Exercise sheet 9 for Friday, Dec 23, 2016

To be handed in either at the beginning of the exercise session, or before Dec 23, 9:55 a.m. at the drop box in front of room 149.

Exercise 31. As shown in the lecture (Theorem 4.5.3), for $r, p > 0$, one has

$$\mathcal{A}_p^r\big((\mathbb{X}_n)_{n\in\mathbb{N}}, \ell_p(\Lambda)\big) = \ell_p^r(\Lambda) := \big\{\mathbf{d} \in \ell_p(\Lambda) \colon \|\mathbf{d}\|_{\ell_p^r} := \big\|(k^r d_{\lambda_k})_{k\in\mathbb{N}}\big\|_{\ell_p(\Lambda)} < \infty\big\}$$

with equivalent norms. Derive a similar norm equivalence for $\mathcal{A}_q^r\big((\mathbb{X}_n)_{n\in\mathbb{N}}, \ell_p(\Lambda)\big)$ for $r, p, q > 0$ with $p \neq q$.

5 points

Exercise 32. Let $p \in (0, \infty)$. Show that $\ell_p(\mathbb{N}) \subsetneq w\ell_p(\mathbb{N})$ and $w\ell_p(\mathbb{N}) \subsetneq \ell_{p+\varepsilon}(\mathbb{N})$ for any $\varepsilon > 0$.

5 points

Exercise 33. Assume that $\mathbf{d} \in w\ell_\tau(\Lambda)$. Define the coarsening operator

$$\mathcal{C}_\varepsilon(\mathbf{d}) = (d_k^*)_{k=1}^{n(\varepsilon)}$$

where

$$n(\varepsilon) = \arg\min_n \left\{ \left(\sum_{k>n} (d_k^*)^p \right)^{1/p} \leq \varepsilon \right\},$$

i.e., $\mathcal{C}_\varepsilon$ coarsens $\mathbf{d}$ back to shortest subsequence—which is a best $n(\varepsilon)$-term approximation—that realizes the target accuracy ε. Then one has

$$n(\varepsilon) \lesssim \varepsilon^{-1/r} \|\mathbf{d}\|_{w\ell_\tau}^{1/r}, \qquad \frac{1}{\tau} = r + \frac{1}{p}.$$

4 points

Exercise 34. Let the assumtions of Theorem 4.6.1 hold.

(i) Show that there exits for each p, $1 \leq p \leq \infty$ (with $C(\bar{\Omega})$ in place of $L_\infty(\Omega)$) a constant $\bar{C}_p$ such that

$$\|f - P_j f\|_{L_p(\Omega)} \leq \bar{C}_p \inf_{g\in V_j} \|f - g\|_{L_p(\Omega)}, \quad j \in \mathbb{N}_0.$$

That is, the P_j provide near-best (linear) approximations from the spaces V_j.

(ii) Give a bound for $\bar{C}_p$.

3+2 points