SIMPLICIAL SURFACES CONTROLLED BY ONE TRIANGLE

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ABSTRACT: Embeddings of combinatorial closed simplicial surfaces in EUCLIDean 3-space with all triangles congruent to one control triangle are investigated, where the control triangle may vary. Definitions and general methods for construction and classification are outlined. For one infinite family of combinatorial surfaces its dihedral symmetry is used to consruct all embeddings and to characterize the possible congruence classes of the control triangle. The investigation is motivated by problems in rigid origami.

Keywords: Simplicial Surfaces, Polytopes, Moduli Spaces, Origami, Tesselations, Symmetry

1. INTRODUCTION

The problem treated in this paper originated from a question raised by the group around Prof. M. Trautz [2], an architect at the RWTH, asking to approximate surfaces by triangular surfaces with very few congruence classes of triangles. Discretizations of surfaces in differential geometry usually lead to many congruence types of triangles, cf. [1]. On the other hand one has the theory of tesselations, which however starts from a given geometric surface.

This paper suggests an approach different from both, namely constructing the surface and the tesselation simultaneously by asking: What do triangular surfaces look like, in case all triangles are congruent? More specifically we give some examples of the following situation: start with a closed simplicial surface in EUCLIDean 3-space whose faces are all congruent. How do the other simplicial surfaces look like which are obtained from the given surface by keeping the combinatorial structure of the underlying simplicial surface fixed but modifying the congruence class of the triangles? What sort of modifications are possible in the first place? To make the problem more manageable we not only keep the combinatorial structure of the simplicial complex fixed, but prescribe what sort of EUCLIDean motions take each triangle to each of its three neighbouring triangles. Roughly speaking two motions are possible, if one does not want to be restricted to isosceles triangles: either a rotation around the common edge or a rotation by an angle of π around an axis perpendicular to the common edge through the midpoint of this edge. The first kind we call *m*-neighbouring, because intrinsically in the surface we have a mirror, the second kind we call *r*-neighbouring, because intrinsically in the surface we have a rotation of π around the midpoint of the common edge. The edges then are called of type *m* resp. *r*. We insist that these operations carry the type of edges over to the neighbouring triangles so that all triangles have the same neighbouring structure, i.e. *mmm* or *mmr* etc. so that we have a control triangle.

In this paper we outline a general method how to construct all embeddings of a given combinatorial structure into EUCLIDean 3-space. Definitions and this outline are contained in Sections 2 and 3. We then treat interesting examples, namely the octahedron and the double hexagon with *mmm*-structure and *mmr*-

structure resp, in Section 4. These examples turn out to be the first instances of infinite families of examples, namely double $2n$-gons with *mmm*-structure, respectively *mmr*-structure. They are also treated in this section. The main result is a linearization result lifting the combinatorial symmetry of the structured combinatorial simplicial surface to an EUCLIDean symmetry of the embedding. Because of transitivity properties of the automorphism group actions in the example treated, the system of quadratic equations, defining the embedding, is reduced to a system of linear equations, thus allowing to enumerate all embeddings in a uniform way. The remaining question concerns the possible congruence classes of the control triangles, which is solved in Section 5.

2. DEFINITIONS

For the purposes of this paper the following definition of simplicial surfaces will suffice.

Definition 2.1. *A* **simplicial surface** *S is a finite set $V := V(S)$ (of vertices), together with a subset $F := F(S) \subseteq \mathrm{Pot}_3(V)$ of three-subsets of V which is called the set of faces, triangles, or two-simplices. We require $V = \bigcup_{x \in F} x$ and call*

$$E := E(S) := \{\{A,B\} \subset x | x \in F, A \neq B\}$$

the set of edges of S. The following conditions must be satisfied:
1.) Any edge $e \in E$ of S belongs to at most two triangles.
2.) For any vertex $A \in V$ the set of all faces of S containing A can be arranged in a sequence $(f_1, \ldots, f_n)$ such that f_i and f_{i+1} have an edge in common.
Triangles with a common edge are called **neighboured**. *The simplicial surface is called* **closed** *if any edge belongs to exactly two triangles.*

Usually our simplial surfaces are closed. If S is closed simplicial surface: the map

$$d : V(S) \mapsto \mathbb{N} : P \mapsto |\{f \in F(S) | P \in f\}|$$

is called the degree map. Quite often S is determined uniquely up to isomorphism by the symbolic product

$$\prod_{i \in \{1,\ldots,|F(S)|\}} i^{|\{P \in V(S) | d(P) = i\}|}$$

So for instance of 3^4 denotes a tetrahedron, 4^6 an octahedron and $4^6 \cdot 6^2$ a double hexagon, which can best be described by using permutation groups as follows:

Definition 2.2. *Let V be a finite set and $G \leq S_V$ be a subgroup of the symmetric group S_V on V. For any three-subset $x \in \mathrm{Pot}_3(V)$ of V let Gx be the orbit of x under the induced action of G on $\mathrm{Pot}_3(V)$. If for $x_1, \ldots, x_k \in \mathrm{Pot}_3(V)$ the union F of the Gx_i satisfies conditions 1.) and 2.) of 2.1, then the resulting simplicial surface complex is denoted by $S := S(G, x_1, \ldots, x_k)$.*

