Existence of global weak solution for the Korteweg system

presented by

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Plan of the talk

1. System and classical energy inequalities.

2. Existing results.

3. Theorems of Global weak solutions.

4. Idea of the proofs.

5. Perspectives.

1 Derivation of Korteweg model

We are concerned with compressible fluids endowed with internal capillarity. The model we consider originates from the XIXth century work by van der Waals and Korteweg and was actually derived by J.- E Dunn and J. Serrin (1985)in its modern form by using the second gradient theory.
Korteweg-type models are based on an extended version of nonequilibrium thermodynamics, which assumes that the energy of the fluid not only depends on standard variables but also on the gradient of the density.
Let us now consider a fluid of density $\rho \geq 0$, velocity field $u \in \mathbb{R}^N$. The conservation of mass and of momentum write :

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \mathrm{div}(\rho u) = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu(\rho)\mathrm{D(u)}) \\ \qquad\qquad - \nabla\big(\lambda(\rho))\mathrm{div}u\big) + \nabla P(\rho) = \mathrm{div}K, \end{cases} \tag{1}$$

where the Korteweg tensor read as following :

$$\begin{aligned} \mathrm{div}K =& \nabla\big(\rho\kappa(\rho)\Delta\rho + \frac{1}{2}(\kappa(\rho) + \rho\kappa^{'}(\rho))|\nabla\rho|^2\big) \\ & - \mathrm{div}\big(\kappa(\rho)\nabla\rho \otimes \nabla\rho\big). \end{aligned}$$

κ is the coefficient of capillarity and is a regular function. P is a general increasing pressure term. $D(u) = \frac{1}{2}[\nabla u +^t \nabla u]$ being the stress tensor, μ and λ are the two Lamé viscosity coefficients depending on the density ρ and satisfying :

$$\mu > 0 \text{ and } 2\mu + N\lambda \geq 0.$$

Form of the non linearities

To simplify the presentation, we assume only that $\kappa(\rho) = \kappa\rho^\alpha$ with $\alpha \in \mathbb{R}$. The stress tensor K reads as following : We get then :

$$K_{i,j} = \big(A^1_\alpha \Delta B(\rho) - A^2_\alpha |\nabla A(\rho)|^2\big)\partial_{i,j} - B_\alpha \partial_i A(\rho)\partial_j A(\rho) \quad \text{if} \quad \alpha \neq -2,$$

$$K_{i,j} = \kappa\big(\Delta \log(\rho) + \frac{1}{2}|\nabla \log(\rho)|^2\big)\partial_{i,j} - \kappa\partial_i \log(\rho)\partial_j \log(\rho) \quad \text{if} \quad \alpha = -2.$$

with :

$$A(\rho) = \rho^{\frac{\alpha}{2}+1}, \quad B(\rho) = \rho^{2+\alpha} \quad \text{and}$$

$$A^1_\alpha = \frac{\kappa}{2+\alpha}, \quad A^2_\alpha = \frac{2\kappa(\alpha+1)}{(\alpha+2)^2}, \quad B_\alpha = \frac{4\kappa}{(\alpha+2)^2}.$$

We remark then that the form of the non linear terms appearing in the tensor K corresponds to quadratic gradient terms $\partial_i A(\rho)\partial_j A(\rho)$ and some terms in $B(\rho)$ of pressure type.

Energy inequalities

Before getting into the heart of mathematical results, one can recall first derive the physical energy bounds of the Korteweg system. Let $\bar{\rho} > 0$ be a constant reference density, and let Π be defined by :

$$\Pi(s) = s\Big(\int_{\bar{\rho}}^{s} \frac{P(z)}{z^2} dz - \frac{P(\bar{\rho})}{\bar{\rho}}\Big),$$

Multiplying the equation of momentum conservation in the system (1) by u and integrating by parts over $\mathbb{R}^N$, we obtain the following estimate :

