Exercise sheet 2 for Friday, Nov 4, 2016

To be handed in either at the beginning of the exercise session, or before Nov 4, 9:55 a.m. at the drop box in front of room 149. You can come to the exercise session on Oct 28, 10:00 a.m. in case you have any questions regarding the lecture or this exercise.

Exercise 6. Let $\Omega = [0,1]$,

$$\Sigma_n := \Big\{ g\colon \exists \mathcal{P} \text{ partition of } \Omega\colon \#\mathcal{P} \le n,\ g = \sum_{I\in\mathcal{P}} c_I \chi_I \Big\}, \tag{1}$$

and $\sigma_n(f) := \inf_{g\in\Sigma_n} \|f-g\|_{L_\infty(\Omega)}$. Show that for $r > 0$, the approximation space

$$\mathcal{A}^r := \mathcal{A}^r\big((\Sigma_n), C(\Omega)\big) := \big\{ f \in C(\Omega)\colon \sup_n n^r \sigma_n(f) < \infty \big\}$$

is a linear space, and that

$$\|f\|_{\mathcal{A}^r} := \|f\|_{L_\infty(\Omega)} + |f|_{\mathcal{A}^r},$$

where $|f|_{\mathcal{A}^r} := \sup_n n^r \sigma_n(f)$, is a quasi-norm on $\mathcal{A}^r$. Recall that to obtain the latter, it needs to be shown that for any $f, g \in \mathcal{A}^r$ and $\lambda \in \mathbb{R}$ we have

$$\|f\|_{\mathcal{A}^r} = 0 \Rightarrow f = 0\,, \quad \|\lambda f\|_{\mathcal{A}^r} = |\lambda|\|f\|_{\mathcal{A}^r}\,, \quad \|f+g\|_{\mathcal{A}^r} \le C\big(\|f\|_{\mathcal{A}^r} + \|g\|_{\mathcal{A}^r}\big)$$

with some $C > 0$ that may depend on r, but not on f, g.

6 points

Exercise 7. Let $\Omega := [0,1]$ and

$$W^1(L_1(\Omega)) := \{ f \in L_1(\Omega)\colon\ f' \in L_1(\Omega)\}\,, \quad \|f\|_{W^1(L_1(\Omega))} := \|f\|_{L^1(\Omega)} + \|f'\|_{L^1(\Omega)}\,.$$

For the following, use without proof that $C^\infty(\Omega)$ is dense in $W^1(L_1(\Omega))$, that is, for each $f \in W^1(L_1(\Omega))$ there exists a sequence (f_n) in $C^\infty(\Omega)$ such that $\|f - f_n\|_{W^1(L_1(\Omega))} \to 0$.

(i) Show that every element of $W^1(L_1(\Omega))$ can be identified with a continuous function, and hence we can define

$$V(f,\Omega) := \sup_{0=x_0<\ldots<x_n=1} \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|$$

for $f \in W^1(L_1(\Omega))$.

Hint: Show first that for $g \in C^1(\Omega)$, we have $\sup_{x\in\Omega}|g(x)| \le \|g\|_{W^1(L_1(\Omega))}$, then use this to show that every $f \in W^1(L_1(\Omega))$ is almost everywhere equal to the uniform limit of a sequence of continuous functions.

(ii) Let $f \in W^1(L_1(\Omega))$. Show that $f \in BV(\Omega)$ and

$$V(f,\Omega) = \int_\Omega |f'|\,dx\,.$$

Hint: Use approximation by smooth functions. Note that for the integral of a continuous function $g \ge 0$, one has

$$\int_\Omega g\,dx = \sup\Big\{ \int_\Omega \varphi\,dx\colon \varphi \in \Sigma_n \text{ for some } n \text{ such that } 0 \le \varphi \le g \Big\}.$$

(iii) Show that $W^1(L_1(\Omega)) \ne BV(\Omega)$ by proving for a suitable element of $BV(\Omega)$ that it cannot have a weak derivative in $L^1(\Omega)$.

3+4+2=9 points

Exercise 8. Let $\Omega := [0,1]$, $0 < s < 1$, and $f(x) := x^s$.

(i) Show that $f \in \mathrm{Lip}(r, C(\Omega))$ if and only if $r \le s$.

(ii) Find the largest possible exponent $\alpha > 0$ such that for some $C > 0$, we have $e_n(f) \le Cn^{-\alpha}$ for all $n \in \mathbb{N}$, and determine the corresponding C. Give an explicit expression for $\sigma_n(f)$ in terms of n.

(iii) For $n \in \mathbb{N}$, determine explicitly a piecewise constant function in Σ_n as in (1) for which the minimum error $\sigma_n(f)$ is attained.

2+2+2=6 points