Exercise sheet 12 for Friday, Jan 27, 2017

To be handed in either at the beginning of the exercise session, or before Jan 27, 9:55 a.m. at the drop box in front of room 149.

Exercise 40.

(i) Let $\Omega = (0,1)$. Consider the variational formulation

$$\begin{cases} \text{Find } u \in H_0^1, \text{ such that} \\ \langle Au, v\rangle = \langle f, v\rangle \qquad \forall v \in \mathbb{U} = H_0^1(\Omega) \end{cases}$$

of the Poisson problem ($-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$). Let $\mathbf{A}_h$ denote the standard FE stiffness matrix for a Galerkin discretization with respect to a finite element space on a uniform mesh with mesh size $h = 2^{-n}$. How does $\kappa_2(\mathbf{A}_h)$ grow with decreasing mesh size h?

(ii) Suppose you have a Riesz basis Ψ for $H_0^1(\Omega)$. For any $\Lambda_h \subset \Lambda$, $\#\Lambda_h = N$, let $\Psi_{\Lambda_h} = \{\psi_\lambda : \lambda \in \Lambda_h\} \subset H_0^1(\Omega)$. Let $\mathbb{U}_{\Lambda_h} = \operatorname{span}\{\psi_\lambda : \lambda \in \Lambda_h\} \subset \mathbb{U}$ and use this as a trial space for the Galerkin scheme. Let $A_h^\Psi = (a(\psi_\nu, \psi_\lambda))_{\lambda,\nu\in\Lambda_h} \in \mathbb{R}^{N\times N}$ be the corresponding stiffness matrix where the bilinear form $a : \mathbb{U}\times\mathbb{U} \to \mathbb{R}$ satisfies the mapping property (MP). Show that

$$\kappa_2(A_h^\Psi) \le \frac{C_a C_\Psi^2}{c_a c_\Psi^2}.$$

$4 + 2$ points

Exercise 41.

(i) Let the nonlinear problem

$$\langle F(u), v\rangle = \langle f, v\rangle \qquad \forall v \in \mathbb{U}$$

be stable, i. e. there exists a neighborhood $\mathcal{N}(u)$ of the solution u, such that for all $w \in \mathcal{N}(u)$

$$c_w \|z\|_{\mathbb{U}} \le \|DF(w)z\|_{\mathbb{V}'} \le C_w \|z\|_{\mathbb{U}} \qquad \forall z \in \mathbb{U}$$

holds. Show that one obtains

$$c_w c_\Psi^2 \|\mathbf{v}\|_{\ell_2} \le \|D\mathbf{F}(\mathbf{w})\mathbf{v}\|_{\ell_2} \le C_w C_\Psi^2 \|\mathbf{v}\|_{\ell_2} \qquad \forall \mathbf{v} \in \ell_2(\Lambda)$$

for all w in the stability region $\mathcal{N}(u)$.

(ii) Identify $D\mathbf{F}(\mathbf{w})\mathbf{v}$ and specify this for $F(v) = v^3$.

$2 + 3$ points

Exercise 42. Let $n \in \mathbb{N}$ and let $A \in \mathbb{R}^{n\times n}$ be symmetric with $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^T$, where $\mathbf{V} \in \mathbb{R}^{n\times n}$ is orthogonal with columns v_i, $i = 1,\ldots,n$, and $\mathbf{D} = \operatorname{diag}(\lambda_i)_{i=1,\ldots,n}$. Let $\mathbf{f}, \mathbf{u}_0 \in \mathbb{R}^n$ and

$$\mathbf{u}_{k+1} := \mathbf{u}_k - \alpha(\mathbf{A}\mathbf{u}_k - \mathbf{f}), \quad k \in \mathbb{N}_0\,.$$

(i) Let $0 < \lambda_1 \le \ldots \le \lambda_n$, and let $\mathbf{u} \in \mathbb{R}^n$ such that $\mathbf{A}\mathbf{u} = \mathbf{f}$. Determine the range of $\alpha > 0$ such that for some $\rho > 0$, we have $\|\mathbf{u} - \mathbf{u}_{k+1}\|_2 \le \rho\|\mathbf{u} - \mathbf{u}_k\|_2$ for all $\mathbf{f}$ and $\mathbf{u}_0$. Find the α for which ρ is minimal.

(ii) For some $0 < m < n$, let $0 = \lambda_1 = \ldots = \lambda_m < \lambda_{m+1} \le \ldots \le \lambda_n$. Let $\mathbb{V}_0 := \operatorname{span}\{v_1,\ldots,v_m\}$, and let $\mathbf{f}, \mathbf{u}_0 \in \mathbb{V}_0^\perp$. Let $u \in \mathbb{R}^n$ such that $\mathbf{A}\mathbf{u} = \mathbf{f}$, and let $\mathbf{u}_\perp$ denote the orthogonal projection of $\mathbf{u}$ onto $\mathbb{V}_0^\perp$. As in (i), find $\alpha > 0$ such that ρ in the estimate $\|\mathbf{u}_\perp - \mathbf{u}_{k+1}\|_2 \le \rho\|\mathbf{u}_\perp - \mathbf{u}_k\|_2$ is minimal.

(iii) For some $0 < m < n$, let $\lambda_1 \le \ldots \le \lambda_m < 0 < \lambda_{m+1} \le \ldots \le \lambda_n$. Show that if $\alpha \ne 0$, there exist $\mathbf{u}_0$ such that $\|\mathbf{u}_k\|_2 \to \infty$.

$2 + 2 + 2 = 6$ points