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big(\frac{1}{2}\rho|u|^2 + (\Pi(\rho) - \Pi(\bar{\rho})) + \frac{1}{2}\kappa(\rho)|\nabla\rho|^2\big)(t)dx \\
&+ \int_0^t \int_{\mathbb{R}^N} \big(\mu(\rho)|D(u)|^2 + \xi(\rho)|\mathrm{div} u|^2\big) dxdt \leq \qquad (2)\\
&\int_{\mathbb{R}^N} \big(\frac{|m_0|^2}{2\rho} + (\Pi(\rho_0) - \Pi(\bar{\rho})) + \frac{\kappa(\rho_0)}{2}|\nabla\rho_0|^2\big)dx.
\end{aligned}$$

with $\xi(\rho) = \mu(\rho) + \lambda(\rho)$. We will note in the sequel :

$$\mathcal{E}(t) = \int_{\mathbb{R}^N} \big(\frac{1}{2}\rho|u|^2 + (\Pi(\rho) - \Pi(\bar{\rho})) + \frac{\kappa(\rho)}{2}|\nabla\rho)|^2\big)(t)dx,$$

It follows that assuming that the initial total energy is finite :

$$\mathcal{E}_0 = \int_{\mathbb{R}^N} \big(\frac{|m_0|^2}{2\rho} + (\Pi(\rho_0) - \Pi(\bar{\rho})) + \frac{\kappa(\rho_0)}{2}|\nabla\rho_0|^2\big)dx < +\infty\,,$$

then we have the a priori following bounds :

$$\Pi(\rho) - \Pi(\bar{\rho}),\ \text{ and }\ \rho|u|^2 \in L^\infty(0,\infty,L^1(\mathbb{R}^N)),$$

$$\sqrt{\kappa(\rho)}\nabla\rho \in L^\infty(0,\infty,L^2(\mathbb{R}^N))^N,\ \text{ and }\ \nabla u \in L^2(0,\infty,\mathbb{R}^N)^{N^2}.$$

In the sequel, we aim at solving the problem of global existence of weak solution for the system (1) in dimension 2.
Let assume now that we can built smooth approximate sequel solutions $(\rho_n, u_n)_{n\in\mathbb{N}}$ of system (1).
One can then observe that the main difficulty concerns the quadratic term $\nabla A(\rho_n)\otimes\nabla A(\rho_n)$ which belongs only to $L^\infty(L^1)$.
Indeed according to the classical theorems on weak topology, $\nabla A(\rho_n)\otimes\nabla A(\rho_n)$ converges up to a sequence to a measure ν.
The difficulty is how to prove that $\nu = \nabla A(\rho)\otimes\nabla A(\rho)$ where ρ is the limit of the sequence $(\rho_n)_{n\in\mathbb{N}}$ in appropriate space?

Existing results on Korteweg system

Results of strong solutions

– Existence of strong solutions in finite time for $N \geq 2$ is known since the works by H. Hattori and D. Li [98].
– R. Danchin and B. Desjardins [2002] existence of strong solution in finite time in critical spaces for the scaling of the equations, the initial data $(\rho_0, \rho_0 u_0)$ belong to $B^{\frac{N}{2}}_{2,1} \times B^{\frac{N}{2}-1}_{2,1}$.
– M. Koschote [2007] existence of strong solution in finite time in a bounded domain.
– B. H [2007] existence in finite time of strong solution in critical spaces for the scaling of the equations for the non-isothermal system, the initial data verify
$(\rho_0, \rho_0 u_0, \theta_0) \in B^{\frac{N}{2}}_{2,1} \times B^{\frac{N}{2}-1}_{2,1} \times B^{\frac{N}{2}-2}_{2,1}$.

Results of global weak solution

– R. Danchin and B. Desjardins [2002] existence of global weak solution in dimension $N = 2$ with small initial data if $\frac{1}{\rho}$ and $\rho \in L^\infty$.
– D. Bresch, B. Desjardins and C-K. Lin existence of global weak solution for $N \geq 2$ when $\mu(\rho) = \mu\rho$ and $\lambda(\rho) = 0$. The test functions depend on the density.

