

# Multiscale and Wavelet Methods for Operator Equations

(1) Examples, motivation, key features of wavelet bases

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- Examples, motivation, key features of wavelet bases
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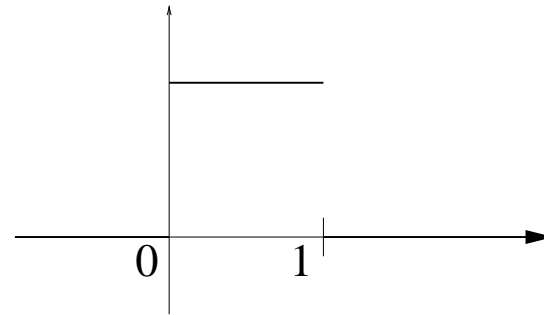
# Sparse Representations of Functions, an Example

## Single scale Approximations

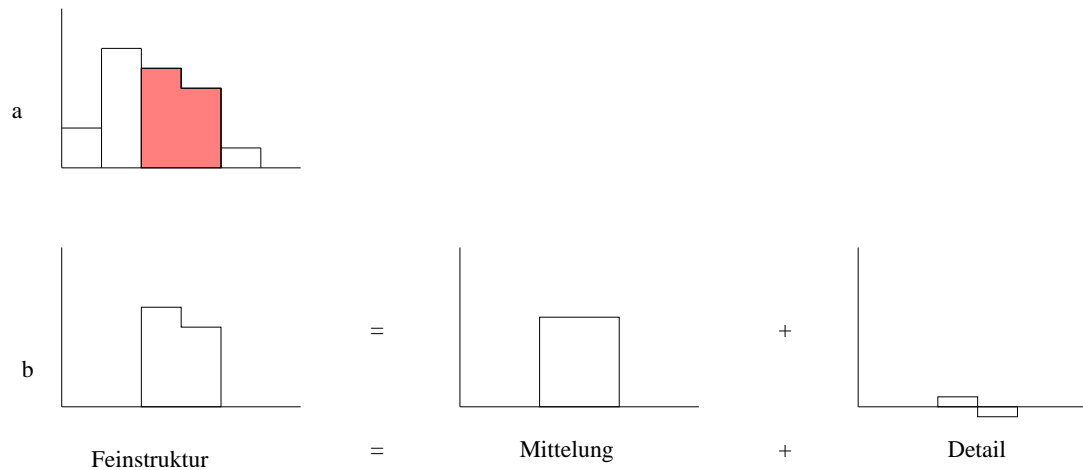
Box function  $\phi(x) = \chi_{[0,1)}(x)$

$$\phi_{j,k} = 2^{j/2} \phi(2^j \cdot -k)$$

$$P_j(f) := \sum_{k=0}^{2^j-1} \langle f, \phi_{j,k} \rangle \phi_{j,k}$$

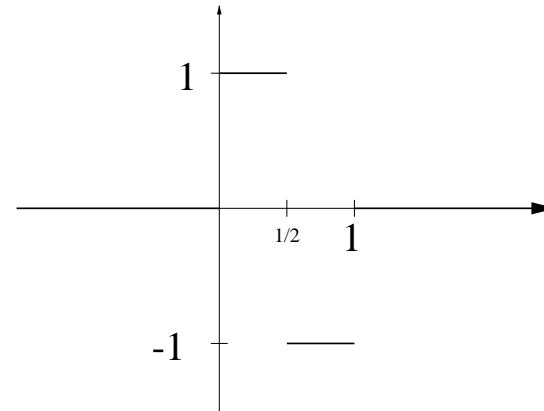


## Splitting averages and details:



## Multiscale Representations: functions – digits

- $\psi(x) := \phi(2x) - \phi(2x - 1)$
- $\psi_{j,k} := 2^{j/2} \psi(2^j \cdot -k)$
- $(P_{j+1} - P_j)f = \sum_{k=0}^{2^j-1} d_{j,k}(f) \psi_{j,k}$
- $d_{j,k}(f) = \langle f, \psi_{j,k} \rangle, d_{-1,0}(f) = \langle f, \phi_{0,0} \rangle$



$$f = P_0 f + \sum_{j=1}^{\infty} (P_j - P_{j-1})f = + \sum_{j=-1}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k}(f) \psi_{j,k} =: \mathbf{d}(f)^T \Psi$$

**Norm Equivalence:**  $f \leftrightarrow \mathbf{d}(f)$

$$\|f\|_{L_2} = \left( \sum_{j=0}^{\infty} \|(P_j - P_{j-1})f\|_{L_2}^2 \right)^{1/2} = \|\mathbf{d}(f)\|_{\ell_2}$$

