$$

$$0 = \int_{I_j} q_h\phi \, dx + \int_{I_j} u_h\phi_x \, dx - \tilde{u}_{j+1/2}\phi(x_{j+1/2}) + \tilde{u}_{j-1/2}\phi(x_{j-1/2})$$

$$0 = \int_{I_j} p_h\phi \, dx + \int_{I_j} q_h\phi_x \, dx - \tilde{q}_{j+1/2}\phi(x_{j+1/2}) + \tilde{q}_{j-1/2}\phi(x_{j-1/2})$$

2. Scalar model problem

$$\boxed{u_t + f(u)_x = \varepsilon u_{xx} + \lambda D^{\varepsilon}_{local/global}[u]_x} \quad \text{in } \mathbb{R} \times (0, T)$$

Unknown: $u = u(x, t) \in \mathbb{R}$
Given parameters: $\lambda > 0$, parameter $\varepsilon > 0$ scales between diffusion and dispersion

Non-local version: $$D^{\varepsilon}_{global}[u] = \gamma(\Phi_{\varepsilon} * u - u)$$

- Derivation of LDG-schemes, especially for the non-local version two different schemes
- Several numerical experiments

2. Scalar model problem

$$\boxed{u_t + f(u)_x = \varepsilon u_{xx} + \lambda D^{\varepsilon}_{local/global}[u]_x} \quad \text{in } \mathbb{R} \times (0, T) \qquad (*)$$

For any smooth solution u of $(*)$ that decays sufficiently fast together with its spatial derivatives as $x \to \pm\infty$ we have

$$\frac{d}{dt} \int_{\mathbb{R}} \frac{u^2}{2} \, dx + \varepsilon \int_{\mathbb{R}} u_x^2 \, dx = 0.$$

Theorem: (L^2-stability)
The LDG-solution u_h of $(*)$ is L^2-stable

$$\frac{d}{dt} \int_{\mathbb{R}} \frac{u_h^2}{2} \, dx \leq 0.$$

2. Scalar model problem

$$\boxed{u_t + f(u)_x = \varepsilon u_{xx} + \lambda D^{\varepsilon}_{local}[u]_x} \quad \text{in } (0,1) \times (0,T) \qquad (*)$$

Theorem: (L^2-error estimate)
Take $f(u) = au,\ a \in \mathbb{R}$. Let $u \in C^{\mathbf{p}+3}([0,1] \times [0,T])$ be a solution of $(*)$ and $u_h(.,t) \in \mathcal{V}_h^{\mathbf{p}}$ be the LDG-solution of $(*)$, where we consider special numerical fluxes. Then there exists a constant $C = C(\mathbf{p}, a, \varepsilon, \lambda, T) > 0$

$$\|u - u_h\|_{L^2(0,1)} \leq Ch^{\mathbf{p}+1/2}$$

holds true for all $t \in [0,T]$.

(Proof similar as in Yan&Shu '02 for $u_t + u_x + u_{xxx} = 0$)

2. Scalar model problem

$$\boxed{u_t + f(u)_x = \varepsilon u_{xx} + \lambda D^{\varepsilon}_{global}[u]_x} \quad \text{in } (0,1) \times (0,T) \qquad (*)$$

Theorem: (L^2-error estimate)
Take $f(u) = au,\ a \in \mathbb{R}$. Let $u \in C^{\mathbf{p}+2}([0,1] \times [0,T])$ be a solution of $(*)$ and $u_h(.,t) \in \mathcal{V}_h^{\mathbf{p}}$ be the LDG-solution of $(*)$, where we consider special numerical fluxes. Furthermore let Φ be an even function from $W^{1,\infty}(\mathbb{R})$ with compact support. Then there exists a constant $C = C(\mathbf{p}, a, \varepsilon, \lambda, \gamma, T) > 0$

$$\|u - u_h\|_{L^2(0,1)} \leq Ch^{\mathbf{p}+1/2}$$

holds true for all $t \in [0,T]$.

