# Exercise sheet 11 for Friday, Jan 20, 2017

To be handed in either at the beginning of the exercise session, or before Jan 20, 9:55 a.m. at the drop box in front of room 149.

**Exercise 36.** Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces, and let $F\colon \mathbb{X} \to \mathbb{Y}'$ be a bounded linear operator.

(i) Show that if there exists $\alpha > 0$ such that $\|Fx\|_{\mathbb{Y}'} \geq \alpha \|x\|_{\mathbb{X}}$ holds for all $x \in \mathbb{X}$, then $\operatorname{range}(F) = \{Fx\colon x \in \mathbb{X}\} \subseteq \mathbb{Y}'$ is closed.

(ii) Show that if $F$ has a bounded inverse $F^{-1}$ and there exists $\alpha > 0$ such that for all $x \in \mathbb{X}$, we have $\|Fx\|_{\mathbb{Y}'} \geq \alpha \|x\|_{\mathbb{X}}$, then $\|F^{-1}\|_{\mathbb{Y}' \to \mathbb{X}} \leq \alpha^{-1}$.

3+2=5 points

**Exercise 37.** Let $\Omega := (0, 1)$ and $\mathbb{X} := H_0^1(\Omega)$, and let $\varepsilon > 0$ and $\beta \in \mathbb{R}$. Let the operator $F\colon \mathbb{X} \to \mathbb{X}'$ be defined by $\langle Fu, v \rangle = a(u, v)$ for all $u, v \in \mathbb{X}$, where

$$a(u, v) := \varepsilon \int_\Omega u'\, v'\, dx + \beta \int_\Omega u'\, v\, dx\,.$$

Show that $F$ is an isomorphism and determine the condition number $\|F\|_{\mathbb{X}\to\mathbb{X}'} \|F^{-1}\|_{\mathbb{X}'\to\mathbb{X}}$.

*Hint:* Use integration by parts to show that $a(u, u) \geq \varepsilon \|u\|_{\mathbb{X}}^2$.

5 points

**Exercise 38.** Let $d \in \mathbb{N}$ and let $\Omega$ be a domain in $\mathbb{R}^d$. Let $a \in L_\infty(\Omega)$ with $a > 0$ a.e. such that $a^{-1} \in L_\infty(\Omega)$.

(i) Show that for any $f \in H^{-1}(\Omega) = (H_0^1(\Omega))'$, the variational problem of finding $[q, u] \in L_2(\Omega)^d \times H_0^1(\Omega)$ such that

$$\int_\Omega a^{-1} q \cdot r\, dx + \int_\Omega r \cdot \nabla u\, dx = 0 \qquad \text{for all } \; r \in L_2(\Omega)^d\,,$$
$$\int_\Omega q \cdot \nabla v\, dx = -\langle f, v \rangle \qquad \text{for all } \; v \in H_0^1(\Omega)$$

has a unique solution that depends continuously on $f$.

(ii) Show that $u$ as in (i) also satisfies

$$\int a \nabla u \cdot \nabla v\, dx = \langle f, v \rangle \quad \text{for all } \; v \in H_0^1(\Omega)\,.$$

4+3=7 points

**Exercise 39.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and $\mathbb{X} := (H_0^1(\Omega))^3 \times L_2(\Omega)$. Furthermore, let $\nu > 0$. Let the nonlinear operator $F$ on $\mathbb{X}$ be defined by the relation

$$\langle F([u, p]), [v, q] \rangle = \int_\Omega \nu \sum_{i=1}^3 \nabla u_i \cdot \nabla v_i + p \operatorname{div} v + \sum_{i=1}^3 (u \cdot \nabla u_i) v_i + q \operatorname{div} u\, dx \qquad \text{for all } [v, q] \in \mathbb{X}.$$

Show that for any $[u, p] \in \mathbb{X}$, that is, for $u = (u_1, u_2, u_3) \in (H_0^1(\Omega))^3$ and $p \in L_2(\Omega)$, we have $F([u, p]) \in \mathbb{X}'$. Find the Fréchet derivative $DF([u, p])\colon \mathbb{X} \to \mathbb{X}'$ and show that $DF([u, p])$ is bounded for any $[u, p] \in \mathbb{X}$.

*Hint:* Use the embedding $H_0^1(\Omega) \hookrightarrow L_4(\Omega)$.

5 points