2 Theorems of existence of global weak solutions

2.1 Gain of derivability on the density

Theorem 1. *Let $N=2$ and (ρ,u) be a smooth approximate solution of the system* (1) *with $\kappa(\rho)=\kappa\rho^{\alpha}$ with $\alpha\in\mathbb{R}$ and $\alpha\neq-2$. We assume that if $\alpha>-2$ then $\frac{1}{\rho}\in L^{\infty}((0,T)\times\mathbb{R}^N)$ else $\alpha<-2$ then $\rho\in L^{\infty}((0,T)\times\mathbb{R}^N)$.*
Then there exists a constant $\eta>0$ depending only on the constant coming from the Sobolev embedding such that if :

$$\|\nabla\rho_0\|_{L^2(\mathbb{R}^2)}+\|\sqrt{\rho_0}|u_0|\|_{L^2(\mathbb{R}^2)}+\|j_{\gamma}(\rho_0)\|_{L^1}\leq\eta$$

then we get for all $\varphi\in C_0^{\infty}(\mathbb{R}^N)$:

$$\|\varphi B(\rho)\|_{L^2_T(H^{1+\frac{s}{2}})}\leq M\qquad\text{with }0\leq s<2,$$

where M depends only on the initial conditions data, on T, on φ, on s and on $\|\frac{1}{\rho}\|_{L^{\infty}}$ or $\|\rho\|_{L^{\infty}}$.
Remark 1. *In fact instead of supposing that $\frac{1}{\rho}\in L^{\infty}$ or $\rho\in L^{\infty}$ in theorem 1, we have just to assume that $\frac{A'(\rho)}{B'(\rho)}\nabla A(\rho)\in L^{\infty}(L^2)$.*

We want now improve this result by extracting a specific structure of the capillarity tensor. By choosing $\kappa(\rho)=\kappa\rho^{-2}$ with $\kappa>0$, we show that we get the same estimate but without any conditions on the vacuum.

Theorem 2. *Let $N = 2$, $\kappa > 0$ and (ρ, u) be a smooth approximate solution of the system (1) with $\kappa(\rho) = \kappa\rho^{-2}$. Then there exists a constant $\eta > 0$ depending only on the constant coming from the Sobolev embedding such that if :*

$$\|\nabla\rho_0\|_{L^2(\mathbb{R}^2)} + \|\sqrt{\rho_0}|u_0|\|_{L^2(\mathbb{R}^2)} + \|j_\gamma(\rho_0)\|_{L^1} \leq \eta$$

then it exists $\alpha > 0$ such that for all $\varphi \in C_0^\infty(\mathbb{R}^N)$:

$$\|\varphi B(\rho)\|^2_{L^2_T(H^{1+\frac{s}{2}})} + \|\varphi\rho^{\alpha-2}\nabla\rho\|^2_{L^2(L^2)} + \|\rho\|^{\gamma+\alpha}_{L^{\gamma+\alpha}((0,T)\times\mathbb{R}^N)} \leq M \quad \text{with } 0 \leq s \leq \varepsilon,$$

where M depends only on the initial conditions data, on T, on φ and on ε. ε depends only of γ the coefficient of the pressure and is small.

We extend this result for capillarity coefficient $\kappa(\rho)$ approximating a constant κ.

Corollary 1. *Let $N = 2$ and $\alpha, M, \kappa \in \mathbb{R}$. (ρ, u) is a smooth approximate solution of the system (1) with the following capillarity coefficient :*

$$\kappa(\rho) = \frac{1}{\rho^2}1_{\{\rho<\alpha\}} + \theta_1(\rho)1_{\{\alpha\leq\rho\leq2\alpha\}} + \kappa 1_{\{2\alpha<\rho\}}.$$

where θ_1, θ_2 are regular function such that κ is a regular function. Then there exists a constant $\eta > 0$ depending only on the constant coming from the Sobolev embedding such that if :

$$\|\nabla\rho_0\|_{L^2(\mathbb{R}^2)} + \|\sqrt{\rho_0}|u_0|\|_{L^2(\mathbb{R}^2)} + \|j_\gamma(\rho_0)\|_{L^1} \leq \eta$$

then we get for all $\varphi \in C_0^\infty(\mathbb{R}^N)$ *:*

$$\|\varphi B(\rho)\|_{L^2_T(\dot{H}^{1+\frac{s}{2}})} \leq M \quad \textit{with } \; 0 < s < 2,$$

where M *depends only on the initial conditions data, on* T*, on* φ *and on* s*.*