2. Scalar model problem

Convergence order due to L^∞-estimates for the projection operators $\mathcal{P}$ and $\mathcal{S}$ defined by

$$\int_{I_j} \mathcal{P}w(x)\phi_h(x)\,dx = \int_{I_j} w(x)\phi_h(x)\,dx \quad \forall \phi_h \in \mathbb{P}^{\mathbf{p}} \quad (L^2\text{-projection}),$$

$$\int_{I_j} \mathcal{S}w(x)\phi_h(x)\,dx = \int_{I_j} w(x)\phi_h(x)\,dx \quad \forall \phi_h \in \mathbb{P}^{\mathbf{p}-1},\ \mathcal{S}w(x_{j-1/2}^+) = w(x_{j-1/2}^+).$$

To prove the non-local error estimate we need:

- $[\Phi_\varepsilon * (\mathcal{S}w - w)](x_{j-1/2}) \le C(\mathbf{p}) \|w^{(\mathbf{p}+1)}\|_{L^\infty(0,1)} \|\Phi_\varepsilon\|_{L^1(\mathbb{R})} h^{\mathbf{p}+1}$
- $\sum_{j=1}^{N} \int_{I_j} [\Phi_\varepsilon * (\mathcal{S}w - w)](x)(\mathcal{S}w - w_h)_x(x)\,dx$
  $\le \frac{1}{2} \int_0^1 (\mathcal{S}w - w_h)^2(x)\,dx + C(\mathbf{p}) \|w^{(\mathbf{p}+1)}\|^2_{L^\infty(0,1)} \|\Phi'_\varepsilon\|^2_{L^\infty(\mathbb{R})} h^{2\mathbf{p}+2}$

(work submitted to Proceedings of Hyp2008)

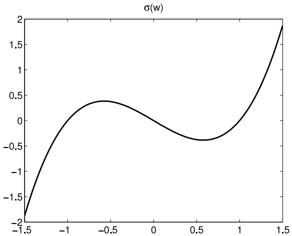
Outline

1. Scalar model problem
2. Elasticity system
3. Navier-Stokes-Korteweg system

3. Elasticity system

$$\boxed{\begin{array}{rcccl} w_t & - & v_x & = & 0 \\ v_t & - & \sigma(w)_x & = & \varepsilon v_{xx} - \lambda D^{\varepsilon}_{local/global}[w]_x \end{array}} \quad \text{in } \mathbb{R} \times (0, T)$$

Unknowns: stress $w = w(x,t) \in \mathbb{R}$, velocity $v = v(x,t) \in \mathbb{R}$
Given: stress-strain-relation $\sigma(w) = w^3 - w$



Local version:

$$D^{\varepsilon}_{local}[w] = \varepsilon^2 w_{xx}$$

Non-local version:

$$D^{\varepsilon}_{global}[w] = \gamma(\Phi_\varepsilon * w - w)$$

3. Elasticity system

$$\boxed{\begin{array}{rcrcl} w_t & - & v_x & = & 0 \\ v_t & - & \sigma(w)_x & = & \varepsilon v_{xx} - \lambda D^{\varepsilon}_{local/global}[w]_x \end{array}} \quad \text{in } \mathbb{R} \times (0, T)$$

Classical solutions satisfy the following energy inequality

$$\frac{d}{dt}\left(\int_{\mathbb{R}} \left(\frac{1}{2}|v(x)|^2 + W(w(x)) + E^{\varepsilon}[w](x) \right) dx \right) \leq 0$$

with

$$E^{\varepsilon}[w](x) = \begin{cases} \frac{1}{2}\lambda\varepsilon^2 |w_x(x)|^2 & \text{(local case)} \\ \frac{1}{4}\lambda\gamma \int_{\mathbb{R}} \Phi_{\varepsilon}(x-y)[w(y) - w(x)]^2 \, dy & \text{(non-local case)} \end{cases}$$

Theorem: (Discrete energy estimate for the non-local elasticity system)
The LDG-solution for the non-local elasticity sytem satisfies

