Exercise sheet 7 for Friday, Dec 9, 2016

To be handed in either at the beginning of the exercise session, or before Dec 9, 9:55 a.m. at the drop box in front of room 149.

Exercise 23. Let $\mathbb{X}$ be a Banach space, and let $\{\psi_\lambda\}_{\lambda\in\Lambda}$ be a Schauder basis of $\mathbb{X}$ for which there exists $C>0$ such that for any finite $\Gamma\subset\Lambda$, any real sequence $(d_\lambda)_{\lambda\in\Lambda}$ and any sequence $(\varepsilon_\lambda)_{\lambda\in\Lambda}$ with $\varepsilon_\lambda\in\{-1,1\}$, we have

$$\Big\|\sum_{\lambda\in\Gamma}\varepsilon_\lambda d_\lambda\psi_\lambda\Big\|_{\mathbb{X}}\le C\Big\|\sum_{\lambda\in\Gamma}d_\lambda\psi_\lambda\Big\|_{\mathbb{X}}.$$

Show that for any finite $\Gamma\subset\Lambda$ and any real sequences (c_λ), (d_λ) with $|c_\lambda|\le|d_\lambda|$,

$$\Big\|\sum_{\lambda\in\Gamma}c_\lambda\psi_\lambda\Big\|_{\mathbb{X}}\le C\Big\|\sum_{\lambda\in\Gamma}d_\lambda\psi_\lambda\Big\|_{\mathbb{X}}.$$

Hint: Write each c_λ as a convex combination of $-d_\lambda$ and d_λ, and repeatedly use convexity of the norm.

5 points

Exercise 24. Let $\Omega:=[0,1]$ and $V_j:=\operatorname{span}\{\chi_{I_{j,k}}\colon k=0,\dots,2^j-1\,,\ I_{j,k}=[\frac{k}{2^j},\frac{k+1}{2^j})\}$, where χ_I denotes the indicator function of the interval I. Use without proof that for any $f\in L_2(\Omega)$, we have

$$\lim_{j\to\infty}\inf_{g\in V_j}\|f-g\|\to 0\,,$$

in order to show that the Haar wavelet basis $\{\psi_\lambda\colon\lambda\in\Lambda\}$, where

$$\phi=\chi_{[0,1]}\,,\quad \psi=\chi_{[0,\frac12)}-\chi_{[\frac12,1]}\,,$$
$$\psi_{-1,0}=\phi\,,\quad \psi_{j,k}=2^{j/2}\psi(2^j\cdot-k)\,,$$
$$\Lambda=\{(-1,0)\}\cup\{(j,k)\colon k=0,\dots,2^j-1\,,\ j=0,1,2,\dots\}\,,$$

is an orthonormal basis of $L_2(\Omega)$.

5 points

Exercise 25. Let $\psi_{j,k}$ be defined as in Exercise 24. Estimate the quantity $|\langle f,\psi_{j,k}\rangle_{[0,1]}|$ when $f\in W^1(L_p((k2^{-j},(k+1)2^{-j}))$, to see that the wavelet coefficient $|\langle f,\psi_{j,k}\rangle_{[0,1]}|$ is “small" when f is smooth on the support of $\psi_{j,k}$.

Hint: Use that $\psi_{j,k}$ is orthogonal to constants.

5 points

Exercise 26. Let $\mathcal{H}$ be a Hilbert space, and let $\{\psi_\lambda\colon\lambda\in\Lambda\}$ be a Riesz basis of $\mathcal{H}$. Show that there exist $\tilde\psi_\lambda\in\mathcal{H}$ such that the following holds: every $f\in\mathcal{H}$ has a unique expansion

$$f=\sum_{\lambda\in\Lambda}\langle f,\tilde\psi_\lambda\rangle\psi_\lambda\,,$$

we have $\langle\psi_\lambda,\tilde\psi_\nu\rangle=\delta_{\lambda\nu}$ for any $\lambda,\nu\in\Lambda$, and $\{\tilde\psi_\lambda\colon\lambda\in\Lambda\}$ is also a Riesz basis of $\mathcal{H}$.

6 points