Proof of theorem 1 : Our goal is to get a gain of derivative on the density by using energy inequalities and by taking advantage of the term of capillarity. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have then by multiplying the momentum equation and applying the operator div (where we use the classical summation index) :

$$\partial_t \text{div}(\varphi\rho u) + \partial_{i,j}(\varphi\rho u_i u_j) - \partial_{i,j}(2\varphi\mu(\rho)Du_{i,j}) - \Delta(\varphi\lambda(\rho)\text{div}u)$$
$$+ \Delta(\varphi P(\rho)) = \Delta\left(\Delta(\varphi B(\rho)) - \varphi|\nabla A(\rho)|^2\right) + R_\varphi$$
$$- \partial^2_{i,j}(\varphi\partial_i A(\rho)\partial_j A(\rho))$$

We can apply to the momentum equation the operator $\Lambda(\Delta)^{-2}$ (where $\mathcal{F}\Lambda f = |\xi|\mathcal{F}f$).

$$\Lambda(\varphi B(\rho)) + \Lambda^{-1}(\varphi|\nabla A(\rho)|^2) + B_\alpha\Lambda^{-1}R_iR_j(\varphi\partial_i A(\rho)\partial_j A(\rho))$$
$$= -\Lambda^{-3}\frac{\partial}{\partial t}\text{div}(\varphi\rho\, u) + \Lambda^{-1}R_iR_j(\varphi\rho\, u_iu_j) - \Lambda^{-1}(\varphi\lambda(\rho)\text{div}u)$$
$$+ \Lambda^{-1}(\varphi P(\rho)) - \Lambda^{-1}R_iR_j(2\mu(\rho)Du_{i,j}) + \Lambda^{-1}(\Delta)^{-1}R_\varphi,$$

where R_i denotes the classical Riesz operator.

We multiply now the previous equality by $\Lambda^{1+s}(\varphi B(\rho))$ and we integrate on space and in time :

$$\int_0^T\int_{\mathbb{R}^N}\big[|\Lambda^{1+\frac{s}{2}}(\varphi B(\rho))|^2+(\varphi|\nabla A(\rho)|^2)\Lambda^s(\varphi B(\rho))\big]dxdt$$
$$+\int_0^T\int_{\mathbb{R}^N}\sum_{i,j}R_iR_j(\varphi\partial_i A(\rho)\partial_j A(\rho))\Lambda^s(\varphi B(\rho))dxdt=$$
$$\int_{\mathbb{R}^N}\Lambda^{-3}\text{div}(\varphi\rho\, u)\Lambda^{1+s}(\varphi B(\rho))(T)$$
$$-\Lambda^{-3}\text{div}(\varphi\rho_0\, u_0)\Lambda^{1+s}(\varphi B(\rho_0))dx$$
$$-\int_0^T\int_{\mathbb{R}^N}\big(\Lambda^{-3}\text{div}(\varphi\rho\, u)\Lambda^{1+s}\frac{\partial}{\partial t}(\varphi B(\rho))$$
$$-\varphi\lambda(\rho)\text{div}u\,\Lambda^s(\varphi B(\rho))\big)dxdt$$
$$+\int_0^T\int_{\mathbb{R}^N}\big(R_{i,j}(2\varphi\mu(\rho)Du_{i,j})\,\Lambda^s(\varphi B(\rho))$$
$$-R_iR_j(\varphi\rho\, u_iu_j)\Lambda^s(\varphi B(\rho))\big)dxdt$$
$$+\int_0^T\int_{\mathbb{R}^N}\big[\varphi P(\rho)\Lambda^s(\varphi B(\rho))+(\Delta)^{-1}R_\varphi\Lambda^s(\varphi B(\rho))\big]dxdt.$$

In order to control $\int_0^T\int_{\mathbb{R}^N}|\Lambda^{1+\frac{s}{2}}(\varphi B(\rho))|^2$, it suffices to bound all the other terms of the previous inequality. It will allow us to get a control on $\Lambda^{1+\frac{s}{2}}(\varphi B(\rho))$ and so a gain of $\frac{s}{2}$ derivative on the gradient of density $\nabla B(\rho)$.