Example 2.3. *1.) The permutation group $C_2 \times C_n := \langle a, b \rangle$ with*
$a := (1, 2n+2)$
$b := (2, 3, \ldots, n+1)(n+2, n+3, \ldots, 2n+1)$
yields the double $2n$-gon
$S(C_2 \times C_n, \{1, 2, n+2\}, \{1, 2, 2n+1\})$
of which the octahedron ($n = 2$) and double hexagon ($n = 3$) are the first examples.
2.) The same simplicial surfaces can also be obtained as $S(C_2 \times D_{2n}, \{1, 2, 2n+1\})$ with $C_2 \times D_{2n} = \langle a, b, c \rangle$ where $c :=$ $(3, n+1)(4, n) \ldots (n+2, 2n+1)(n+3, 2n) \ldots$ inverts b by conjugation.

Definition 2.4. *Let S be a simplicial surface and $\lambda : E(S) \mapsto \mathbb{R}_{>0}$ a map called* **edge-valuation**. *A* **realization** *of (S, λ) in* EUCLID*ean 3-space $(\mathbb{R}^{3\times 1}, |\ |)$ is a map $\rho : V(S) \mapsto \mathbb{R}^{3\times 1}$ such that*

$$|A - B| = \lambda(\{A,B\}) \text{ for all } \{A,B\} \in E(S).$$

ρ induces a map from $E(S)$ mapping $\{A,B\} \in E(S)$ to the convex hull of $\{\rho(A), \rho(B)\}$ and a

map from $F(S)$ mapping $\{A,B,C\} \in F(S)$ to the convex hull of $\{\rho(A),\rho(B),\rho(C)\}$. These two maps are also denoted by ρ. The realization is called **vertex-faithful** *if ρ is injective. It is called* **edge-faithful** *if it is vertex-faithful and $\rho(e)\cap\rho(f) = \rho(e\cap f)$ for all $e, f \in E(S)$. It is called* **faithful** *or an* **embedding** *if it is edge-faithful and $\rho(x)\cap\rho(y) = \rho(x\cap y)$ for all $x, y \in F(S)$.*

In realizations where all triangles are congruent, the edge-valuation λ takes at most three values. More precisely we have the following setup.

Definition 2.5. *A* **(neighbouring) structure** *Σ on a simplicial surface complex S is a surjective map $\tau : E(S) \mapsto \{1,2,3\}$ such that τ takes three different values on the set of edges of any face. The fibre of $w \in \{1,2,3\}$ under τ is called an m-class (for mirror) resp. an r-class (for rotation), if and only if for any edge $\{A,B\}$ with $\tau(\{A,B\}) = w$ and being edge to two different triangles $\{A,B,C\},\{A,B,D\} \in F(S)$ of S, then $\tau(\{A,C\}) = \tau(\{A,D\})$ resp. $\tau(\{A,C\}) = \tau(\{B,D\})$. The structure Σ is called an $s_1s_2s_3$*-**structure** *for $s_i \in \{m,r\}$, if $\tau^{-1}(\{i\})$ is an s_i-class.*

Example 2.6. *1.) The tetrahdron $S(S_4,\{1,2,3\})$ allows an rrr-structure.*
2.) $S(C_2\times D_{2n},\{1,2,n+2\})$, cf. Example 2.3, has an mmm-structure with $\tau(g\{1,2\}) = 1$, $\tau(g\{1,n+2\}) = 2$,$\tau(g\{2,n+2\}) = 3$ for all $g \in C_2\times D_{2n}$.
3.) It further has an mmr-structure with $\tau(g\{1,2\}) = \overline{g}(1)$, $\tau(g\{1,n+2\}) = \overline{g}(2)$, $\tau(g\{2,n+2\}) = \overline{g}(3)$ for all $g \in C_2\times D_{2n}$, where $^- : C_2\times D_{2n} \to S_3$ mapping the three generators a,b,c to $(1,2),(),()$.

A realization of a structured simplicial surface (S,Σ) is selfexplanatory, if one assigns a length to each class of edges. However, more can be done in the case where the abstract simplicial surface is defined by a group as in the examples above. Whereas it is a difficult problem to find all realizations since one has to solve a large system of quadratic equations (one equation for each edge), there might be a type of realization constructable from a subgroup G of the automorphism group which we will call G-equivariant. They have the advantage that the number of quadratic equations is reduced to the number $|E(S)/G|$ of orbits of G on $E(S)$ by solving certain linear equations. In the main examples of this paper all embeddings will be equivariant for the full automorphism group and therefore easily computable because $|E(S)/G| = 3$ is as small as possible.