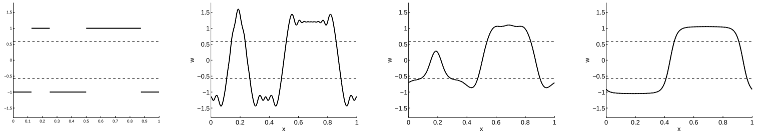
$$\frac{d}{dt}\left(\int_{\mathbb{R}} \left(\frac{1}{2}|v_h(x)|^2 + W(w_h(x)) + \frac{1}{4}\lambda\gamma \sum_{k \in \mathbb{Z}} h\Phi^h_{\varepsilon}(x - x_k)[w(x_k) - w(x)]^2 \right) dx \right) \leq 0.$$

(see also Dressel&Rohde '07)

3. Elasticity system

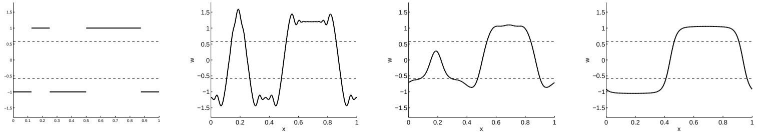
Test: Phase separation

Local case



$t = 0$ $t = 0.01$ $t = 0.1$ $t = 5$

Non-local case



$t = 0$ $t = 0.01$ $t = 0.1$ $t = 5$

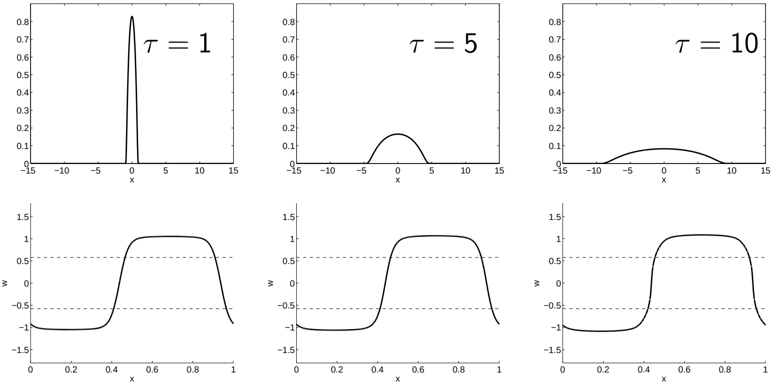
3. Elasticity system

Test: Phase separation, $\gamma(\Phi_\varepsilon * w - w) \approx \varepsilon^2 w_{xx}$

3. Elasticity system

Test: Phase separation, $\gamma(\tau)(\Phi_\varepsilon^\tau * w - w) \approx \varepsilon^2 w_{xx}$, $\gamma(\tau) = \frac{2}{\int_{\mathbb{R}} \Phi^\tau(x) x^2 \, dx}$

$$\boxed{\begin{array}{cccl} w_t & - & v_x & = 0 \\ v_t & - & \sigma(w)_x & = \varepsilon v_{xx} - \lambda\gamma(\tau)(\Phi_\varepsilon^\tau * w - w)_x \end{array}}$$



$t = 5$ $t = 5$ $t = 5$

Outline

1. Scalar model problem
2. Elasticity system
3. Navier-Stokes-Korteweg system

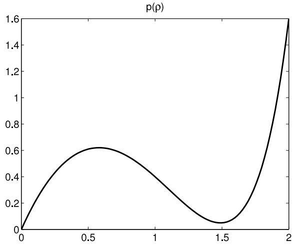
4. Navier-Stokes-Korteweg system

$$\boxed{\begin{array}{ccccc} \rho_t & + & \nabla\cdot(\rho\mathbf{v}) & = & 0 \\ (\rho\mathbf{v})_t & + & \nabla\cdot(\rho\mathbf{v}\mathbf{v}^T) + \nabla p(\rho) & = & \varepsilon\nabla\cdot\boldsymbol{\tau} + \lambda\rho\nabla(D^{\varepsilon}_{local/global}[\rho]) \end{array}}$$