1) $\int_0^T \int_{\mathbb{R}^N} (\varphi|\nabla A(\rho)|^2)\Lambda^s(\varphi B(\rho))$ **:**

We have by induction $\Lambda^s(\varphi B(\rho)) \in L^2(\dot{H}^{1-\frac{s}{2}})$ and by Sobolev embedding $\Lambda^s(\varphi B(\rho)) \in L^2(H^{1-\frac{s}{2}}) \hookrightarrow L^2(L^q)$ with $\frac{1}{q} = \frac{s}{4}$.
We have :

$$\varphi \nabla A(\rho) \simeq \frac{A^{'}(\rho)}{B'(\rho)} \nabla(\varphi B(\rho)).$$

We get $\varphi \nabla A(\rho) \in L^2(L^p)$. Finally by Hölder inequality $\varphi|\nabla A(\rho)|^2 \Lambda^s(\varphi B(\rho))$ is in $L^1_T(L^1(\mathbb{R}^N))$ because $\frac{1}{2} + \frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{2} - \frac{s}{4} + \frac{s}{4} = 1$ and we get more precisely :

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^N} (\varphi|\nabla A(\rho)|^2)\Lambda^s(\varphi B(\rho)) dxdt \lesssim \\ &\lesssim \|\frac{1}{\rho}\|_{L^\infty_T(L^\infty)} \|\Lambda^{1+\frac{s}{2}}(\varphi B(\rho))\|^2_{L^2_T(L^2)} \|\nabla A(\rho)\|_{L^\infty_T(L^2)}.\end{aligned}$$

By the hypothesis of smallness on the initial data we can conclude. $\square$

Proof of theorem 2 : The proof follow the same line as the proof of theorem 1 except concerning the bounds of estimate coming from the capillarity term and the fact that we lost the control of ρ in $L^\infty(L^p_{loc})$ for all $1 \leq p < +\infty$. We need then to get a gain of integrability on the density to treat the term coming of the pressure.

We apply to the momentum equation the operator $(\Delta)^{-1}\text{div}$ and we multiply par ρ^α with $\alpha > 0$ and we integrate on $(0,T)\times\mathbb{R}^N$, we get then :

$$\int_{(0,T)\times\mathbb{R}^N} \varphi\rho^{\gamma+\alpha}(x,t)dxdt + A_\alpha \int_{(0,T)\times\mathbb{R}^N} \rho^{\alpha-2}|\nabla\rho|^2 dxdt =$$
$$\int_{(0,T)\times\mathbb{R}^N} \varphi\partial_i \ln(\rho)\partial_j \ln(\rho) R_{i,j}\rho^\alpha dxdt$$
$$- \int_{\mathbb{R}^N} (\Delta)^{-1}\text{div}(\varphi\rho\, u)\rho^\alpha(T,x)dx + \cdots$$

Our goal is now to have $\rho \in L^{\gamma+\alpha}_{loc}((0,T)\times\mathbb{R}^N)$, for this we have to control all the terms on the right handside. The new difficulty consists only in the following term :

$$\int_{(0,T)\times\mathbb{R}^N} \varphi\partial_i \ln(\rho)\partial_j \ln(\rho) R_{i,j}\rho^\alpha dxdt,$$

We conclude by setting :

$$A(T) = \int_{(0,T)\times\mathbb{R}^N} \left[\varphi\rho^{\gamma+\alpha}(x,t) + \rho^{\alpha-2}|\nabla\rho|^2\right] dxdt,$$
$$B(T) = \int_0^T \int_{\mathbb{R}^N} |\Lambda^{1+\frac{s}{2}}(\varphi B(\rho))|^2 dxdt.$$

We have then the following final estimate :

$$A(T)+B(T) \lesssim \varepsilon B(T)+\varepsilon A(T)^{\frac{\alpha}{\alpha+\gamma}} B(T)^{\frac{1}{2}}+CA(T)^{\beta_1}+C^{'} B(T)^{\beta_2},$$

with $0 < \beta_1, \beta_2 < 1$. By boosbstrap we can conclude.

2.1.1 What happens when we choose a coefficient of capillarity κ large

Theorem 3. *Let $N = 2$ and $\alpha, M \in \mathbb{R}$. (ρ, u) is a smooth approximate solution of the system* (1) *with initial data in the energy space and with the following capillarity coefficient :*

$$\kappa(\rho) = \frac{1}{\rho^2} 1_{\{\rho<\alpha\}} + \theta_1(\rho) 1_{\{\alpha\leq\rho\leq 2\alpha\}} + \kappa 1_{\{2\alpha<\rho\}}.$$

where θ_1, θ_2 are regular function such that κ is a regular function. Then there exists a constant $\eta > 0$ and κ enough big depending only on the constant coming from the Sobolev embedding and on the initial data such that if :

$$\|\nabla A(\rho_0)\|_{L^2(\mathbb{R}^2)} \leq \eta$$

then we get for all $\varphi \in C_0^\infty(\mathbb{R}^N)$:

$$\|\varphi B(\rho)\|_{L^2_T(\dot{H}^{1+\frac{s}{2}})} \leq M \quad \text{with } 0 < s < 2,$$

where M depends only on the initial conditions data, on T, on φ and on s.

