A Geometric Description of Feasible Singular Values in Tree Tensor Formats

Sebastian Krämer

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Feasibility and Compatibility

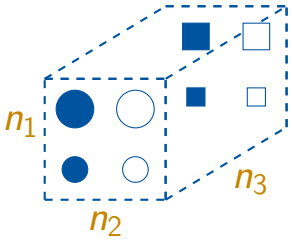
Tensor Tree Decompositions / SVDs

Sums of Hermitian Matrices and Honeycombs

Feasibility of Pairs

TT-Feasibility

Feasibility and Compatibility — Reshaping

$A =$ 

$\in \mathbb{K}^{n_1 \times n_2 \times n_3}$

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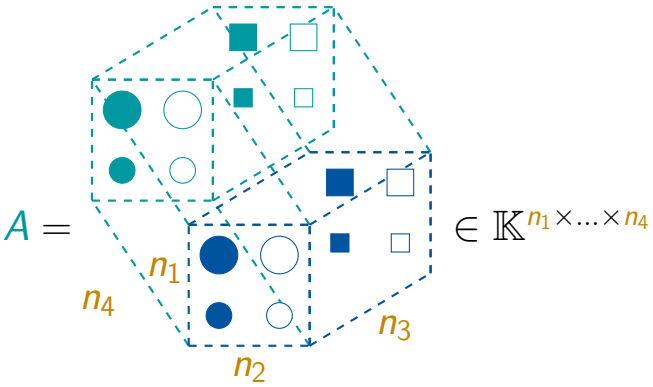
Feasibility and Compatibility — Reshaping



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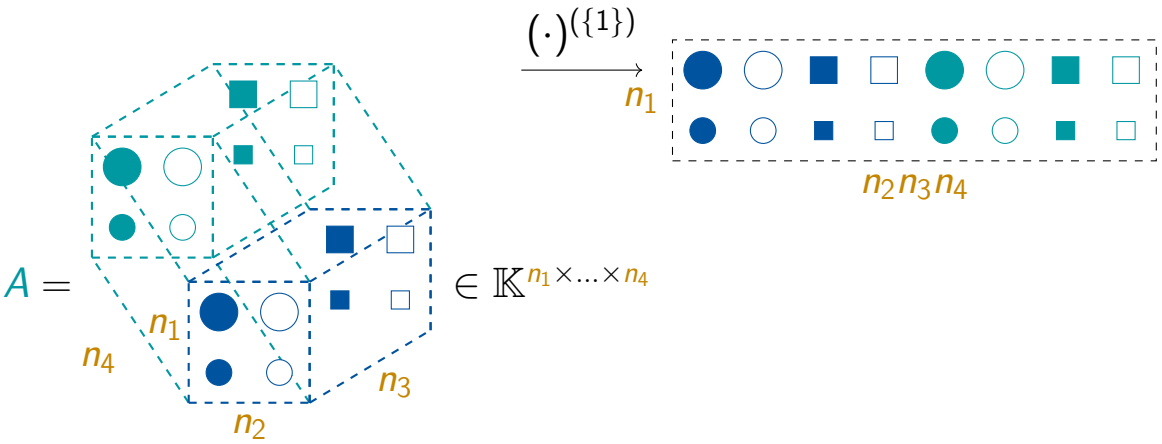

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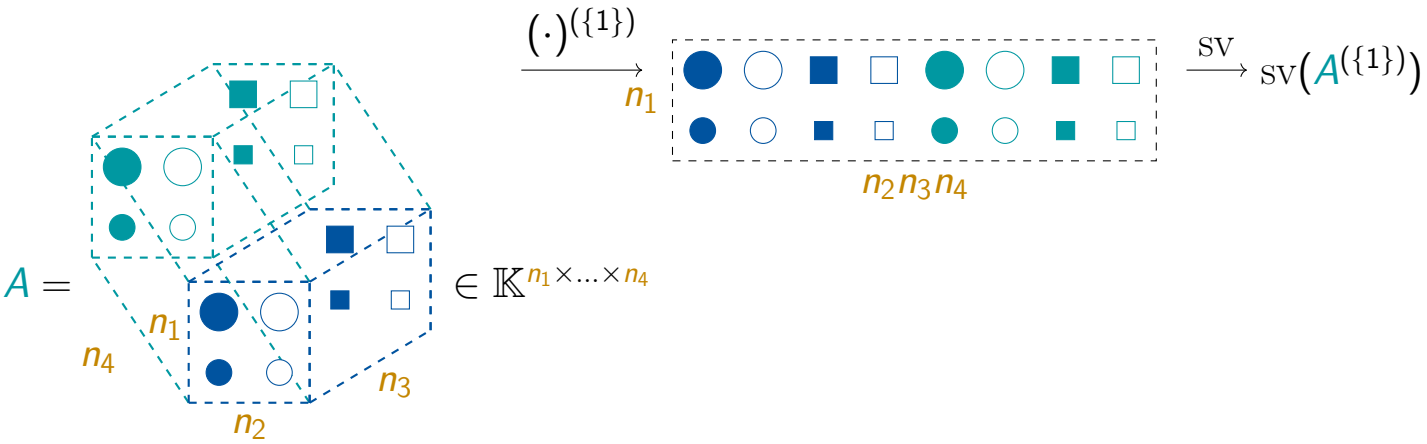

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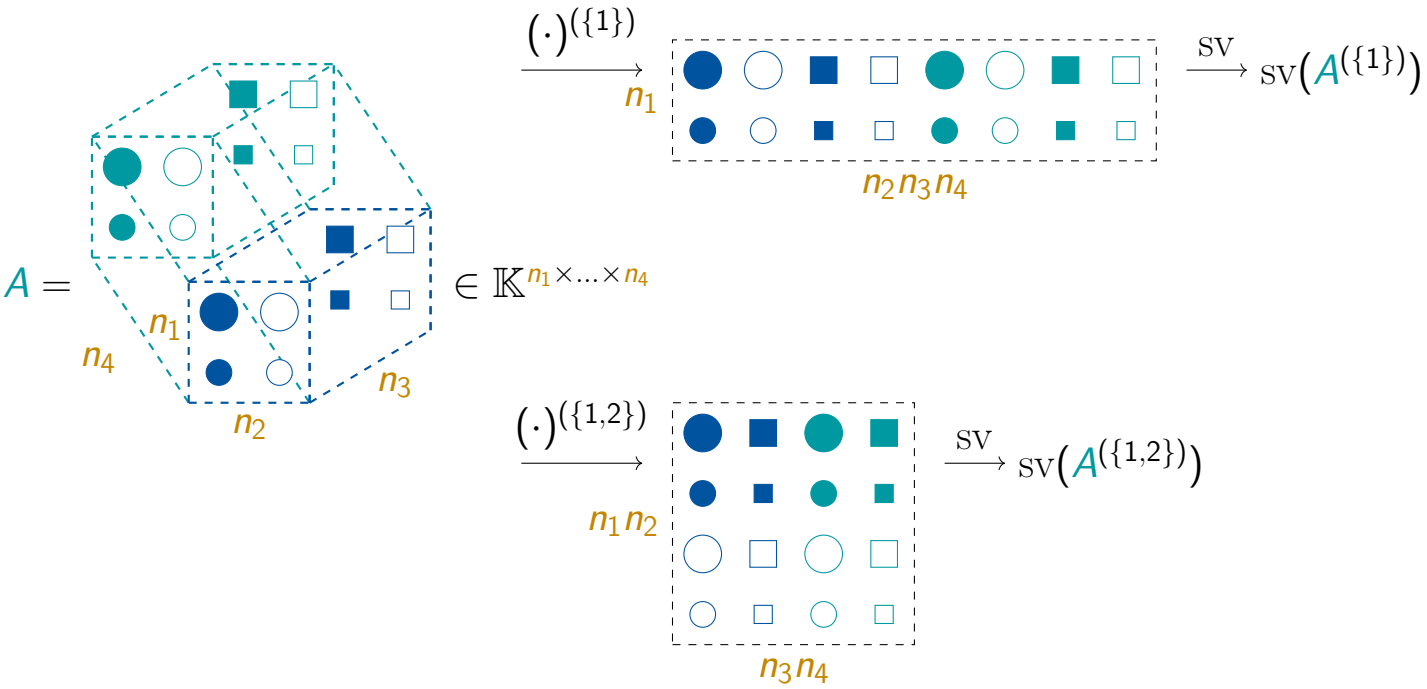