Unknowns: density $\rho = \rho(x,t) > 0$, velocity $\mathbf{v} = \mathbf{v}(x,t) \in \mathbb{R}^d$

Given: van-der-Waals pressure $p = p(\rho) = \frac{RT^*\rho}{b-\rho} - a\rho^2$

Navier-Stokes tensor $\boldsymbol{\tau} = \mu(\nabla\mathbf{v} + \nabla\mathbf{v}^T) + \lambda(\nabla\cdot\mathbf{v})\mathbf{I}$



Local version:

$$D^{\varepsilon}_{local}[\rho] = \varepsilon^2\Delta\rho$$

Non-local version:

$$D^{\varepsilon}_{global}[\rho] = \gamma(\Phi_{\varepsilon} * \rho - \rho)$$

4. Navier-Stokes-Korteweg system

Dennis Diehl: LDG-code for the local NSK-system

Convolution $\Phi_\varepsilon * \rho$ in the LDG-scheme:

$$\int_{\Delta_j} [\Phi_\varepsilon * \rho_h(., t)](\mathbf{x})\phi_h(\mathbf{x})\, d\mathbf{x} = \int_{\Delta_j} \left(\int_{\mathbb{R}^d} \Phi_\varepsilon(\mathbf{x} - \mathbf{y})\rho_h(\mathbf{y}, t)\, d\mathbf{y} \right) \phi_h(\mathbf{x})\, d\mathbf{x}$$

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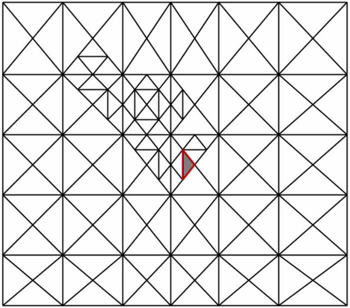
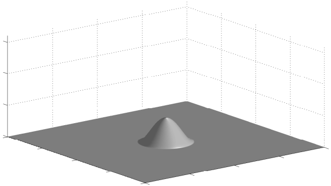
- Identify the essential triangles





4. Navier-Stokes-Korteweg system

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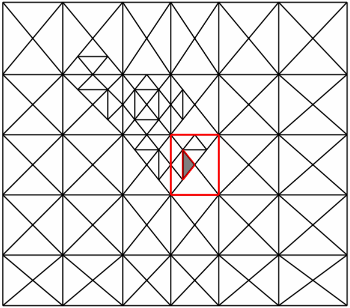

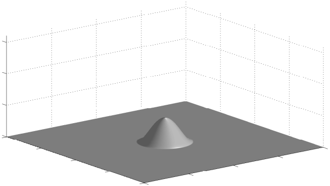

4. Navier-Stokes-Korteweg system

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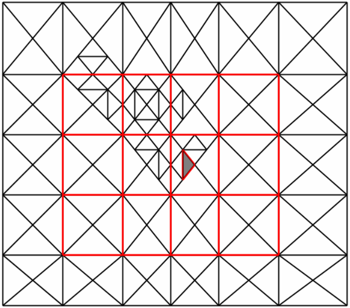
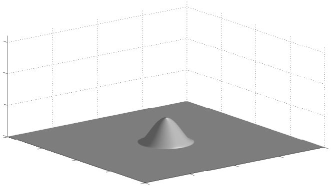
- Identify the essential triangles





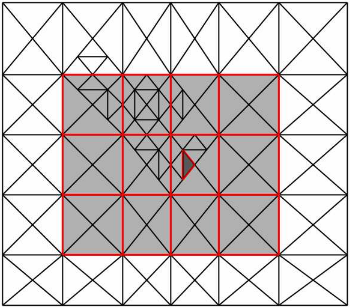
4. Navier-Stokes-Korteweg system

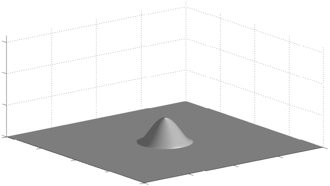
Dennis Diehl: LDG-code for the local NSK-system