Lemma 2.7. *Let (S,Σ) be a structured simplicial surface, $G \le \mathrm{Aut}(S,\Sigma)$ a subgroup of the automorphism group of S respecting Σ.*
1.) If $\Delta : G \to \mathrm{Isom}(\mathbb{R}^{3\times1})$ is a representation of G then a Δ-linear realization of (S,Σ) is a realization $\rho : V(S) \to \mathbb{R}^{3\times1}$ satisfying

$$\rho(gP) = \Delta(g)\rho(P) \text{ for all } P \in V(S),\ g \in G.$$

Note, these equations are linear equations for the tuple $(\rho(P))_{P\in V(S)} \in \mathbb{R}^{3\times|V(S)|}$.
2.) The relevant representations Δ can be chosen to take values in the orthogonal group $\mathrm{O}(\mathbb{R}^{3\times1})$ and is a non neccessarily irreducible constituent of degree 3 of the linearized permutation representation of G on $V(S)$ (or in terms of modules an epimorphic image of the permutation $\mathbb{R}G$-module $\mathbb{R}V$). Note that an $\mathbb{R}G$-homomorphism ρ from $\mathbb{R}V$ to $\mathbb{R}^{3\times1}$ yields the condition above for G-equivariance automatically. The realization property is satisfied if and only

$$|\rho(A)-\rho(B)| = d_i, \quad \text{for all } \{A,B\} \in \tau^{-1}(i) \subseteq E(S),$$

where $d_1,d_2,d_3 \in \mathbb{R}_{>0}$ are the assigned λ-values for the three classes, cf. Definition 2.4.

Proof. 2.) Since G is finite it fixes a point in $\mathbb{R}^{3\times1}$ so that G is conjugate to a linear group under a translation. Note also, two finite subgroups of the orthogonal group are conjugate under the full linear group, if and only if they are conjugate under the orthogonal group. The rest is clear. □

As a special property of combinatorial simplicial surfaces relevant in this paper we define G-linearizability.

Definition 2.8. *A structured simplicial surface* (S,Σ) *is called* **linearizable** *for a subgroup* $G \le \mathrm{Aut}(S,\Sigma)$ *if there is an orthogonal representation* $\Delta G \to \mathrm{Isom}(\mathbb{R}^{3\times 1})$ *of G, such that every embedding of* (S,Σ) *is* Δ*-linear.*

3. PRELIMINARIES

As a general reference for elementary geometric facts adjusted to the use of computer algebra systems [3] is quite helpful. The following simple and well known lemma will be the key to our uniqueness proofs.

Lemma 3.1. *Let* $(P_1,P_2,P_3) \in (\mathbb{R}^{3\times 1})^3$ *be three non collinear points in* EUCLIDean *3-space and* $r_1,r_2,r_3 \in \mathbb{R}_{>0}$*. Then there are at most two points* $P \in \mathbb{R}^{3\times 1}$ *with* $|P_i - P| = r_i$ *for* $i = 1,2,3$*. The point P is unique if and only if it lies in the plane spanned by* P_1,P_2,P_3*. In the case of two solutions the orthogonal reflection fixing this plane interchanges the two solutions.*

The way we proceed is based on this lemma and follows the tetrahedron philosophy:

Lemma 3.2. *Let* $(P_0,P_1,P_2,P_3) \in (\mathbb{R}^{3\times 1})^4$ *be four non coplanar points in* EUCLIDean *3-space* $(\mathbb{R}^{3\times 1},\Phi)$ *and*

$$\Gamma := (\Phi(\overrightarrow{P_0P_i},\overrightarrow{P_0P_j}))_{1\le i,j\le 3}$$

the GRAM*-Matrix of the scalar product* Φ *with respect to the vector space basis* $(\overrightarrow{P_0P_1},\dots,\overrightarrow{P_0P_3})$ *of* $\mathbb{R}^{3\times 1}$.
1.) Γ *determines and is uniquely determined by the six values of* $|\overrightarrow{P_iP_j}|$ *for* $0 \le i < j \le 3$.
2.) Any $P \in \mathbb{R}^{3\times 1}$ *is determined by its "dual" coordinates* $\Phi(\overrightarrow{P_0P_i},\overrightarrow{PP_i})$ *for* $i = 1,2,3$.
3.) Let $P \in \mathbb{R}^{3\times 1}$ *in the situation of Lemma 3.1. Let* $X := |\overrightarrow{P_0P}|^2$*. Then X satisfies the quadratic equation*

$$\det\begin{pmatrix}\Gamma & y^{tr}\\ y & X\end{pmatrix} = 0 \text{ with } y_i := \frac{1}{2}(X + r_i^2 - \Gamma_{ii})$$

for $i = 1,2,3$.

The following remark, probably dating back to ARCHIMEDES, cf. [3], is an alternative to the usual characterization $l_i + l_j > l_k$ for all $\{i,j,k\} = \{1,2,3\}$ of the lengths triple (l_1,l_2,l_3) in the EUCLIDean space.