2.2 Control of the density without any conditions of smallness on the initial data

We are interesting in getting new energy inequalities which take in account the support of the initial data. In particular we want investigate what happened when we choose initial data localize in small ball. So for making we let $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi = 1$ on $B(x_0, R)$ and supp$\varphi \subset B(x_0, 2R)$. Multiplying the equation of momentum conservation in the system (1) by φu and integrating by parts over $(0,t) \times \mathbb{R}^N$, we obtain the following estimate :

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N} \varphi(t,x)\big(\rho|u|^2(t,x) + \frac{\kappa(\rho)}{2}|\nabla\rho|^2\big)(t,x)dx \\
&+ \int_{R^N} \varphi(x)\big(\Pi(\rho) - \Pi(\bar{\rho})\big)(t,x)dx + \mu\int_0^t \int_{\mathbb{R}^N} |\nabla u|^2 \varphi dx \\
&+ (\lambda+\mu)\int_0^t \int_{\mathbb{R}^N} |\text{div} u|^2 \varphi dx \leq \int_{\mathbb{R}^N} \frac{1}{2}\varphi(x)(\rho_0 u_0^2 \\
&+ \kappa(\rho_0)|\nabla\rho_0|^2)(x)dx + \int \varphi(x)(\Pi(\rho_0) - \Pi(\bar{\rho}))(x)dx
\end{aligned}$$

$$+\int_0^t\int_{\mathbb{R}^N}(u\cdot\nabla u).\nabla\varphi dxdt+\int_0^t\int_{\mathbb{R}^N}P(\rho)u\cdot\nabla\varphi\, dxdt$$
$$+\int_{\mathbb{R}}\int_{\mathbb{R}^N}\kappa(\rho)\nabla\rho\cdot\nabla\varphi(u\cdot\nabla\rho+\rho\mathrm{div}u)\,dx\,dt$$
$$+\int_{\mathbb{R}}\int_{\mathbb{R}^N}\big(\kappa(\rho)+\frac{1}{2}\rho\kappa^{'}(\rho)\big)|\nabla\rho|^2u\cdot\nabla\varphi dxdt+\cdots$$

Proposition 1. *Let $\alpha>0$, $\beta>0$, $\varepsilon\geq 0$, $\kappa>0$, $M>0$, θ a regular function such that :*

$$\kappa(\rho)=\frac{1}{\rho^2}1_{\{\rho<\alpha\}}+\theta(\rho)1_{\{\alpha\leq\rho<2\alpha\}}+\kappa 1_{\{2\alpha\leq\rho\leq M\}}+\frac{1}{\rho^2}1_{\{\rho>M\}}.$$

Let (ρ,u) a regular approximate solution of system (1) with large initial data in the energy space. Assuming that $u\in L^{1+\beta}_T(L^\infty)\cap L^{2+\beta}_T(L^2)$ then it exist a time $T_0>0$ such that for all compact K we have :

$$\|\varphi B(\rho\|_{L^2_{T_0}(H^{1+\frac{s}{2}})}\leq M,$$

where M depends only of the initial data, K, $\|u\|_{L^{1+\beta}_T(L^\infty)}$ and $\|u\|_{L^{2+\beta}_T(L^2)}$.

Remark 2. *In a similar way, we could show that if we control the high frequencies of $\nabla A(\rho)$ in $L^\infty(L^2)$ then we can get a gain of derivative with large initial data in the energy space.*

3 Existence of global weak solution for the Korteweg system

We may now turn to our compactness result. First, we assume that a sequence $(\rho_n, u_n)_{n\in\mathbb{N}}$ of approximate weak solutions has been constructed by a mollifying process, which have suitable regularity to justify the formal estimates like the energy estimate and the previous theorem.