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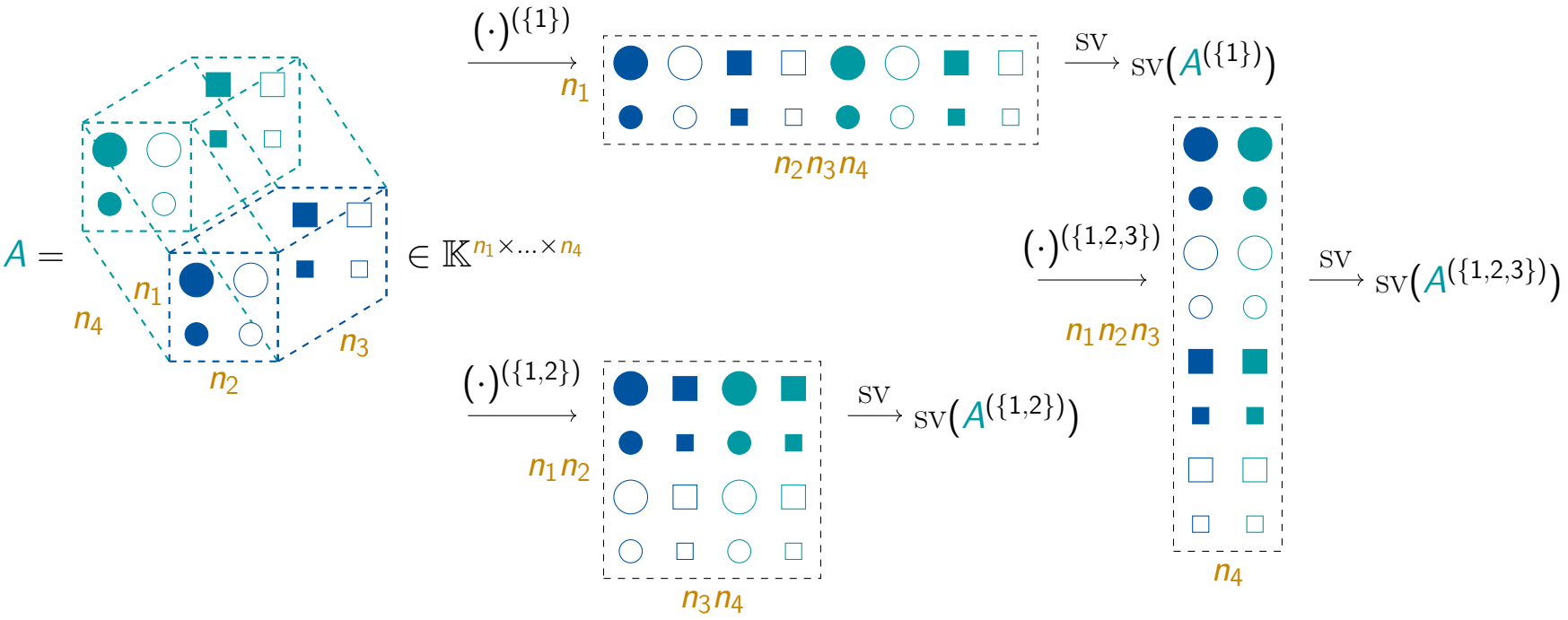

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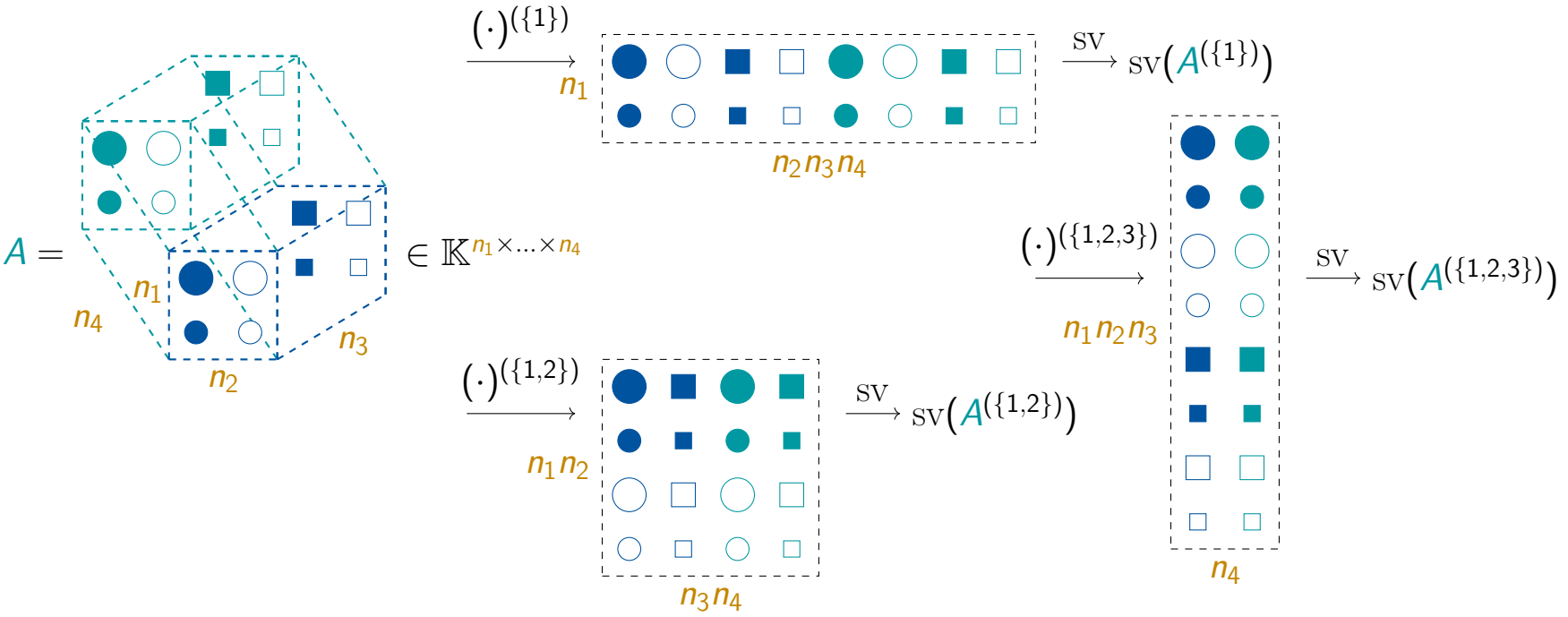

$$\mathbb{K}^{n_1 \times n_2 \times n_3 \times n_4} \cong \mathbb{K}^{n_1 \times (n_2 n_3 n_4)} \cong \mathbb{K}^{(n_1 n_2) \times (n_3 n_4)} \cong \mathbb{K}^{(n_1 n_2 n_3) \times n_4}$$

$$\cong \mathbb{K}^{(n_1 n_3) \times (n_2 n_4)} \cong \ldots$$

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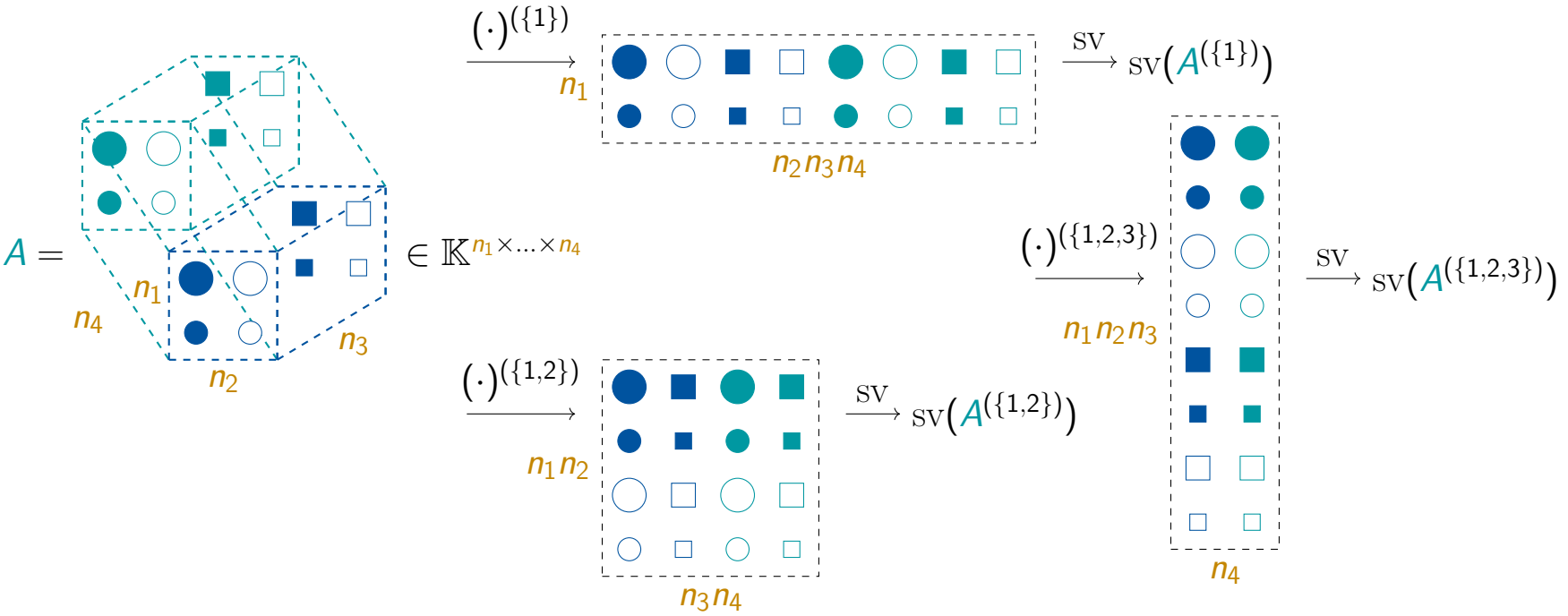