Remark 3.3. *Let* $(L_r,L_b,L_g) \in \mathbb{R}^3_{>0}$*. There exist a nondegenerate triangle in the* EUCLIDean *plane with squared side lengths* L_i*, if and only if*

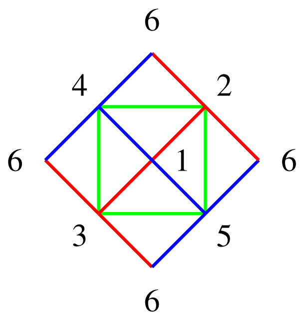
$$(L_r + L_g + L_b)^2 - 2(L_r^2 + L_g^2 + L_b^2) > 0.$$

(In that case the left side of the inequality is sixteen times the square of the area of the triangle.)

4. EMBEDDINGS FOR THE DOUBLE 2N-GONS

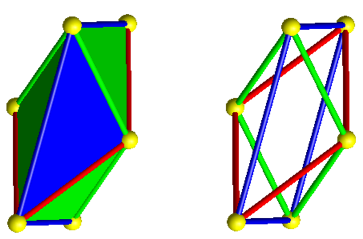
In the following tables we summarize the central data for constructing all G-equivariant embeddings into EUCLIDean 3-space where G is the combinatorial automorphism group of the structured simplicial surface $2n$-gons with *mmm* and *mmr* structure for $n = 2$ and $n = 3$. We then proceed to prove that there are no further embeddings. Finally we treat the case of general n.

Octahedron 4^6 with *mmm*-structure :



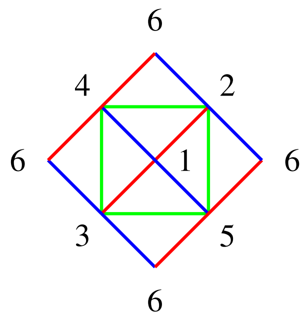
mmm-structure for 4^6

Group of automorphisms:
$C_2 \times D_4 := \langle a,b,c \rangle \cong C_2^3$ with
$a := (1,6)$, $b := (2,3)(4,5)$, $c := (4,5)$.
Representation:
$a \mapsto \text{diag}(-1,1,1)$,
$b \mapsto \text{diag}(1,-1,1)$,
$c \mapsto \text{diag}(1,1,-1)$.
orthogonal w.r.t. GRAM matrix I_3.
Affine centralizer as linear matrix group:
$\{\text{diag}(\alpha,\beta,\gamma) | \alpha,\beta,\gamma \in \mathbb{R}^*\}$.
Coordinates of control triangle:
$1 \mapsto \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}, 2 \mapsto \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}, 5 \mapsto \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$ with $x,y,z \in \mathbb{R}^*$.
Edge lengths:
$l_r^2 = y^2 + z^2, l_b^2 = x^2 + z^2, l_g^2 = x^2 + y^2$.
Sample picture: (Note all embeddings are affinely equivalent.)



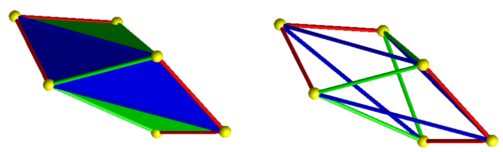
mmm-structure for 4^6 with triangle of edge lengths $1, \frac{131}{100}, \frac{102}{100}$

Octahedron 4^6 with *mmr*-structure :

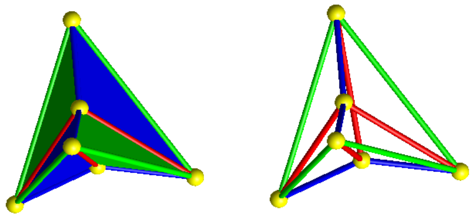


mmr-structure for 4^6

Group of automorphisms:
$D_8 := \langle a,b,c \rangle$ with
$a := (1,6)(2,5)(4,3)$, $b := (2,3)(4,5)$, $c := (4,5)$.
Representation:
$a \mapsto \text{diag}(1,-1,-1)$,
$b \mapsto \text{diag}(-1,-1,1)$,
$c \mapsto \text{diag}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1)$
orthogonal w.r.t. GRAM matrix I_3.
Affine centralizer as linear matrix group:
$\{\text{diag}(\alpha,\alpha,\beta) | \alpha,\beta \in \mathbb{R}^*\}$.
Coordinates of control triangle:
$1 \mapsto \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}, 2 \mapsto \begin{pmatrix} x \\ x \\ y \end{pmatrix}, 5 \mapsto \begin{pmatrix} x \\ -x \\ -y \end{pmatrix}$
with $x,y,z \in \mathbb{R}, x,z \neq 0$
Edge lengths:
$l_r^2 = 2x^2 + (z-y)^2, l_b^2 = 2x^2 + (z+y)^2, l_g^2 = 4x^2 + 4y^2$.
Sample pictures: (Note all embeddings with the same value $\zeta := \frac{y}{z}$ are affinely equivalent).

