Exercise sheet 6 for Friday, Dec 2, 2016

To be handed in either at the beginning of the exercise session, or before Dec 2, 9:55 a.m. at the drop box in front of room 149.

Exercise 19. Let $\mu \geq 0$ and

$$E_\mu(\mathcal{T}) := e(\mathcal{T}) + \mu\#(\mathcal{L}(\mathcal{T})).$$

Show that if for some $\mathcal{T}_\mu \neq \{R\}$ one has $E_\mu(\mathcal{T}_\mu) = \min_{\tilde{\mathcal{T}} \preceq \mathcal{T}} E_\mu(\tilde{\mathcal{T}})$, then for each $T' \in \mathcal{C}(R)$ the branch $\mathcal{B}(T'; \mathcal{T}_\mu)$ is a μ-optimally pruned subtree of $\mathcal{B}(T'; \mathcal{T})$.

3 points

Exercise 20. Let $\mathcal{T}' \preceq \mathcal{T}$. Show that

$$\mathcal{T}_\mu(\mathcal{T}) \preceq \mathcal{T}' \quad \Rightarrow \quad \mathcal{T}_\mu(\mathcal{T}) = \mathcal{T}_\mu(\mathcal{T}'),$$

i.e. a μ-optimally pruned minimal subtree of a given $\mathcal{T}$ stays optimal in any subtree $\mathcal{T}'$ of $\mathcal{T}$ containing it.

2 points

Exercise 21. Let $\zeta(\cdot; \mathcal{T}) : \mathcal{T} \setminus \mathcal{L}(\mathcal{T}) \to \mathbb{R}_+$ be defined by

$$\zeta(T; \mathcal{T}) := \frac{e(T) - e(\mathcal{B}(T; \mathcal{T}))}{\#(\mathcal{L}(\mathcal{B}(T; \mathcal{T}))) - 1}.$$

Show that for each $T \in \mathcal{T} \setminus \mathcal{L}(\mathcal{T})$ one has

$$\begin{aligned} \mu \leq \zeta(T; \mathcal{T}) \quad &\Leftrightarrow \quad E_\mu(T) \geq E_\mu(\mathcal{B}(T; \mathcal{T})), \\ \mu < \zeta(T; \mathcal{T}) \quad &\Leftrightarrow \quad E_\mu(T) > E_\mu(\mathcal{B}(T; \mathcal{T})). \end{aligned}$$

2 points

Exercise 22. Show that as long as $\mu < \mu_1 := \min_{T \in \mathcal{T} \setminus \mathcal{L}(\mathcal{T})} \zeta(T; \mathcal{T})$, $\mathcal{T}$ is the minimal μ-optimally pruned subtree of itself. Show further that for $\mu = \mu_1$ the tree $\mathcal{T}$ is still μ_1-optimally pruned but no longer minimal and that

$$\mathcal{T}_{\mu_1}(\mathcal{T}) = \{T \in \mathcal{T} : \zeta(\hat{T}; \mathcal{T}) > \mu_1 \text{ for all ancestors } \hat{T} \text{ of } T\} \cup \{R(\mathcal{T})\}.$$

5 points