Exercise sheet 5 for Friday, Nov 25, 2016

To be handed in either at the beginning of the exercise session, or before Nov 25, 9:55 a.m. at the drop box in front of room 149.

Exercise 16. Let $\mathcal{T}$ be a complete tree, $\mathcal{L}(\mathcal{T})$ the set of its leaves and for each node $T \in \mathcal{T}^*$ let $\mathcal{C}(T) \geq 2$ be the set of its children.

(i) Show that the number of nodes in $\mathcal{T}$ is bounded by the number of leaves as

$$\#\mathcal{L}(\mathcal{T}) \leq \#\mathcal{T} \leq 2\#\mathcal{L}(\mathcal{T}).$$

(ii) When the sets of children have a fixed cardinality $\#\mathcal{C}(T) = M$ for each $\mathcal{T} \subset \mathcal{T}^*$ we have after m refinements

$$\#\mathcal{T} = 1 + mM \quad \text{and} \quad \#\mathcal{L}(\mathcal{T}) = m(M-1) + 1.$$

3 + 1 = 4 points

Exercise 17. Let $R(\mathcal{T})$ denote the root of the tree $\mathcal{T}$ and

$$\mathfrak{T}_n := \{\mathcal{T} \subset \mathcal{T}^* : R(\mathcal{T}) = R(\mathcal{T}^*), \#\mathcal{L}(\mathcal{T}) \leq n\}.$$

Show that a complete search through $\mathfrak{T}_n$ has exponential cost in n, i. e., $a^n \lesssim \#\mathfrak{T}_n \lesssim b^n$ for some $a, b > 1$.

5 points

Exercise 18. Let ρ be a measure on a space $Z := X \times Y$, $\mathrm{d}\rho(x,y) = \mathrm{d}\rho(y|x)\mathrm{d}\rho_X(x)$ and

$$f_\rho(x) := \mathbb{E}(y|x) = \int_Y y \,\mathrm{d}\rho(y|x).$$

Show that the *risk functional*

$$\mathcal{E}(f) := \int_Z (y - f(x))^2 \,\mathrm{d}\rho$$

can be decomposed as

$$\mathcal{E}(f) = \mathcal{E}(f_\rho) + \|f - f_\rho\|^2_{L_2(X,\rho_X)}.$$

3 points