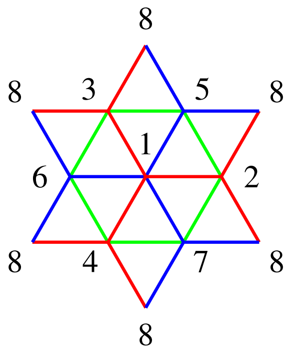


mmr-structure for 4^6 convex with triangle of edge lengths $1, \frac{184}{100}, \frac{192}{100}$



mmr-structure for 4^6 non convex with triangle of edge lengths $1, \frac{67}{100}, \frac{139}{100}$

Double hexagon $4^6 \cdot 6^2$ with *mmm*-structure:



mmm-structure for $4^6 \cdot 6^2$

Group of automorphisms:
$C_2 \times D_6 := \langle a,b,c \rangle$ with
$a := (1,8)$, $b := (2,3,4)(5,6,7)$, $c := (3,4)(5,7)$
Representation:
$a \mapsto \text{diag}(1,1,-1)$,

$b \mapsto \text{diag}(\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, 1)$,

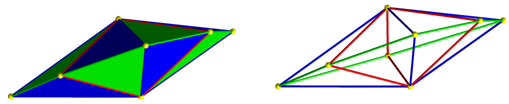
$c \mapsto \text{diag}(\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, 1)$.

orthogonal w.r.t. GRAM matrix $\text{diag}(\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, 1)$.
Affine centralizer as linear matrix group:
$\{\text{diag}(\alpha,\alpha,\beta) | \alpha,\beta \in \mathbb{R}^*\}$.
Coordinates of control triangle:

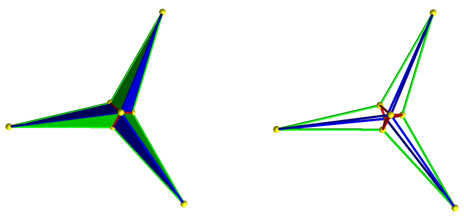
$$1 \mapsto \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}, 2 \mapsto \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, 7 \mapsto \begin{pmatrix} 0 \\ -y \\ 0 \end{pmatrix}$$

with $x,y,z \in \mathbb{R}^*, xy > 0$

Edge lengths:
$l_r^2 = 2x^2 + z^2, l_b^2 = 2y^2 + z^2, l_g^2 = 2(x^2 - xy + y^2)$.
Sample picture: (Note all embeddings with the same value $\zeta := \frac{x}{y}$ are affinely equivalent.)

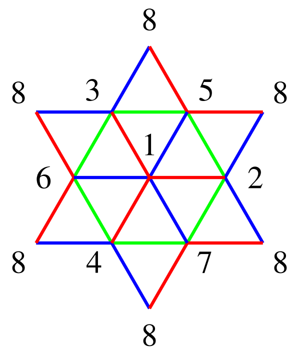


mmm-structure for $4^6 \cdot 6^2$ convex with triangle of edge lengths $1, 1, \frac{15}{16}$



mmm-structure for $4^6 \cdot 6^2$ non convex with triangle of edge lengths $1, 7, \frac{20}{3}$

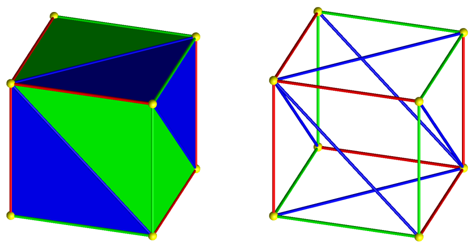
Double hexagon $4^6 \cdot 6^2$ with *mmr*-structure:



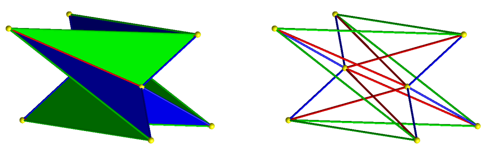
mmr-structure for $4^6 \cdot 6^2$

Group of automorphisms:
$D_{12} := \langle a,b,c \rangle \cong C_2 \times D_6$ with
$a := (1,8)(2,6)(3,7)(4,5)$,
$b := (2,3,4)(5,6,7)$,
$c := (3,4)(5,7)$

Representation:
$a \mapsto \mathrm{diag}(-1,-1,-1)$,
$b \mapsto \mathrm{diag}(\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, 1)$,
$c \mapsto \mathrm{diag}(\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, 1)$.
orthogonal w.r.t. GRAM matrix $\mathrm{diag}(\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, 1)$.
Affine centralizer as linear matrix group: $\{\mathrm{diag}(\alpha,\alpha,\beta)|\alpha,\beta \in \mathbb{R}^*\}$.
Coordinates of control triangle:
$1 \mapsto \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}, 2 \mapsto \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}, 7 \mapsto \begin{pmatrix} 0 \\ -x \\ -y \end{pmatrix}$
with $x,y,z \in \mathbb{R}, x,z \neq 0$
Edge lengths:
$l_r^2 = 2x^2 + (z-y)^2, l_b^2 = 2x^2 + (z+y)^2, l_g^2 = 2x^2 + 4y^2$.
Sample picture: (Note all embeddings with the same value $\zeta := \frac{y}{z}$ are affinely equivalent.



mmr-structure for $4^6 \cdot 6^2$ convex with triangle of edge lengths $1, 1, \sqrt{2}$



mmr-structure for $4^6 \cdot 6^2$ non convex with triangle of edge lengths $1, \frac{17}{28}, \frac{195}{56}$

Comments: In case $(4^6 \cdot 6^2, (m,m,m))$ the control triangle with $xy < 0$ lead to realizations which are not embeddings. in case $(4^6 \cdot 6^2, (m,m,r))$ a different representation is possible in principle, however it only leads to realizations that are not embeddings.