Convolution $\Phi_\varepsilon * \rho$ in the LDG-scheme:

$$\int_{\Delta_j} [\Phi_\varepsilon * \rho_h(.,t)](\mathbf{x})\phi_h(\mathbf{x})\, d\mathbf{x} = \int_{\Delta_j} \left(\int_{\mathbb{R}^d} \Phi_\varepsilon(\mathbf{x}-\mathbf{y})\rho_h(\mathbf{y},t)\, d\mathbf{y} \right) \phi_h(\mathbf{x})\, d\mathbf{x}$$

- Identify the essential triangles





4. Navier-Stokes-Korteweg system

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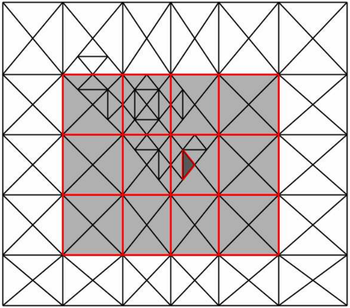
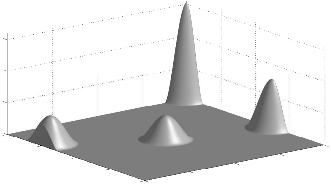
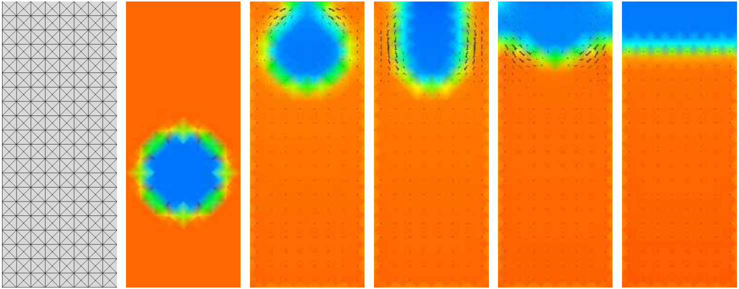
- Identify the essential triangles
- Steepen kernel next to boundaries





4. Navier-Stokes-Korteweg system

Test 1: Rising bubble in liquid (grid convergence)



1600 cells $t = 0$ $t = 32.2$ $t = 32.8$ $t = 35$ $t = 50$

4. Navier-Stokes-Korteweg system

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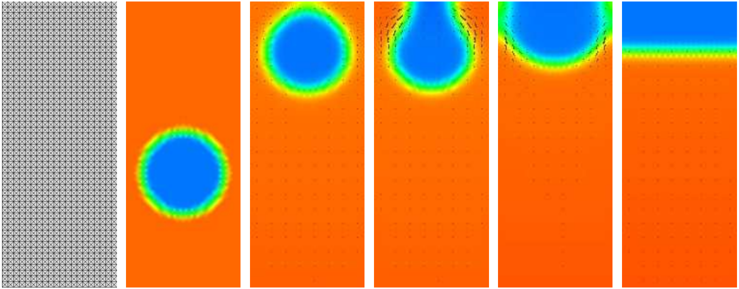
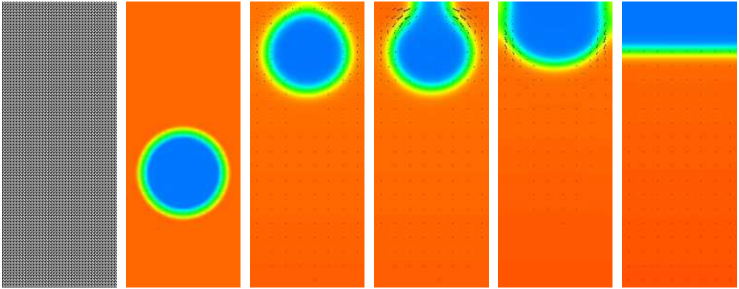