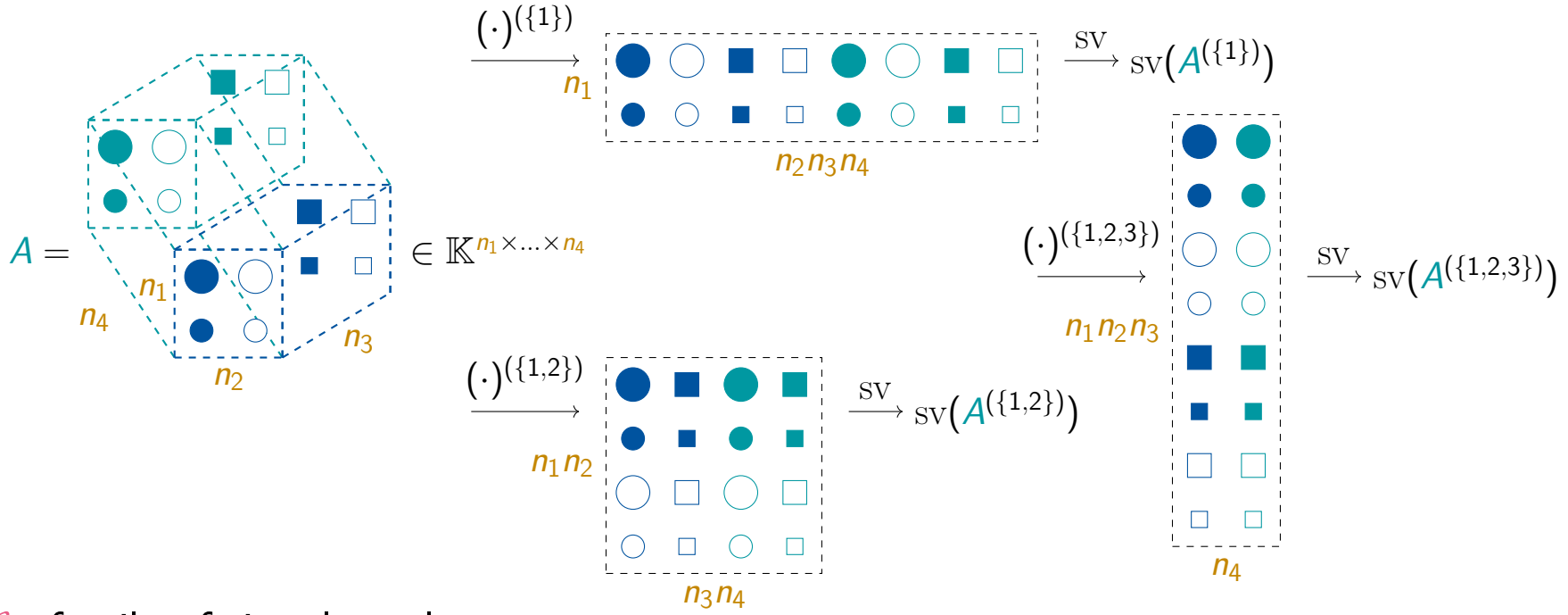
Infinite sequence:

$$\mathrm{sv}(B) = \sigma = (\sigma_+, 0, 0, \ldots) \in \ell_2(\mathbb{R}_{\geq 0}), \quad \sigma_1 \geq \ldots \geq \sigma_r > 0$$

Positive part:

$$\sigma_+ = (\sigma_1, \ldots, \sigma_r) \in \mathbb{R}^r, \quad r = \deg(\sigma) := \mathrm{length}(\sigma_+)$$

Feasibility and Compatibility — Reshaping



$\mathcal{K}$—family of singular values:

$$\{\mathrm{SV}(A^{(J)})\}_{J \in \mathcal{K}} \quad \text{for} \quad J \subset \{1, \ldots, d-1\}$$

In the picture:

$$\mathcal{K} = \{\ \{1\},\ \{1,2\},\ \{1,2,3\}\ \} \qquad \text{(tensor train for } d = 4\text{)}$$

Tensor feasibility problem (TFP):

Definition (Feasibility)

$\{\sigma^{(J)}\}_{J\in\mathcal{K}}$ is feasible for $n \in \mathbb{N}^d \Leftrightarrow \exists\, A \in \mathbb{K}^{n_1\times\ldots\times n_d}$, $\mathrm{sv}(A^{(J)}) = \sigma^{(J)}$ for all $J \in \mathcal{K}$

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Trace condition for feasibility: $\|\sigma^{(I)}\| = \|\sigma^{(J)}\|$ for all $I, J \in \mathcal{K}$

proof: $\mathrm{sv}(A^{(J)}) = \sigma^{(J)} \quad \Rightarrow \quad \sum_{i\in\mathbb{N}}(\sigma_i^{(J)})^2 = \|A\|_F$

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Example: $\mathcal{K} = \{\ \{1\},\ \{1,2\},\ \{1,2,3\}\ \}$

- $(\sigma_+^{(\{1\})})^2 = (9,\ 4.5)$
- $(\sigma_+^{(\{1,2\})})^2 = (10,\ 2,\ 1,\ 0.25,\ 0.25)$
- $(\sigma_+^{(\{1,2,3\})})^2 = (4,\ 3,\ 2.5,\ 2,\ 2)$

Question: σ feasible for some $n \in \mathbb{N}^4$? / can we construct such a tensor A?

Why ask: complexity analysis / similarity of decay / better intuition

Results for the Tucker-format, $\mathcal{K} = \{\{1\}, \{2\}, \ldots, \{d\}\}$:

- Introduction and first contributions to the Tucker TFP:
  - [*HaKrUs17*] : *W. Hackbusch, D. Kressner, A. Uschmajew, Perturbation of higher-order singular values, SIAGA (2017)*
  - [*HaUs17*] : *W. Hackbusch, A. Uschmajew, In the interconnection between the higher-order singular values of real tensors, Numerische Mathematik (2017)*

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- Solution for largest singular values:
  - [*DoStLa17*] : *I. Domanov, A. Stegeman, L. De Lathauwer, On the largest multilinear singular vallues of higher-order tensors, SIMAX (2017)*
- Independent solution for binary tensors:
  - [*Se18*] : *A. Seigal, Gram determinants of real binary tensors, Linear Algebra and its Applications (2018)*

(a version of the) Quantum marginal problem (QMP) [Kl06][DaHa05][Sc14]:

Definition (Compatibility)

$\{\lambda^{(J)}\}_{J\in\mathcal{K}}$ is compatible for $n \in \mathbb{N}^{d-1}$ $\Leftrightarrow$ there exists a hermitian, positive semi-definite matrix $\rho \in \mathbb{C}^{(n_1 \dots n_{d-1})\times(n_1 \dots n_{d-1})}$ such that

$$\mathrm{ev}\big(\mathrm{trace}_{\{1,\dots,d-1\}\setminus J}(\rho)\big) = \lambda^{(J)}$$

for all $J \in \mathcal{K}$.

Pure quantum marginal problem (pure QMP) adds $\mathrm{rank}(\rho) = 1$.
The partial traces are called *marginals* of the *density operator* ρ.

The QMP is much older than the TFP !

[Kl06] : A. A. Klyachko, Quantum marginal problem and N-representability, Journal of Physics (2006)
[DaHa05] : S. Daftuar and P. Hayden, Quantum state transformations and the Schubert calculus, Annals of Physics, (2005)
[Sc14] : C. Schilling, Quantum marginal problem and its physical relevance, PhD thesis (2014)

Feasibility and Compatibility — Equivalence of TFP and QMP

Tensor feasibility (TFP) and quantum marginal problem (QMP) are equivalent:

Theorem (Equivalence)

$$\{(\sigma^{(J)})^2\}_{J\in\mathcal{K}} \text{ is compatible} \quad \Leftrightarrow \quad \{\sigma^{(J)}\}_{J\in\mathcal{K}} \text{ is feasible (for } n_d = \operatorname{rank}(\rho))$$

proof: $\rho = A^{(I)} A^{(I)^H}$, $I = \{1, \dots, d-1\}$

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proof: $\rho = A^{(I)} A^{(I)^H}$, $I = \{1, \dots, d-1\}$

Solutions for three dimensions:

- $\mathcal{K} = \{\{1\},\{2\},\{3\}\}$: pure QMP [KI06] ($\rightsquigarrow$ Tucker-feasibility for $d = 3$)
- $\mathcal{K} = \{\{1\},\{1,2\}\}$: QMP [DaHa05] ($\rightsquigarrow$ Tensor Train-feasibility for $d = 3$)

These sets of compatible values form convex, polyhedral cones.

QMP results allow to calculate H-description for each fixed n_1, n_2,
but do not (seem to) yield universal inequalities for arbitrary n.

Tensor Tree Decompositions / SVDs —

Feasibility and Compatibility

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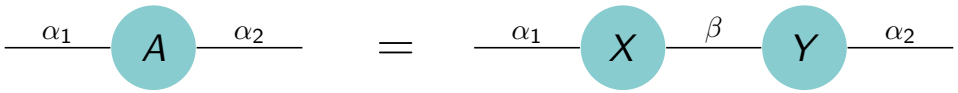
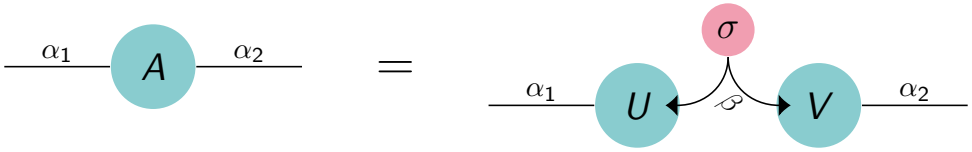
$d = 2$: we can visualize the low-rank decomposition of $A \in \mathbb{K}^{n_1 \times n_2}$, $r = \operatorname{rank}(A)$,

$$A_{\alpha_1,\alpha_2} = \sum_{\beta=1}^{r} X_{\alpha_1,\beta} \; Y_{\alpha_2,\beta}$$

as *tensor network*:

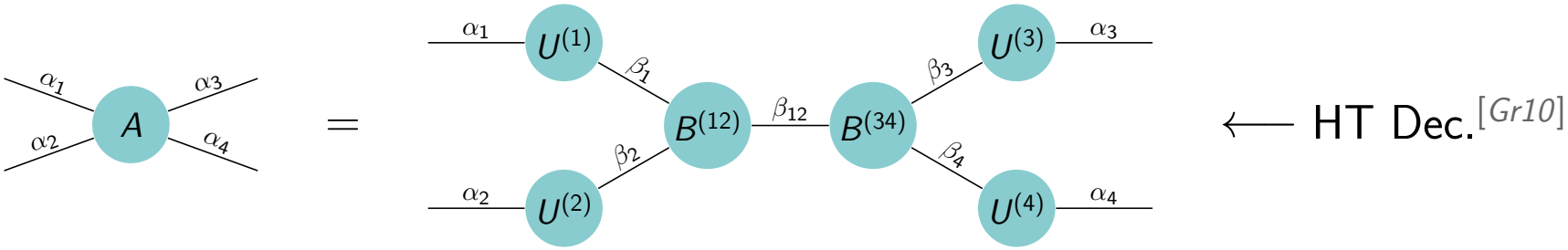


$$A_{\alpha_1,\alpha_2} = \sum_{\beta=1}^{r} U_{\alpha_1,\beta} \; \sigma_\beta \; V_{\alpha_2,\beta} \qquad \text{(SVD)}$$



