Exercise sheet 8 for Friday, Dec 16, 2016

To be handed in either at the beginning of the exercise session, or before Dec 16, 9:55 a.m. at the drop box in front of room 149.

Exercise 27. Let $\mathcal{H}$ be a Hilbert space, let $\Psi := \{\psi_\lambda : \lambda \in \Lambda\}$ be a Riesz basis of $\mathcal{H}$ with constants $C_\Psi \geq c_\Psi > 0$, and let $\tilde{\Psi} := \{\tilde{\psi}_\lambda : \lambda \in \Lambda\}$ be the corresponding dual basis.

(i) Let $(\lambda_k)_{k\in\mathbb{N}}$ be an enumeration of Λ, and let $e_n(f)_\mathcal{H} := \inf\{\|f-g\| : g \in \operatorname{span}\{\psi_{\lambda_k} : 1 \leq k \leq n\}\}$ and $P_n f := \sum_{k=1}^n \langle f, \tilde{\psi}_{\lambda_k}\rangle \psi_{\lambda_k}$. Show that

$$e_n(f)_\mathcal{H} \leq \|f - P_n f\|_\mathcal{H} \leq \frac{C_\Psi}{c_\Psi} e_n(f)_\mathcal{H}\,.$$

(ii) Let $\Sigma_n := \{\sum_{\lambda\in\Gamma} d_\lambda \psi_\lambda : \#\Gamma \leq n\,,\, d_\lambda \in \mathbb{R}\}$ and $\sigma_n(f) := \inf_{g\in\Sigma_n} \|f-g\|_\mathcal{H}$. For $f \in \mathcal{H}$, let $(\lambda_k(f))_{k\in\mathbb{N}}$ be such that for any $k \in \mathbb{N}$, $|\langle f, \tilde{\psi}_{\lambda_k(f)}\rangle| \geq |\langle f, \tilde{\psi}_{\lambda_{k+1}(f)}\rangle|$. Show that for $G_n(f) := \sum_{k=1}^n \langle f, \tilde{\psi}_{\lambda_k(f)}\rangle \psi_{\lambda_k(f)}$, we have

$$\sigma_n(f)_\mathcal{H} \leq \|f - G_n(f)\|_\mathcal{H} \leq \frac{C_\Psi}{c_\Psi} \sigma_n(f)_\mathcal{H}\,.$$

3+3=6 points

Exercise 28. Let

$$\mathcal{H} := L_{2,2\pi} = \{f \in L_2(-\pi,\pi) : f(\cdot + 2\pi k) = f \text{ a.e. for all } k \in \mathbb{Z}\}$$

with inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f\,\overline{g}\,dx$, let $\Lambda := \mathbb{Z}$, $\Psi := \{\phi_k(x) := \frac{1}{\sqrt{2\pi}} e^{-ikx} : k \in \Lambda\}$, and $\Lambda_n := \{k \in \mathbb{Z} : |k| \leq n\}$. For

$$\mathbb{X}_n := \{\mathbf{d} = (d_k)_{k\in\Lambda} : \operatorname{supp}\mathbf{d} \subset \Lambda_n\}\,, \qquad e_n(\mathbf{d})_{\ell_2} := \Big(\sum_{|k|>n} |d_k|^2\Big)^{\frac{1}{2}}\,,$$

and $r > 0$, recall the definition of the approximation spaces

$$\mathcal{A}^r_\infty\big((\mathbb{X}_n)_{n\in\mathbb{N}}, \ell_2(\Lambda)\big) := \{\mathbf{d} \in \ell_2(\Lambda) : \sup_{n\in\mathbb{N}} n^r e_n(\mathbf{d})_{\ell_2} < \infty\}\,.$$

(i) Verify that $\langle \phi_k, \phi_l\rangle = \delta_{kl}$ for $k, l \in \mathbb{Z}$.

(ii) For $m \in \mathbb{N}$, let

$$H^m_{2\pi} := \{f \in L_{2,2\pi} : D^m f \in L_{2,2\pi}\}\,.$$

Show that $f \in H^m_{2\pi}$ implies

$$\big(\hat{f}(k)\big)_{k\in\Lambda} \in \mathcal{A}^m_\infty\big((\mathbb{X}_n)_{n\in\mathbb{N}}, \ell_2(\Lambda)\big)\,.$$

where $\hat{f}(k) := \langle f, \phi_k\rangle$ for $k \in \Lambda$.

2+5 = 7 points

Exercise 29. Let Ψ be a Riesz basis for the Hilbert space $\mathcal{H}$. Show that

$$\mathbf{v} \in \mathcal{A}^r((\Sigma_n(\Lambda)), \ell_2(\Lambda)) \quad\Leftrightarrow\quad \mathbf{F}(\mathbf{v}) \in \mathcal{A}^r((\Sigma_n(\Psi)), \mathcal{H})$$

and

$$C_\Psi^{-1} \sigma_n(f)_\mathcal{H} \leq \sigma_n((\langle f, \tilde{\psi}_\lambda\rangle_\mathcal{H}))_{\ell_2(\Lambda)} \leq c_\Psi^{-1} \sigma_n(f)_\mathcal{H}.$$

4 points

Exercise 30. Let $\mathbb{X}$ be a quasi-Banach space, let $\Sigma_n \subset \mathbb{X}$ for $n \in \mathbb{N}$ with $\Sigma_n \neq \emptyset$ and $\Sigma_n \subset \Sigma_{n+1}$, and let $\sigma_n(f)_\mathbb{X} := \inf_{g\in\Sigma_n} \|f-g\|_\mathbb{X}$. Show that for $r, q > 0$, we have

$$\sum_{n\in\mathbb{N}} \big(n^r \sigma_n(f)_\mathbb{X}\big)^q \frac{1}{n} \sim \sum_{j\in\mathbb{N}_0} \big(2^{jr} \sigma_{2^j}(f)_\mathbb{X}\big)^q\,.$$

5 points