10000 cells $\quad t = 0 \quad t = 32.2 \quad t = 32.8 \quad t = 35 \quad t = 50$

4. Navier-Stokes-Korteweg system

Test 1: Rising bubble in liquid (grid convergence)

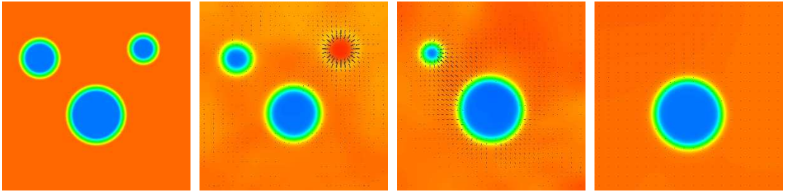


40000 cells $t = 0$ $t = 32.2$ $t = 32.8$ $t = 35$ $t = 50$

4. Navier-Stokes-Korteweg system

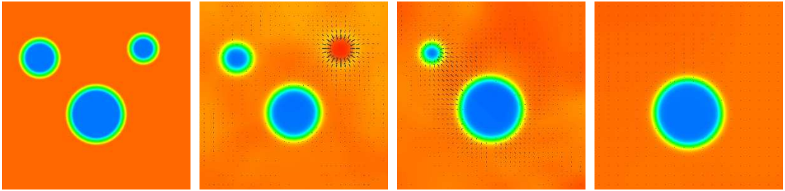
Test 2: Three bubbles in liquid

Local NSK



$t = 0$ $\quad$ $t = 0.55$ $\quad$ $t = 1.45$ $\quad$ $t = 10$

Non-local NSK

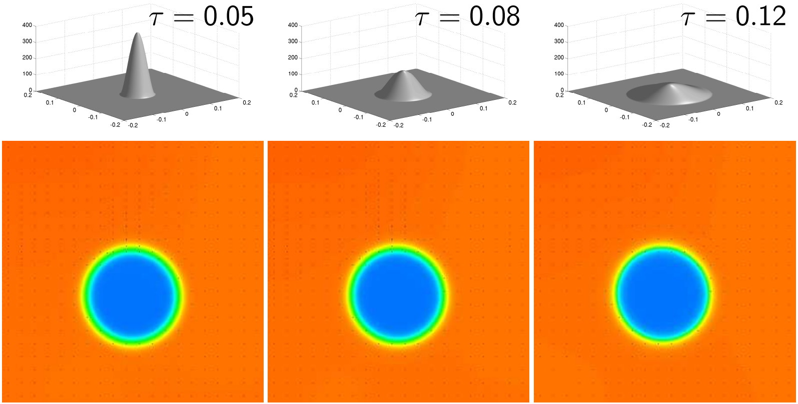


$t = 0$ $\quad$ $t = 0.55$ $\quad$ $t = 1.45$ $\quad$ $t = 10$

4. Navier-Stokes-Korteweg system

$$\gamma(\tau)(\Phi_\varepsilon^\tau * \rho - \rho) \approx \varepsilon^2 \Delta\rho$$

$$\Phi^\tau(\mathbf{x}) = \begin{cases} \frac{3}{\pi\tau^4}\left(\tau - \frac{|\mathbf{x}|^2}{\tau}\right)^2 & \text{for } |\mathbf{x}| < \tau, \\ 0 & \text{otherwise.} \end{cases} \quad \Rightarrow \quad \gamma(\tau) = \frac{16}{\tau^2}$$



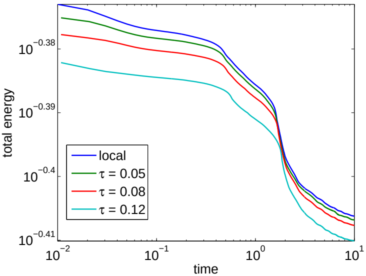
4. Navier-Stokes-Korteweg system

Energy inequality:

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d} \left(\frac{1}{2}\rho(\mathbf{x})|\mathbf{v}(\mathbf{x})|^2 + W(\rho(\mathbf{x})) + E^\varepsilon[\rho](\mathbf{x}) \right) \, d\mathbf{x} \right) \leq 0$$

with

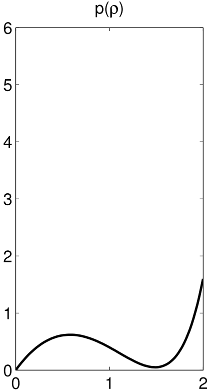
$$E^\varepsilon[\rho](\mathbf{x}) = \begin{cases} \frac{1}{2}\lambda\varepsilon^2|\nabla\rho(\mathbf{x})|^2 & \text{(local NSK)} \\ \frac{1}{4}\lambda\gamma \int\limits_{\mathbb{R}^d} \Phi_\varepsilon^\tau(\mathbf{x}-\mathbf{y})[\rho(\mathbf{y}) - \rho(\mathbf{x})]^2 \, d\mathbf{y} & \text{(non-local NSK)} \end{cases}$$



4. Navier-Stokes-Korteweg system

Momentum balance equation:

$$(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \mathbf{v}^T) + \nabla p(\rho) = \varepsilon \nabla \cdot \boldsymbol{\tau} + \lambda \gamma \rho \nabla (\Phi_\varepsilon^\tau * \rho - \rho)$$

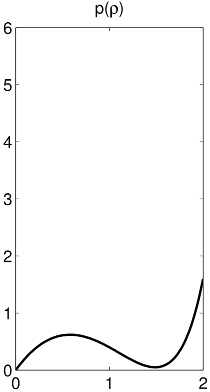


Remark: The Euler-part of the NSK-system is hyperbolic for increasing branches of p.

4. Navier-Stokes-Korteweg system

Momentum balance equation:

$$(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v}\mathbf{v}^T) + \nabla p(\rho) = \varepsilon \nabla \cdot \boldsymbol{\tau} + \lambda \gamma \rho \nabla (\Phi_\varepsilon^\tau * \rho - \rho)$$

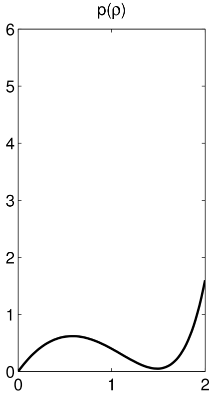


Remark: The Euler-part of the NSK-system is hyperbolic for increasing branches of p.

4. Navier-Stokes-Korteweg system

Momentum balance equation:

$$(\rho \mathbf{v})_t + \nabla\cdot(\rho \mathbf{v}\mathbf{v}^T) + \nabla\Big(p(\rho) + \frac{1}{2}\lambda\gamma\rho^2\Big) = \varepsilon\nabla\cdot\boldsymbol{\tau} + \lambda\gamma\rho\nabla(\Phi_\varepsilon^\tau * \rho)$$

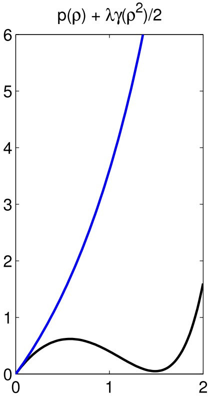


Remark: The Euler-part of the NSK-system is hyperbolic for increasing branches of p.

4. Navier-Stokes-Korteweg system

Momentum balance equation:

$$(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v}\mathbf{v}^T) + \nabla \Big(p(\rho) + \frac{1}{2}\lambda\gamma\rho^2 \Big) = \varepsilon \nabla \cdot \boldsymbol{\tau} + \lambda\gamma\rho\nabla(\Phi_\varepsilon^\tau * \rho)$$



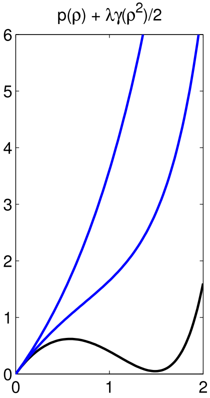
τ	$\gamma(\tau) = \frac{16}{\tau^2}$
0.05	6400

Remark: The Euler-part of the NSK-system is hyperbolic for increasing branches of p.