- U and V are column-unitary

$r_J := \operatorname{rank}(A^{(J)}) = \deg(\sigma^{(J)}), \quad \beta_J = 1, \ldots, r_J$:



⟵ HT Dec.[Gr10]

[Gr10] : *L. Grasedyck, Hierarchical SVD of Tensors, SIAM J. Matrix Anal. & Appl (2010)*

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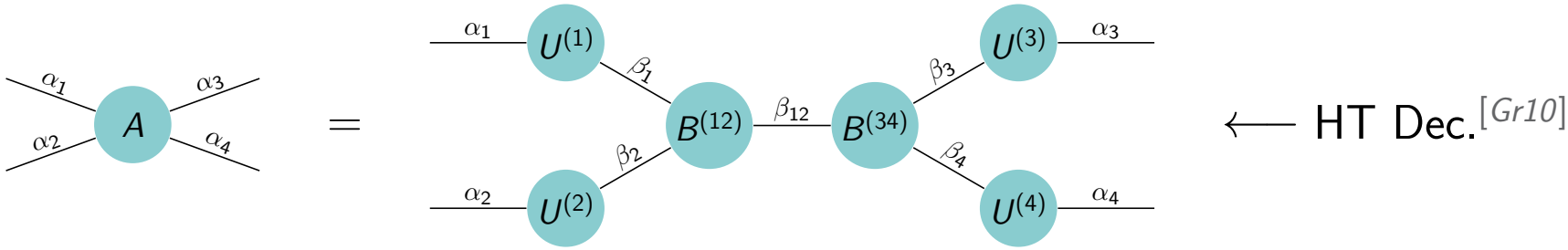
$r_J := \operatorname{rank}(A^{(J)}) = \deg(\sigma^{(J)}), \quad \beta_J = 1, \dots, r_J$:

$$A_{\alpha_1,\dots,\alpha_4} = \sum_{\beta_1=1}^{r_{\{1\}}} \cdots \sum_{\beta_4=1}^{r_{\{4\}}} \sum_{\beta_{12}=1}^{r_{\{1,2\}}} U^{(\{1\})}_{\alpha_1,\beta_1} \cdots U^{(\{4\})}_{\alpha_4,\beta_4} \cdot B^{(12)}_{\beta_1,\beta_2,\beta_{12}} \cdot B^{(34)}_{\beta_3,\beta_4,\beta_{12}}$$



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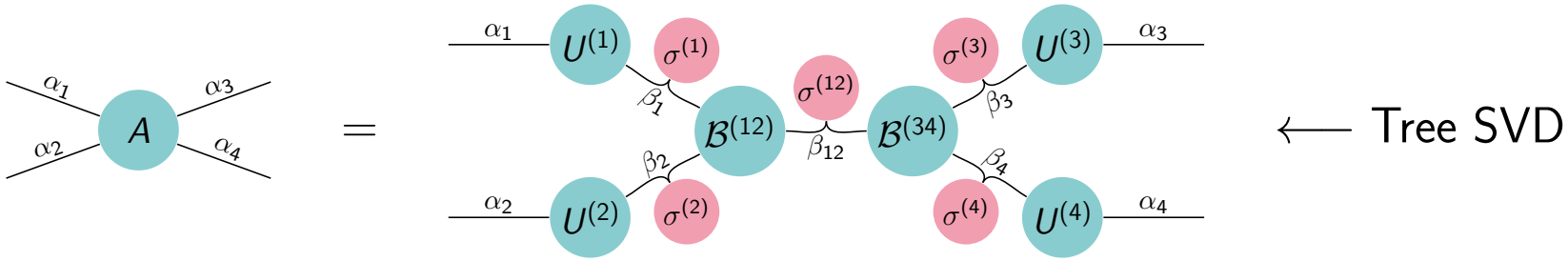

⟵ Tree SVD

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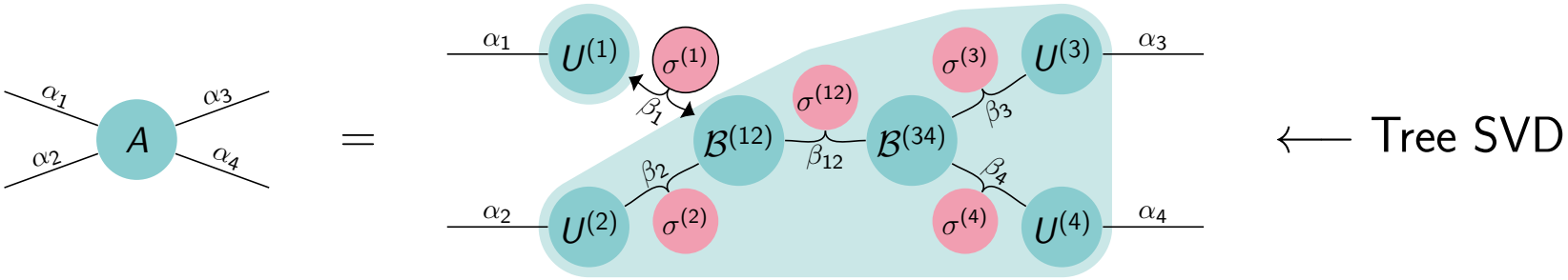

⟵ Tree SVD

- $\left(U^J,\ \sigma^{(J)},\ V^{(J)}\right) = \mathrm{SVD}\left(A^{(J)}\right),\ J \in \mathcal{K}$
- $\mathcal{K} = \{\ \{1,2\},\ \{\mathbf{1}\},\ \{2\},\ \{3\},\ \{4\}\ \} \rightarrow$ binary hierarchical Tucker format

The tree SVD puts gauge conditions on inner nodes:

Theorem (Decoupling)

Let $\mathcal{K}$ fulfill the hierarchy condition [Gr10] (it corresponds to a tree graph)

$$J \cap \widetilde{J} \in \{\emptyset,\ J,\ \widetilde{J}\}, \quad \text{for all } J, \widetilde{J} \in \mathcal{K}.$$

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Then the conditions for feasibility can be decoupled into two and three dimensional subproblems. With [Kl06][DaHa05], we thereby know that squared feasible values for such $\mathcal{K}$ form closed, convex, polyhedral cones.

Example: $\mathcal{K} = \{\{1,2\},\{1\},\{2\},\{3\},\{4\}\}$

$$\{\sigma^{(J)}\}_{J\in\mathcal{K}} \text{ feasible for } n \in \mathbb{N}^4 \Leftrightarrow \{\sigma^{(\{1\})}, \sigma^{(\{2\})}, \sigma^{(\{1,2\})}\} \text{ feasible for } (n_1, n_2, r_{\{1,2\}})$$
$$\text{and } \{\sigma^{(\{3\})}, \sigma^{(\{4\})}, \sigma^{(\{1,2\})}\} \text{ feasible for } (n_3, n_4, r_{\{1,2\}})$$

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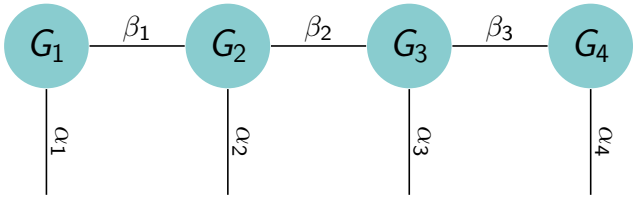
Tensor Tree Decompositions / SVDs — TT and Tucker

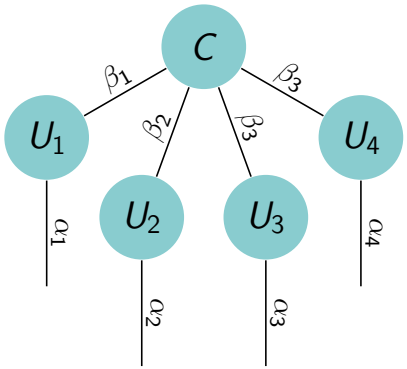
Tensor Train (TT)



$$\mathcal{K} = \{ \{1\}, \{1,2\}, \{1,2,3\} \},$$

TT tree dec. $\equiv$ TTSVD[Os11] / TT tree SVD $\equiv$ canonical MPS[Vi03]

Tucker

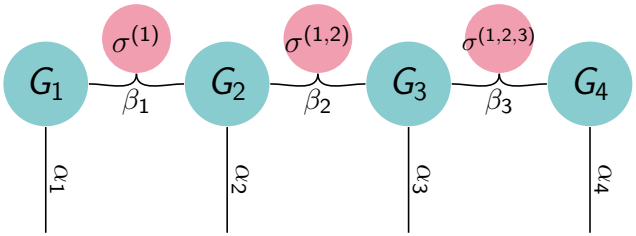


$$\mathcal{K} = \{ \{1\}, \{2\}, \{3\}, \{4\} \}$$

Tucker tree dec. $\equiv$ HOSVD / Tucker tree SVD $\rightarrow$ all-orthogonality[LaMoVa00]

[*Os11*] : *I. Oseledets, Tensor-Train Decomposition, SIAM J. Sci. Comput. (2011)*
[*Vi03*] : *G. Vidal, Efficient Classical Simulation of Slightly Entangled Quantum Computations, Physical Review Letters (2003)*
[*LaMoVa00*] : *L. Lathauwer, B. Moor, J. Vandewalle, A multilinear SVD, Siam J. Matrix Anal. Appl. (2000)*

Tree SVD for the tensor train format:



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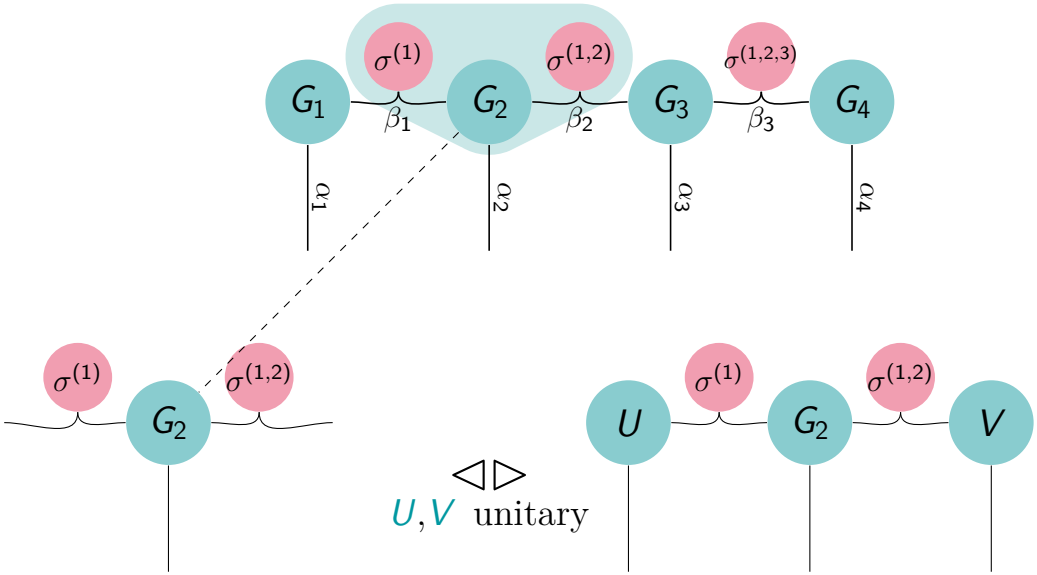

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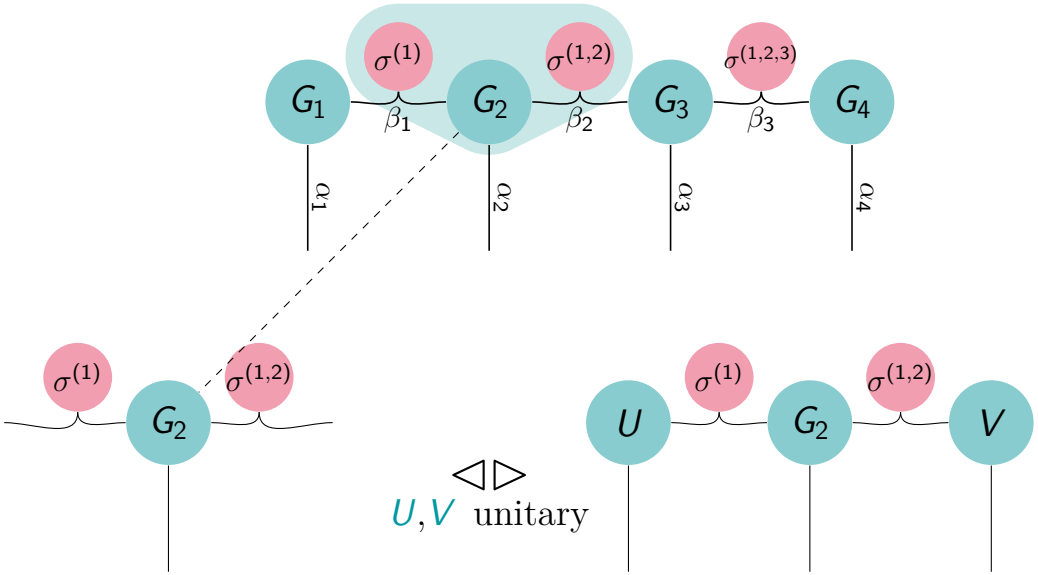

$G_2 = G_2(\alpha_2)_{\beta_1,\beta_2}$ fulfills simple constraints:

$$\sum_{\alpha_2} G_2(\alpha_2)^H \operatorname{diag}(\sigma^{(\{1\})})^2 \, G_2(\alpha_2) = I, \quad \sum_{\alpha_2} G_2(\alpha_2) \operatorname{diag}(\sigma^{(\{1,2\})})^2 \, G_2(\alpha_2)^H = I$$

Corollary (TT-feasibility)

Let $\mathcal{K} = \{\{1\}, \{1,2\}, \dots, \{1,\dots,d-1\}\}$. *Then* $\{\sigma^{(J)}\}_{J\in\mathcal{K}}$ *is feasible*

if and only if

each neighboring pair $(\sigma^{(\{1,\dots,\mu-1\})}, \sigma^{(\{1,\dots,\mu\})})$ *is feasible for* $(r_{\{1,\dots,\mu-1\}}, n_\mu, r_{\{1,\dots,\mu\}})$ *and* $\mathcal{K} = \{\{1\}, \{1,2\}\}$, $\mu = 2,\dots,d-1$ *(as well as* $r_{\{1\}} \leq n_1$ *and* $r_{\{1,\dots,d-1\}} \leq n_d$*).*

Corollary (TT-feasibility)

Let $\mathcal{K} = \{\{1\}, \{1,2\}, \dots, \{1,\dots,d-1\}\}$. *Then* $\{\sigma^{(J)}\}_{J\in\mathcal{K}}$ *is feasible*

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For pairs, we just write *feasible for* m $(= n_\mu) \in \mathbb{N}$. The decoupling is constructive.

We will turn the feasibility of pairs into an interlinked eigenvalue relation.
Later yields: conditions for TT-feasibility are the same for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

Sums of Hermitian Matrices and Honeycombs —

Feasibility and Compatibility

Tensor Tree Decompositions / SVDs

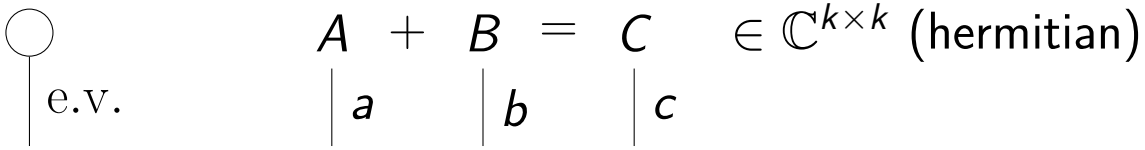
Sums of Hermitian Matrices and Honeycombs

Feasibility of Pairs

TT-Feasibility

Sums of Hermitian Matrices and Honeycombs — Weyl's Problems

- Weyls problem [We1912]: Given $(a, b, c) \in \mathbb{R}^{3k}$, do there exist hermitian matrices $A, B, C \in \mathbb{R}^{k \times k}$ with $A + B = C$ and $a = \mathrm{ev}(A),\ \ldots,\ c = \mathrm{ev}(C)$:



[We1912] : *H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (...), Math. Annalen (1912)*
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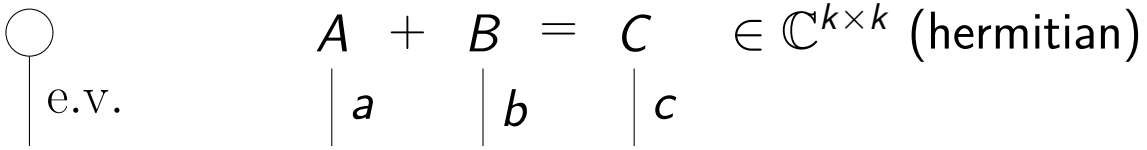

- Horn Conjecture [Ho62] (verified [KnTa01]): the set $F \subset \mathbb{R}^{3k}$ of such (a, b, c) is a polyhedral cone (governed by simple, but recursively defined inequalities).

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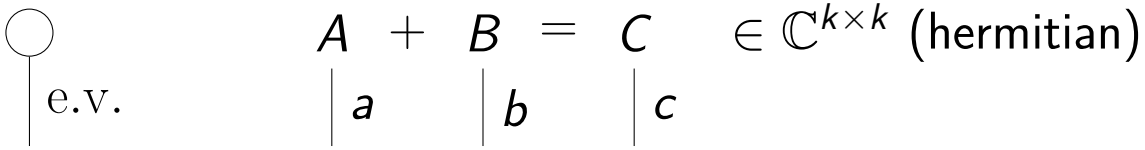

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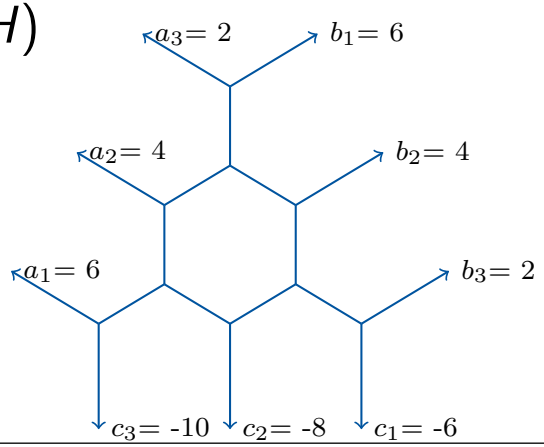
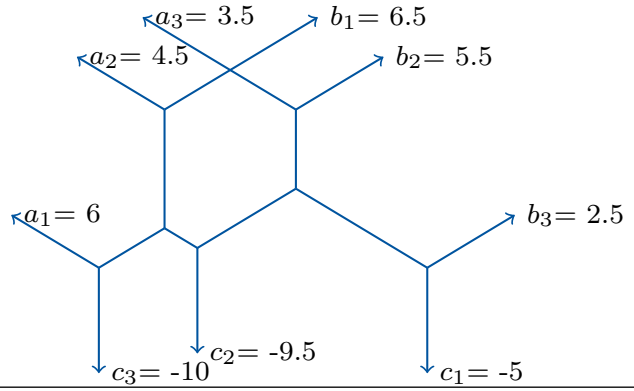
- $(a, b, c) \in F \Leftrightarrow$ there is a *honeycomb* H with boundary $(a, b, -c)$ [KnTa01]: $F = \mathrm{Proj}_{\text{boundary}}(H)$