Theorem 4. *Here we assume that $P(\rho) = \rho^\gamma$ with $\gamma > \frac{N}{2}$. Let $N = 2$ and assume that $\kappa(\rho) = \frac{\kappa}{\rho^2}$ with $\kappa > 0$. There exists $\eta > 0$ such that if :*

$$\|\nabla(\ln \rho_0^n)\|_{L^2} + \|\sqrt{\rho_0^n}|u_0^n|\|_{L^2} + \|j_\gamma(\rho_0^n)\|_{L^1} \leq \eta$$

then, up to a subsequence (ρ_n, u_n) converges strongly to a weak solution (ρ, u) of the system (1). Moreover we have $\nabla \log(\rho_n) \otimes \nabla \log(\rho_n)$ converges strongly in $L^1_{loc}(\mathbb{R} \times \mathbb{R}^N)$. In addition ρ check for all $\varphi \in C^0_\infty$:

$$\|\varphi \ln(\rho)\|^2_{L^2_T(H^{1+\frac{s}{2}})} + \|\varphi \rho^{\alpha-2} \nabla \rho\|^2_{L^2(L^2)} + \|\rho\|^{\gamma+\alpha}_{L^{\gamma+\alpha}((0,T)\times\mathbb{R}^N)} \leq M,$$

with s small where M depends only on the initial conditions data, on T, on φ and on s.

The next theorem treat of global weak solution for capillarity coefficients approximating a constant.

Theorem 5. *Let $N = 2$, $\alpha > 0$, $\varepsilon \geq 0$ and the following capillary coefficient :*

$$\kappa(\rho) = \frac{1}{\rho^{2+\varepsilon}} 1_{\{\rho<\alpha\}} + \theta_1(\rho) 1_{\{\alpha\leq\rho\leq 2\alpha\}} + \kappa 1_{\{\rho>2\alpha\}}.$$

where θ_1, θ_2 are regular function ssuch that κ is a regular function. There exists $\eta > 0$ such that if :

$$\|\nabla(A(\rho_0^n))\|_{L^2} + \|\sqrt{\rho_0^n}|u_0^n|\|_{L^2} + \|j_\gamma(\rho_0^n)\|_{L^1} \leq \eta$$

then, up to a subsequence (ρ_n, u_n) converges strongly to a weak solution (ρ, u) of the system (1). Moreover we have $\nabla A(\rho_n) \otimes \nabla A(\rho_n)$ converges strongly in $L^1_{loc}(\mathbb{R} \times \mathbb{R}^N)$. In addition ρ check for all $\varphi \in C^0_\infty$:

$$\|\varphi A(\rho)\|^2_{L^2_T(H^{1+\frac{s}{2}})} \leq M \quad \text{with } 0 \leq s \leq 2,$$

where M depends only on the initial conditions data, on T, on φ and on s.

Proof : The proof is based on classical Lions-Aubin theorems of compactness and in particular compact embedding of the Sobolev spaces.

We can now prove a result of existence of global weak solution in dimension $N = 1$ by using the previous gain of derivative. Let $(\rho_n, u_n)_{n\in\mathbb{N}}$ a sequel of approximate weak solutions of system (1).

Theorem 6. *Let (ρ_0^n, u_0^n) initial data of the system (1) in the energy space what it means that :*

$$\int_{\mathbb{R}} \rho_0^n |u_0^n|^2 + j_\gamma(\rho_0^n) + |\partial_x \rho_0^n|^2 dx \leq M$$

with $M > 0$.
Let $\kappa(\rho) = \frac{1}{\rho^{2+\varepsilon}} 1_{\{\rho\leq\alpha\}} + \theta(\rho) 1_{\{\alpha<\rho\leq 2\alpha\}} + \kappa 1_{\{<\rho>2\alpha\}}$ with $\varepsilon \geq 0$ and θ a regular function. Then up to a subsequence, (ρ_n, u_n) converges strongly to a weak solution (ρ, u) on $(0, T) \times \mathbb{R}$ in the sense of the distribution.
Moreover $\partial_x A(\rho_n)$ converges strongly in $L^2(\mathbb{R} \times \mathbb{R})$ to $\partial_x A(\rho)$.

4 Perspectives

1. Existence of weak solution in finite time with large initial data.

2. Existence of global weak solution with large initial data.

3. What happened for the case of capillarity constant.

4. Improve the results of Bresch, Desjardins and Lin .

5. What happened for $N \geq 3$?