The embedding treated in the tables above are equivariant embeddings for the combinatorial automorphism group. We now prove that there are no other embeddings in the four cases treated.

Theorem 4.1. *The simplicial surfaces $S := S(C_2 \times D_{2n}, \{1,2,2n+1\})$ with mmm and mmr structure, cf. 2.3 and 2.6, are G-linearizable, where G is the full automorphism group of the structured simplicial surface.*

Proof. Let ρ be an embedding of the *mmm* structured simplicial surface S. Clearly $C_2 \times D_{2n}$ is the full automorphism group of this structured surface. ρ restricts to an embedding of the tetrahedron $1,2,2n+1,2n+2$ with edge lengths l_r, l_b, l_g, d for some $d > 0$ where $d = |\rho(1) - \rho(2n+2)|$ is the distance between the two points 1 and $2n+2$ which are not connected by an edge of the surface. The tetrahedron $\{1,2,2n+1,2n+2\}$ shares a common tetrahedron face $\{1,2,2n+2\}$ (which is not a face of the surface) with the tetrahedron $\{1,2,n+2,2n+2\}$. By Lemma 3.1 $\rho(n+2)$ can take two values, one of which is $\rho(2n+1)$. As ρ is an embedding $\rho(n+2) \neq \rho(2n+1)$ and hence uniquely determined by applying the orthogonal reflection σ_2 fixing $\rho(1), \rho(2), \rho(2n+2)$. Repeating the same argument the orthogonal reflection σ_{n+2} yields $\sigma_{n+2}(\rho(2)) = \rho(3)$. Similary $\sigma_3(\rho(n+2)) = \rho(n+3)$ etc.. Note $\sigma_3 = \sigma_{n+2}\sigma_2\sigma_{n+2}$ etc. and $\sigma_{n+2}\sigma_2$ is a rotation around the edge $\rho(1)\rho(2n+2)$. Therefore we get $(\sigma_{n+2}\sigma_2)^n = \mathrm{Id}$, more precisely $\sigma_{n+2}\sigma_2$ is a rotation by an angle $\frac{2\pi}{n}$ since ρ is an embedding and not just a realization. Let σ be the orthogonal reflection fixing all $\rho(i)$ for $1 < i < 2n+2$, which is well defined because $|\rho(1) - \rho(i)| = |\rho(2n+2) - \rho(i)|$ for $1 < i < 2n+2$. Then $a \mapsto \sigma, b \mapsto \sigma_{n+2}\sigma_2, c \mapsto \sigma_2$ defines an orthogonal representation of $C_2 \times D_{2n}$ for which ρ is Δ-linearizable.

In the *mmr* case the full automorphism group is a dihedral group of order $4n$ generated by $f := (1,2n+2)(2,n+2,3,n+3,\ldots,n+1,2n+1)$ together with b,c as in 2.3. Note, $f^2 = b$. The construction and properties of σ_i for $1 < i < 2n+2$ are the same as in the *mmm* case so that we obtain a representation Δ of $U := \langle b,c \rangle$ via $b \mapsto \sigma_{n+2}\sigma_2, c \mapsto \sigma_2$ as above. To construct the image of f under the desired representation note that U has two orbits on the set $\{2,3,\ldots,2n+1\}$ according to the corresponding cycles of b. One easily checks that each of the two orbits lies in a plane orthogonal to $\overline{\rho(1)\rho(2n+2)}$ equidistant to its midpoint and forms a regular n-gon. Their orthogonal projections in the central vertical M of $\overline{\rho(1)\rho(2n+2)}$ yields a regular $2n$-gon as convex hull. Hence one can map f onto the rotation by the angle of π/n multiplied by the orthogonal reflection fixing M and interchanging $\rho(1)$ and $\rho(2n+2)$. The rest is as above. □

Though we have used a slightly different notation in the tables than in the proof, the uniqueness of the orthogonal representations up to equivalence follows from the proof without further calculations. Note also that in all cases considered the automorphism group is transitive on each of the three types of edges. Therefore we get the following corollary.