Sums of Hermitian Matrices and Honeycombs — Hives

- The triplets (a, b, c) are the same for (complex, hermitian) $\leftrightarrow$ (real, symmetric) matrices [*Fu00*]

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Sums of Hermitian Matrices and Honeycombs — Hives

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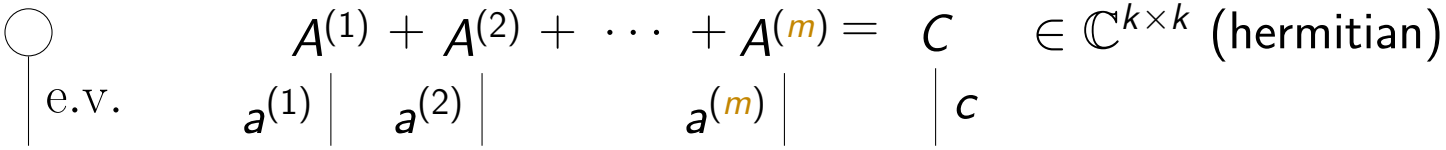

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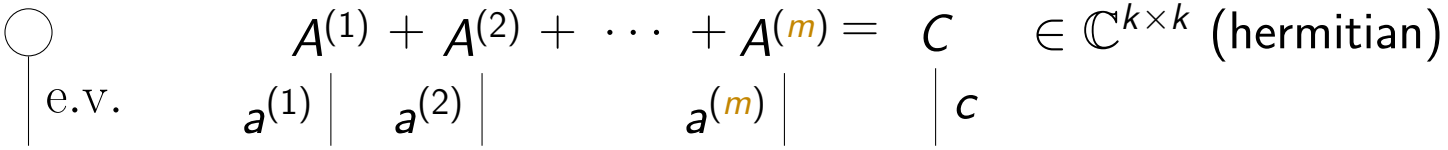
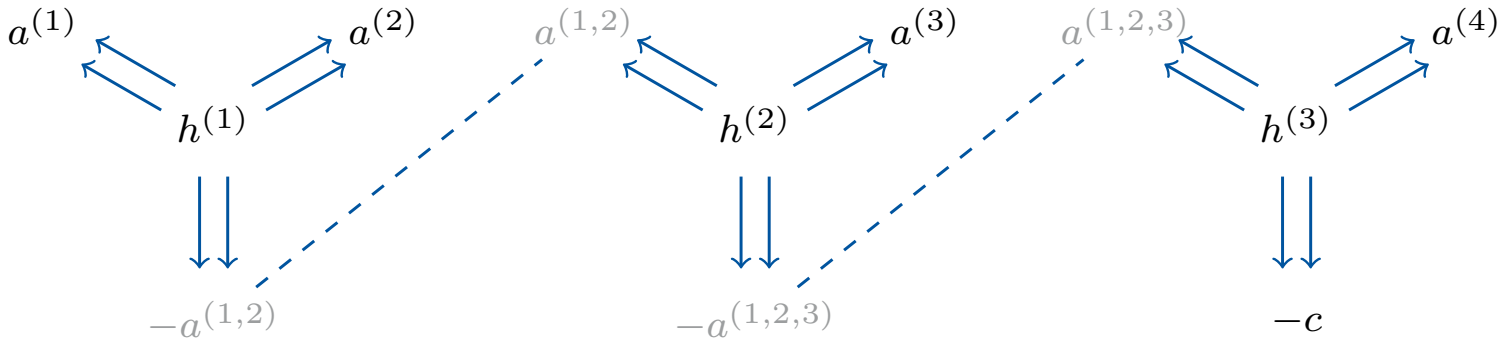


- Form chains of honeycombs $H = (h^{(1)}, h^{(2)}, \ldots, h^{(m)})$ (*hives*) [KnTa01]:



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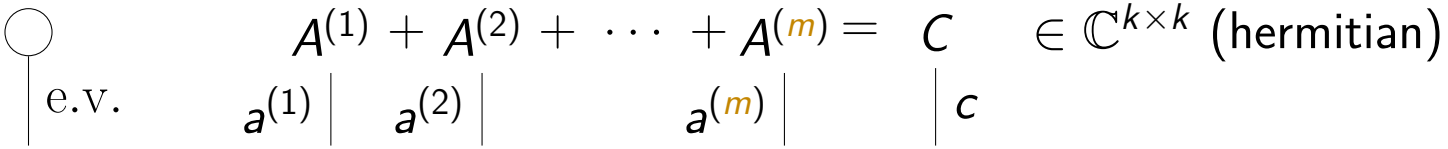

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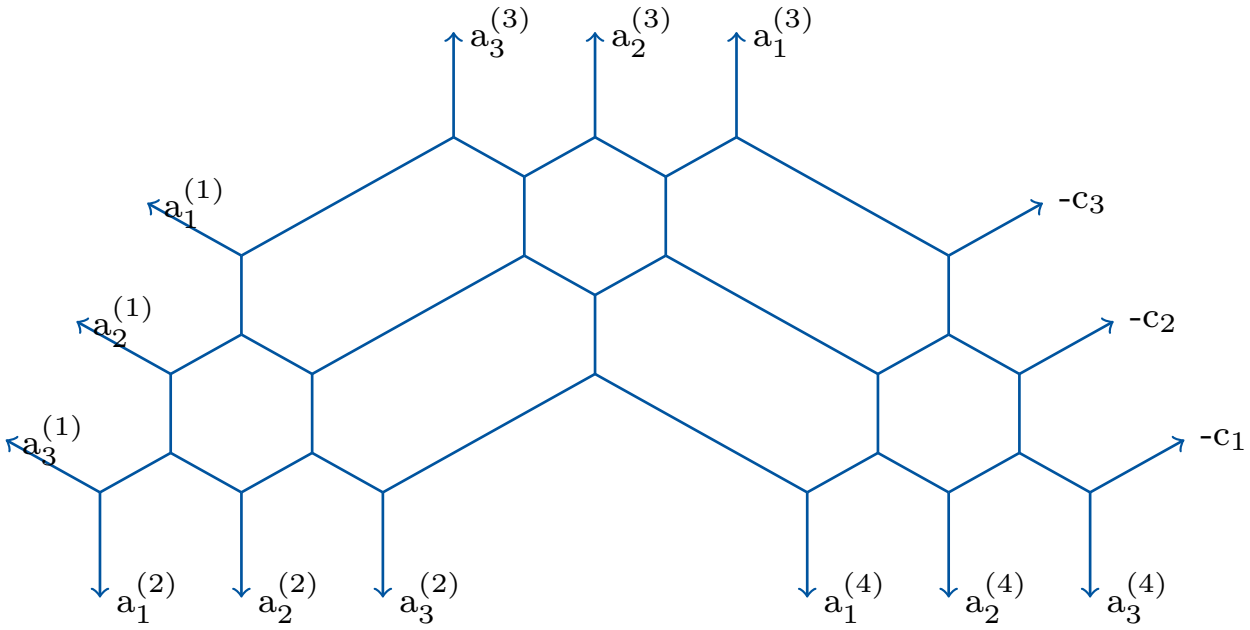

Feasibility of Pairs —

Feasibility and Compatibility

Tensor Tree Decompositions / SVDs

Sums of Hermitian Matrices and Honeycombs

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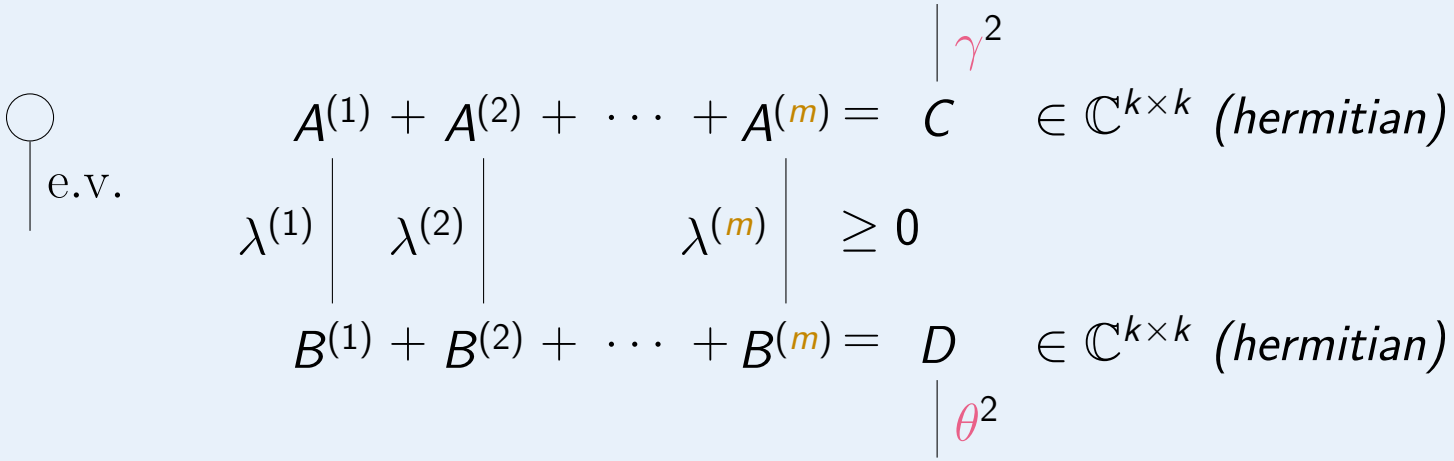
TT-Feasibility

Theorem (Conversion to eigenvalue problem (constructive))

(γ, θ) feasible for m if and only if for some $k \geq \deg(\gamma), \deg(\theta)$ there exist hermitian, positive semi-definite matrices $A^{(i)}, B^{(i)}, C, D$ in following relation:



Feasibility of Pairs — Connection to Hives

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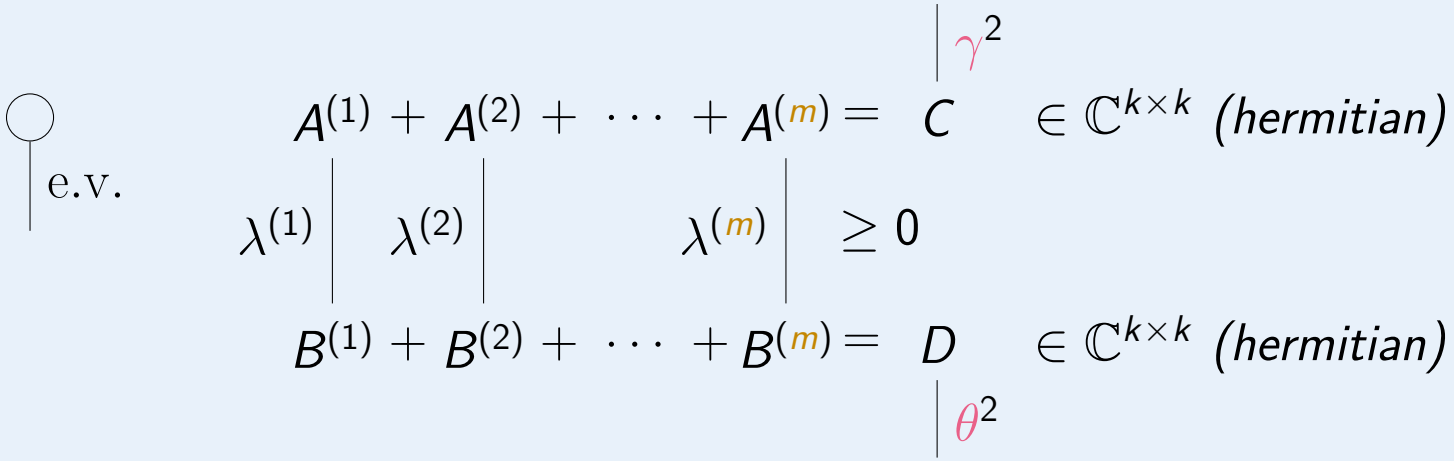
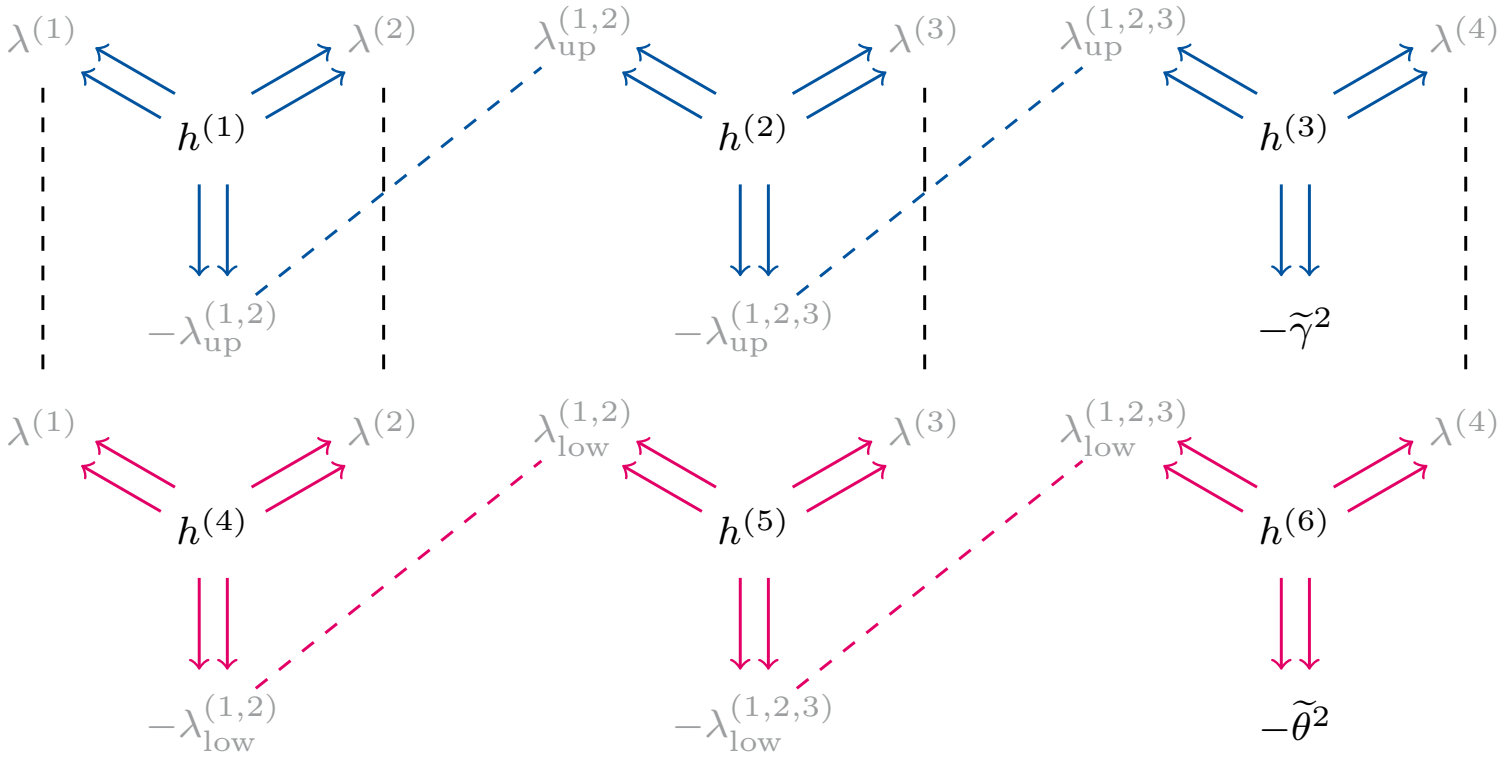


Lemma (Restriction to diagonal matrices)

If $\deg(\gamma), \deg(\theta) \leq m$, then we can always construct such matrices, as well as a corresponding node.

Feasibility of Pairs — Connection to Hives

- Form coupled chains of honeycombs $H = (h^{(1)}, h^{(2)}, \ldots, h^{(2m)})$ (*hives*):



- Condition are given by a system of linear inequalities (also proving the cone property).

We can derive classes of necessary inequalities, including:

Lemma

If (γ, θ) *is feasible for* m*, then for every* k

$$\sum_{i=1}^{k} \gamma_i^2 \leq \sum_{i=1}^{mk} \theta_i^2 \qquad \text{as in [DaHa05]} \tag{1}$$

and

$$\gamma_{km+1}^2 \leq \sum_{i=k+1}^{k+m} \theta_i^2.$$

A Geometric Description of Feasible Singular Values in Tree Tensor Formats — Sebastian Krämer, kraemer@igpm.rwth-aachen.de — RWTH Aachen University — 05.04.19 — Leipzig

Feasibility of Pairs — Necessary Inequalities

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A special case has been conjectured in QMP literature[DaHa05], and then proven by:

Theorem ([*LiPoWa14*])

If $\deg(\gamma) \leq m$ *and* (1) *holds, then the pair* (γ, θ) *is feasible for* m*.*

[*LiPoWa14*] : *C.-K. Li, Y.-T. Poon, and X. Wang, Ranks and ev. of states with prescribed reduced states, Electronic J. of Linear Algebra (2014)*

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[DaHa05] shows that one further inequality completes the list for $\deg(\gamma) \leq 3$, $m = 2$: $\gamma_2^2 + \gamma_3^2 \leq \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_6^2$. We can generalize this to (the possibly redundant)

$$\gamma_2^2 + \gamma_3^2 \leq \sum_{i=1}^{2m-1} \theta_i^2 + \theta_{2m+2}^2$$

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Which rates $v, w > 0$ of exponential decay are possible based on (1)?:

$$\gamma_i \propto e^{-iv}, \quad \theta_i \propto e^{-iw}$$
$$(\gamma, \theta) \text{ feas. for } m \Rightarrow v \in \left[\tfrac{w}{m}, wm\right]$$

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Feasibility of Pairs — Linear Programming Approach

- Let $\gamma_+^2 = (10, 2, 1, 0.25, 0.25)$, $\theta_+^2 = (4, 3, 2.5, 2, 2)$

Feasibility of Pairs — Linear Programming Approach

- Let $\gamma_+^2 = (10, 2, 1, 0.25, 0.25)$, $\theta_+^2 = (4, 3, 2.5, 2, 2)$
- (γ, θ) not feasible for $m \leq 3$ since $10 \not\leq 4 + 3 + 2.5$