Small changes in  $\mathbf{d} \iff$  small changes in  $f$

**Vanishing Moments:**

$$|d_{j,k}| = \inf_{c \in \mathbb{R}} |\langle f - c, \psi_{j,k} \rangle| \leq \inf_{c \in \mathbb{R}} \|f - c\|_{L_2(I_{j,k})} \leq 2^{-j} \|f'\|_{L_2(I_{j,k})},$$

$d_{j,k}$  is small when  $f|_{I_{j,k}}$  is smooth

## Two-Scale Relations:

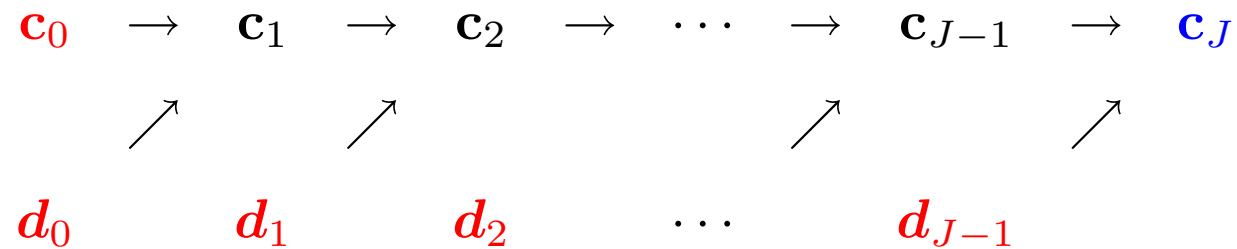
$$\begin{aligned}\phi_{j,k} &= \frac{1}{\sqrt{2}}(\phi_{j+1,2k} + \phi_{j+1,2k+1}), & \psi_{j,k} &:= \frac{1}{\sqrt{2}}(\phi_{j+1,2k} - \phi_{j+1,2k+1}) \\ \phi_{j+1,2k} &= \frac{1}{\sqrt{2}}(\phi_{j,k} + \psi_{j,k}), & \phi_{j+1,2k+1} &= \frac{1}{\sqrt{2}}(\phi_{j,k} - \psi_{j,k})\end{aligned}$$

**Change of Bases:**  $\Phi_{j+1}^T \mathbf{c}_{j+1} = \Phi_j^T \mathbf{c}_j + \Psi^T \mathbf{d}_j$

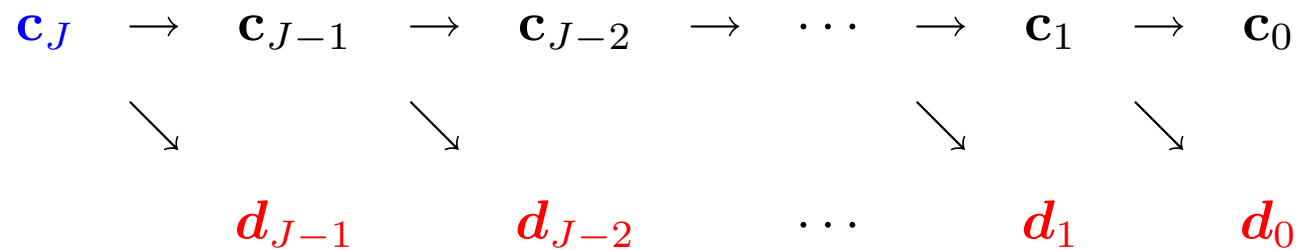
$$\begin{aligned}c_{j,k} &= \frac{1}{\sqrt{2}}(c_{j+1,2k} + c_{j+1,2k+1}), & d_{j,k} &= \frac{1}{\sqrt{2}}(c_{j+1,2k} - c_{j+1,2k+1}) \\ c_{j+1,2k} &= \frac{1}{\sqrt{2}}(c_{j,k} + d_{j,k}), & c_{j+1,2k+1} &= \frac{1}{\sqrt{2}}(c_{j,k} - d_{j,k})\end{aligned}$$

## Fast (Orthogonal) Transform (linear time):

$$\mathbf{T}_J : \mathbf{d}^J := (\mathbf{c}_0, \mathbf{d}_0, \dots, \mathbf{d}_{J-1}) \rightarrow \mathbf{c}_J$$



$$\mathbf{T}^{-1} = \mathbf{T}^T : \mathbf{c}_J \rightarrow \mathbf{d}^J$$



## (Quasi-) Sparse Representation of Operators

### The Hilbert Transform

$$(\mathcal{L}f)(x) := \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(y)}{x - y} dy$$