Corollary 4.2. *For the simplicial surfaces $S := S(C_2 \times D_{2n}, \{1,2,2n+1\})$ with mmm and mmr structure, with $C_2 \times D_{2n} = \langle a,b,c \rangle$ as in 2.3, all embeddings in* EUCLID*ean 3-space can be calculated from the full automorphism group by solving linear equations only.*

5. CONTROLLING THE CONTROL TRIANGLES

The question left over in the last section is to decide for a given triangle wether and how it occurs as a control triangle for a structured simplicial surface $(S,(s_1,s_2,s_3))$. In other words, given the lengths (l_1,l_2,l_3) of some triangle, find all embeddings of S with the edges of type s_i of length l_i.

Proposition 5.1. *Let $(S,(m_r,m_b,m_g))$ denote the simplicial surface 4^6 with mmm-structure. The following four statements for a triangle T are equivalent.*
1.) The three side lengths l_i of T satisfy

$$l_i^2 + l_j^2 > l_k^2 \text{ for } \{i,j,k\} = \{r,b,g\}.$$

2.) All three angles of T are smaller than $\pi/2$.
3.) There exists an embedding of $(S,(m_r,m_b,m_g))$ into EUCLID*ean 3-space with control triangle T.*
4.) Up to EUCLID*ean motion there exists a unique embedding of $(S,(m_r,m_b,m_g))$ into* EUCLID*ean 3-space with control triangle T.*

Proof. The equivalence of 1.) and 2.) is elementary. The implication 3.) ⇒ 1.) is an immediate consequence of the table for $(4^6,(mmm))$ in Section 4. It remains to show 1.) ⇒ 4.). But this follows by a a simple linear elimination of x^2,y^2,z^2 from the equations for the l_i^2 in the table for $(4^6,(mmm))$. □

For $(4^6,(m_r,m_b,r_g))$ with *mmr*-structure the situation is slightly more complicated.

Proposition 5.2. *Let $(S,(m_r,m_b,r_g))$ denote the simplicial surface 4^6 with mmr-structure. The following four statements for a triangle T are equivalent.*
1.) The three side lengths l_i of T satisfy

$$l_r \neq l_b \text{ or } l_r^2 + l_b^2 > l_g^2 \text{ in case } l_r = l_b.$$

2.) T is an arbitrary triangle with the restriction that in the iscosceles case $l_r = l_b$ the apex angle is smaller than $\pi/2$.
3.) There exists an embedding of $(S,(m_r,m_b,r_g))$ into EUCLID*ean three*

space with control triangle T.
4.) Up to EUCLID*ean motion there exists a unique embedding of $(S,(m_r,m_b,r_g))$ into* EUCLID*ean three space with control triangle T.*

Proof. The equivalence of 1.) and 2.) is elementary. The implication 3.) $\Rightarrow$ 1.) is an immediate consequence of the table for $(4^6,(mmr))$ in Section 4. It remains to show 1.) $\Rightarrow$ 4.). Assume first that $l_r \neq_b$. Then from the equations for the l_i^2 in the table for $(4^6,(mmr))$ we obtain equations for x,y,z and we have to show that one has a unique solution with $x>0, z>0$. Substituting $y=(l_b^2-l_r^2)/(4z)$ into right hand side of $1/2(l_g^2-l_r^2)=3y^2-z^2+2yz$ yields the following biquadratic equations for z.

$$z^4+\frac{1}{2}(l_g^2-l_b^2-l_r^2)z^2-\frac{1}{16}(l_b^2-l_r^2)^2=0,$$

of which the discriminant with respect to z^2 is

$$\frac{1}{4}((l_g^2-l_r^2-l_b^2)^2+(l_r^2-l_b^2)^2).$$

This shows that there is excactly one positive solution for z. Going backwards one finds $y=(l_b^2-l_r^2)/(4z)$ and finally x. The final case $l_r=l_b$ is left to the reader. □

Theorem 5.3. *For the structured simplicial surface $(4^{2n}\cdot(2n)^2, mmm)$, i.e. the double $2n$-gon as defined in 2.3 with mmm-structure, a triangle T with side squared lengths $(L_r,L_b,L_g)\in\mathbb{R}^3_{>0}$, i. e.*

$$(L_r+L_g+L_b)^2-2(L_r^2+L_g^2+L_b^2)>0,$$

is admissible for an embedding, if and only if the following conditions hold:
a.) $w:=\frac{L_b-L_r}{L_g}$ satisfies $w^2<\frac{1}{\sin(\pi/n)^2}$.
b.) For $w\neq 0$ there is a solution of $\frac{\zeta^2-1}{\zeta^2-2\cos(\pi/n)\zeta+1}-w=0$ with $\zeta>0$ satisfying

$$\frac{L_r\zeta^2-L_b}{\zeta^2-1}>0,\ \frac{L_b-L_r}{\zeta^2-1}>0,$$

and in case $w=0$, i.e. $L_r=L_b$, one has $L_g<4\sin^2(\frac{\pi}{2n})L_r$.