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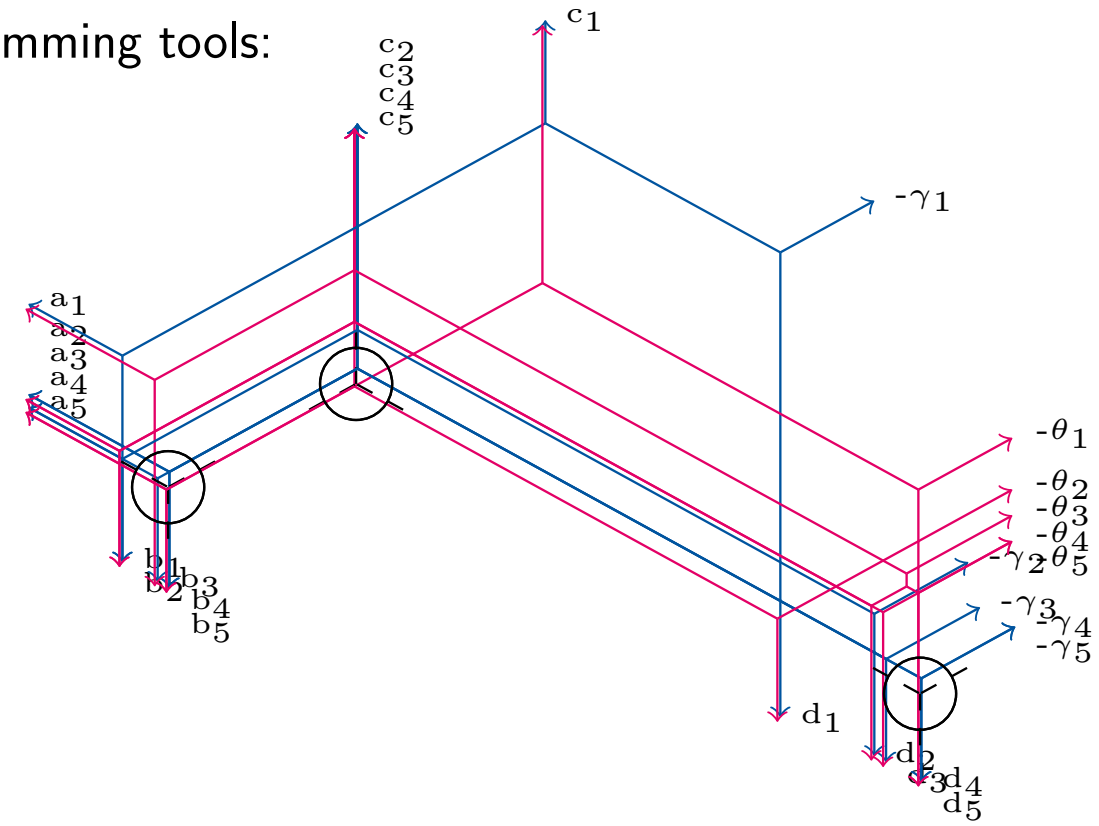
- (γ, θ) not feasible for $m \leq 3$ since $10 \not\leq 4 + 3 + 2.5$
- (constructively) feasible for $m = 5$
- $m = 4$? $\rightarrow$ use linear programming tools:



TT-Feasibility —

Feasibility and Compatibility

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TT-Feasibility

Example

We return to the introduction:

- $(\sigma_+^{(\{1\})})^2 = (9,\ 4.5)$
- $(\sigma_+^{(\{1,2\})})^2 = (10,\ 2,\ 1,\ 0.25,\ 0.25)$
- $(\sigma_+^{(\{1,2,3\})})^2 = (4,\ 3,\ 2.5,\ 2,\ 2)$

Regarding the single pairs:

- $n_1 = \deg(\sigma^{(\{1\})}) = 2$.
- $(\sigma^{(\{1\})}, \sigma^{(\{1,2\})})$ is feasible for $n_2 = 3$ as provided by the earlier special case.
- $(\sigma^{(\{1,2\})}, \sigma^{(\{1,2,3\})})$ is feasible for $n_3 = 4$ as provided by the hive.
- $n_4 = \deg(\sigma^{(\{1,2,3\})}) = 5$.

Thereby σ is (TT-)feasible for and no less than $n = (2, 3, 4, 5)$.

TT-Feasibility — Overview of Methods

Given possible TT-singular values σ and $n \in \mathbb{N}^d$, check and construct pairwise $(\gamma, \theta) = (\sigma^{(\{1,\dots,\mu-1\})}, \sigma^{(\{1,\dots,\mu\})})$, $\mu = 2, \dots, d-1$:

	prove feas.	rule out feas.	construct G_μ

Combine constructed nodes $G = (G_1, \dots, G_d)$ to obtain a TT-SVD of the desired tensor A, $\mathrm{sv}(A) = \sigma$. The construction is possible in parallel.

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classes of nec. ineq.		✓	

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classes of nec. ineq.		✓	
linear progr. hive (if well conditioned)	✓	✓	if lucky: ✓

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(1) and ($\deg(\gamma) \leq n_\mu$ or $\deg(\theta) \leq n_\mu$)[LiPoWa14]	✓	✓	

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derive H-descr. (if manageable)[DaHa05]	✓	✓	
classes of nec. ineq.		✓	
linear progr. hive (if well conditioned)	✓	✓	if lucky: ✓
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$\deg(\gamma) \leq n_\mu$ and $\deg(\theta) \leq n_\mu$	✓	✓	✓
iteratively enforce sv (if convergent)	✓		✓

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public TT-Feasibility-Toolbox on RWTH Git-Lab
Matlab code and practical introduction with live-scripts (also as pdf)

Feasibility of singular values in the Tensor Train format

A 4th-order example

We will go through one example of feasibility for a $d = 4$th-order tensor using different algorithms. Let σ_ν be as follows:

```
d = 4;
sigma = cell(1,5);
sigma{1} = sqrt(13);
sigma{2} = [3,2];
sigma{3} = [sqrt(10),sqrt(2),1];
sigma{4} = [2,sqrt(3),sqrt(2.5),sqrt(2),sqrt(1.5)];
sigma{5} = sqrt(13);
sigma{:}
```

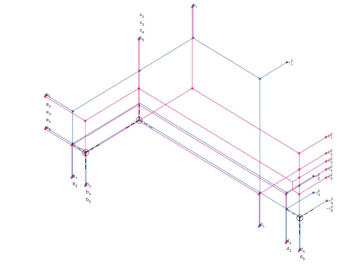
```
ans = 3.6056
ans =
     3     2
ans =
    3.1623    1.4142    1.0000
ans =
    2.0000    1.7321    1.5811    1.4142    1.2247
ans = 3.6056
```

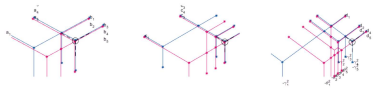
1. Using linear programming and hives:

For each pair $(\sigma^{(\mu)}, \sigma^{(\mu+1)})$ we can use the linear programming algorithm `linear_programming_check` construct hives and determine the minimal required more size n such that the pair is feasible for that value.

```
help linear_programming_check
```

```
  linear_programming_check  investigate the feasibility of a pair (gamma,theta)

    linear_programming_check(n,gamma,theta,drawoptions) finds the smallest
    mode size m <= n for which (gamma,theta) is feasible and plots the hive
    obtained through the linear programming algorithm.

    gamma and theta must be arrays of length n but may contain zeros.

    Please note that this code uses honeycomb notation (not tensor
    notation). n is the size of the hermitian matrices.

    Options :

    drawoptions.compact == 1: less distance between honeycombs
    drawoptions.single == 1: also draw single honeycombs

    Example:
         gamma = sqrt([10,2,1]);
         theta = sqrt([4,3,2.5,2,1.5]);

         drawoptions = struct;
         drawoptions.compact = 0;
```





Thereby we know that σ is feasible for $n = [2, 2, 4, 5]$ and no less (except if instabilities should have corrupted our results, which is not to be expected for such small examples).

```
n
```

```
n =
     2     2     4     5
```

2. Using the alternating iteration

For each pair we may also use the second algorithm and enforce these singular values iteratively to actually construct a representation. In order to use `alternating_core_generation` we need to specify a suspected n_μ.

The cell `G` now contains a representation of a tensor $A = \tau_\nu(G)$ which has singular values σ.

Additionally, $\mathcal{G} = G$ is the standard representation of A. Without previously finding n, we can also use `construct_tensor_with_sv` to find n and construct all cores in G in parallel.

```
help construct_tensor_with_sv
```

```
  construct_tensor_with_sv  constructs a tensor with prescribed TT-singular
  values in parallel

    construct_tensor_with_sv(n,sigma) expects a starting mode size n with
    n(mu) >= 1 for all mu and a cell sigma, containing TT-singular values,
    s.t. length(sigma) = length(n) + 1, and returns a TT-representation G
    with size(G{mu}) = [r(mu),r(mu)+1,n_plus(mu)] for the possible
    increased mode size n_plus and r(mu) = length(sigma{mu})

    construct_tensor_with_sv(n,sigma,opts) specifies options for printing
    and tol

    Example (from the talk):
         sigma = cell(1,5);
         sigma{1} = sqrt(13);
         sigma{2} = [3,2];
         sigma{3} = [sqrt(10),sqrt(2),1];
         sigma{4} = [2,sqrt(3),sqrt(2.5),sqrt(2),sqrt(1.5)];
         sigma{5} = sqrt(13)
         n = [1,1,1,1];
         [G,n_plus] = construct_tensor_with_sv(n,sigma)
         % returns n_plus = [2,2,4,5]


    See also:
         test_construct_tensor_with_sv, alternating_core_generation
```

We may specify a minimal mode size to start from. We choose $n = 1$ here. The first call may require much more time since Matlab needs to connect to the workers in order to proceed with the `parfor` loop.

```
construct_tensor_with_sv([1,1,1,1],sigma);
```

```
  Starting parallel pool (parpool) using the 'local' profile ...
  connected to 4 workers.
  sigma is (numerically) feasible for
  n =     2     2     4     5
```

In this case, the alternating iteration converges in all cases to the desired result. Note that none of the calls of `alternating_core_generation` need to communicate. In this sense, the algorithm runs completely parallel.

3. Diagonal core generation

Based on the Theorem which provides the feasibility of a pair (γ, θ) for $n = \max(\deg(\gamma), \deg(\theta))$, the algorithm `diagonal_core_generation` constructs the required core of length n.

```
G{1} = diagonal_core_generation(sigma{1},sigma{2});
```

Summary — Overview

Thank you for your attention!

(If interested, just ask or email me for the citations, slides and the current, so far revised paper.)

Feasibility and Compatibility
- Reshaping
- Tensor Feasibility Problem (TFP)
- Feasibility for the Tucker Format
- Equivalence of TFP and QMP

Tensor Tree Decompositions / SVDs
- Matrix Decompositions
- Tensor Decompositions
- Decoupling for Hierarchical Tucker
- TT and Tucker
- Decoupling for TT

Sums of Hermitian Matrices and Honeycombs
- Weyl's Problems
- Hives

Feasibility of Pairs
- Connection to Hives
- Necessary Inequalities
- Linear Programming Approach

TT-Feasibility
- Example from the Introduction
- Overview of Methods
- TT-Feasibility-(Mini)Toolbox