4. Navier-Stokes-Korteweg system

Momentum balance equation:

$$(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v}\mathbf{v}^T) + \nabla \Big(p(\rho) + \frac{1}{2}\lambda\gamma\rho^2 \Big) = \varepsilon \nabla \cdot \boldsymbol{\tau} + \lambda\gamma\rho\nabla(\Phi_\varepsilon^\tau * \rho)$$



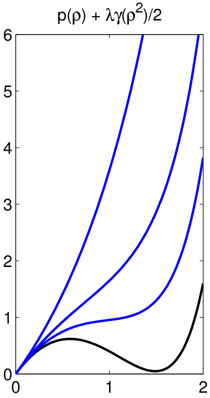
τ	$\gamma(\tau) = \frac{16}{\tau^2}$
0.05	6400
0.08	2500

Remark: The Euler-part of the NSK-system is hyperbolic for increasing branches of p.

4. Navier-Stokes-Korteweg system

Momentum balance equation:

$$(\rho \mathbf{v})_t + \nabla\cdot(\rho \mathbf{v}\mathbf{v}^T) + \nabla\Big(p(\rho) + \frac{1}{2}\lambda\gamma\rho^2\Big) = \varepsilon\nabla\cdot\boldsymbol{\tau} + \lambda\gamma\rho\nabla(\Phi_\varepsilon^\tau * \rho)$$



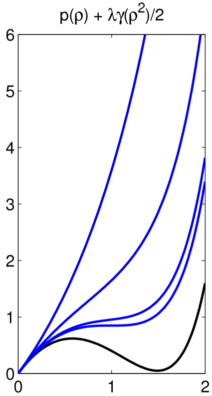
τ	$\gamma(\tau) = \frac{16}{\tau^2}$
0.05	6400
0.08	2500
0.12	1111

Remark: The Euler-part of the NSK-system is hyperbolic for increasing branches of p.

4. Navier-Stokes-Korteweg system

Momentum balance equation:

$$(\rho \mathbf{v})_t + \nabla\cdot(\rho \mathbf{v}\mathbf{v}^T) + \nabla\Big(p(\rho) + \frac{1}{2}\lambda\gamma\rho^2\Big) = \varepsilon\nabla\cdot\boldsymbol{\tau} + \lambda\gamma\rho\nabla(\Phi_\varepsilon^\tau * \rho)$$



τ	$\gamma(\tau) = \frac{16}{\tau^2}$
0.05	6400
0.08	2500
0.12	1111
$0.1\overline{3}$	900

Remark: The Euler-part of the NSK-system is hyperbolic for increasing branches of p.

4. Navier-Stokes-Korteweg system

Energy inequality:

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d}\left(W(\rho(\mathbf{x}))+\frac{1}{2}\lambda\gamma(\rho(\mathbf{x}))^2-\frac{1}{2}\lambda\gamma[\Phi_\varepsilon*\rho](\mathbf{x})\rho(\mathbf{x})\right)\,d\mathbf{x}\right)\leq 0$$

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d}\left(W(\rho(\mathbf{x}))+\frac{1}{4}\lambda\gamma\int_{\mathbb{R}^d}\Phi_\varepsilon(\mathbf{x}-\mathbf{y})[\rho(\mathbf{y})-\rho(\mathbf{x})]^2\,d\mathbf{y}\right)\,d\mathbf{x}\right)\leq 0$$

$\Rightarrow$ Energy density function W "convexified" to $W^* := W + \frac{1}{2}\lambda\gamma\rho^2$ (see also Brandon&Lin&Rogers '95)

Remark: This does not work for the local NSK-system

Remarks about future work

- Numerical tests for interfaces as thin as possible (increase τ)
- Comparison of numerical results and experimental data
- Investigate the contact angle in the non-local model
- Existence of weak solutions for the boundary value problem of the NSK-system (with Boris Haspot)