Proof. Assume there is an embedding of the structured simplicial surface for a triangle with squared sidelengths (L_r,L_g,L_b). According to Theorem 4.1 the vertices of the triangle can be chosen to be

$$\begin{aligned} 1 &\mapsto (0,0,z),\\ 2 &\mapsto (x,0,0),\\ 2n+1 &\mapsto x\zeta\cdot(\cos(\frac{\pi}{n}),\sin(\frac{\pi}{n}),0)\end{aligned}$$

for $x,z,\zeta>0$ in standard CARTESean coordinates so that the squared lengths can be written as

$$\begin{aligned} L_r &= x^2+z^2,\\ L_b &= (x\zeta)^2+z^2,\\ L_g &= x^2+(x\zeta)^2-2x^2\zeta\cos(\frac{\pi}{n}).\end{aligned}$$

Hence $\zeta^2L_r-L_b=(1-\zeta^2)z^2$ yielding the first condition in b.). Similarly $L_b-L_r=(1-\zeta^2)x^2$ yields the second condition in b.). Also by substituting the squared lengths in the defintion of w leads to

$$w=\frac{\zeta^2-1}{\zeta^2-2\cos(\pi/n)\zeta+1}.$$

The right hand side is easily seen to be bounded by $|\frac{1}{\sin(\pi/n)}|$. (Note, in principle there there might be two values ζ for a given w.) The case $w=0$ corresponds to the isosceles triangle with apex in vertex 1 which obviously satisfy the above condition.
Conversely if all the conditions are satisfied, the above analysis leads to unique expressions for $x,y,z>0$ which leads to a triangle with the squared side lengths L_r,L_b,L_g and the claim follows by Theorem 4.1. □

Theorem 5.4. *For the structured simplicial surface $(4^{2n}\cdot(2n)^2, mmr)$, i.e. the double $2n$-gon as defined in 2.3 with mmr-structure, a triangle T with side squared lengths $(L_r,L_b,L_g)\in\mathbb{R}^3_{>0}$, i. e.*

$$(L_r+L_g+L_b)^2-2(L_r^2+L_g^2+L_b^2)>0,$$

is admissible for an embedding, if and only if the following conditions hold:

$$L_r \neq L_b \text{ or } L_g < 4\sin^2(\frac{\pi}{2n})L_r \text{ in case } L_r = L_b.$$

Proof. Assume there is an embedding of the structured simplicial surface for a triangle with squared sidelengths (L_r, L_g, L_b). According to Theorem 4.1 the vertices of the triangle can be chosen to be

$$\begin{aligned} 1 &\mapsto (0,0,z), \\ 2 &\mapsto (x,0,\zeta z), \\ 2n+1 &\mapsto (x\cos(\frac{\pi}{n}), x\sin(\frac{\pi}{n}), -\zeta z) \end{aligned}$$

for $x, z, \zeta > 0$ in standard CARTESean coordinates so that the squared lengths can be written as

$$\begin{aligned} L_r &= x^2 + z^2(1-\zeta)^2, \\ L_b &= x^2 + z^2(1+\zeta)^2, \\ L_g &= x^2 - 2x^2\cos(\frac{\pi}{n}) + 4\zeta^2 z^2. \end{aligned}$$

Substituting $z^2 = (L_b - L_r)/(4\zeta)$ into the right hand side of $L_g - (2-2\cos(\pi/n))L_r$ yields the following quadratic equation for ζ:

$$\zeta^2 + \frac{2((L_b+L_r)(1-\cos(\frac{\pi}{n})) - L_g))}{(\cos(\frac{\pi}{n})+1)(L_b - L_r)}\zeta + \frac{(\cos(\frac{\pi}{n})-1)}{(\cos(\frac{\pi}{n})+1)} = 0$$

which has excactly one positive solution. The rest including the isosceles case completely analogous to the previous proof. □

Of the various questions which can now be answered such as convexity, rigidity and so on, we finish this paper with the following remark.

Remark 5.5. *For the simplicial surface* $S := (4^{2n} \cdot (2n)^2)$, *i.e. the double* $2n$*-gon as defined in 2.3, with either structures* Σ, *an mmm-structure or an mmr-structure, the following holds: The function* $T \mapsto V^2_{S,T,\Sigma}/A^3_{S,T,\Sigma}$ *with* T *ranging over all possible similarity classes of control triangles for embeddings takes its unique maximum at the class of isosceles triangles with apex at the vertex of degree* $2n$ *and apex angle* $2\arcsin(\sqrt{\frac{1}{1+\cos(\pi/n)}})$. *Here* $V_{S,T,\Sigma}$ *denotes the enclosed volume and* $A^3_{S,T,\Sigma}$ *the area of the embedding of the structured surface* (S,Σ).

6. CONCLUSIONS

Finding the embeddings of a structured simplicial surface into EUCLIDean 3-space leads to a large system of quadratic equations. This system is simplified if one only looks for embeddings with symmetry. For the case of double $2n$-gons it turns out that all combinatorial symmetries carry over to EUCLIDean symmetries of the embeddings. In other words all embedings are highly symmetric and can therefore already be found by solving linear equations only. The admissable control triangles are characterized.

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