### Wavelet (standard) representation:

$$\mathcal{L}f = \sum_{j,k} \langle f, \psi_{j,k} \rangle \mathcal{L}\psi_{j,k} = \sum_{j,k} \left( \sum_{l,m} \langle \mathcal{L}\psi_{j,k}, \psi_{l,m} \rangle \psi_{l,m} \right) \langle f, \psi_{j,k} \rangle = \Psi^T \mathbf{L} \mathbf{d}$$

where

$$\mathbf{L} := (\langle \psi_{j,k}, \mathcal{L}\psi_{l,m} \rangle)_{(j,k),(l,m)} =: \langle \Psi, \mathcal{L}\Psi \rangle, \quad \mathbf{d} := (\langle \psi_{j,k}, f \rangle)_{(j,k)} =: \langle \Psi, f \rangle$$



$$\begin{aligned}
\pi \left| \mathbf{L}_{(j,k),(l,m)} \right| &= \left| \int_{2^{-j}k}^{2^{-j}(k+1)} \left\{ \int_{2^{-l}m}^{2^{-l}(m+1)} \left( \frac{1}{x-y} - \frac{1}{x-2^{-l}m} \right) \psi_{l,m}(y) dy \right\} \psi_{j,k}(x) dx \right| \\
\text{Taylor } \rightsquigarrow &= \left| \int_{2^{-l}m}^{2^{-l}(m+1)} \left\{ \int_{2^{-j}k}^{2^{-j}(k+1)} \frac{(y-2^{-l}m)}{(x-y_{l,m})^2} \psi_{j,k}(x) dx \right\} \psi_{l,m}(y) dy \right| \\
&= \left| \int_{2^{-l}m}^{2^{-l}(m+1)} \left\{ \int_{2^{-j}k}^{2^{-j}(k+1)} \left( \frac{(y-2^{-l}m)}{(x-y_{l,m})^2} - \frac{(y-2^{-l}m)}{(2^{-j}k-y_{l,m})^2} \right) \psi_{j,k}(x) dx \right\} \psi_{l,m}(y) dy \right|
\end{aligned}$$

$$\int_{\mathbb{R}} |\psi_{j,k}(x)| dx \leq 2^{-j/2} \quad (l \geq j) \rightsquigarrow \pi |\mathbf{L}_{(j,k),(m,l)}| \lesssim 2^{-(l+j)\frac{3}{2}} |2^{-j}k - 2^{-l}m|^{-3} = \frac{2^{-\frac{3}{2}|j-l|}}{|k - 2^{j-l}m|^3}$$

## Preconditioning

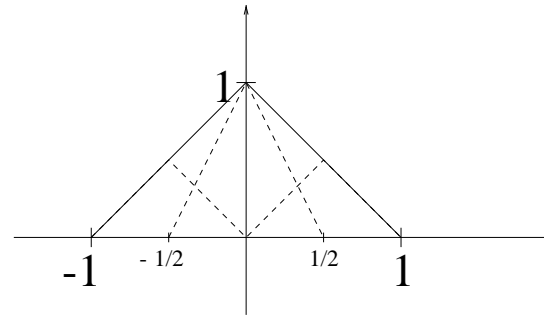
**A simple example:**  $-u'' = f$  on  $[0, 1]$ ,  $u(0) = u(1) = 0$

**Weak formulation:**  $\langle u', v' \rangle = \langle f, v \rangle$ ,  $v \in H_0^1([0, 1])$

**Hat-function:**  $\phi(x) := (1 - |x|)_+$

$$\phi_{j,k} = 2^{j/2} \phi(2^j \cdot -k)$$

$$S_j := \text{span} \{ \phi_{j,k} : k = 1, \dots, 2^j - 1 \}$$



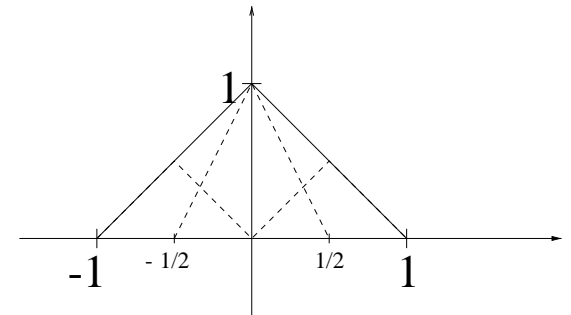
**Galerkin scheme**  $\langle u'_j, v' \rangle = \langle f, v \rangle$ ,  $v \in S_j$ ,

$$u_J = \sum_{k=1}^{2^j-1} u_{J,k} \phi_{J,k} \rightsquigarrow \mathbf{A}_J \mathbf{u}_J = \mathbf{f}_J, \quad \mathbf{A}_J := \langle \Phi'_J, \Phi'_J \rangle, \quad \mathbf{f}_J := \langle \Phi_j, f \rangle$$

**Multilevel splitting:**

$$\phi(x) = \frac{1}{2}\phi(2x+1) + \phi(2x) + \frac{1}{2}\phi(2x-1)$$

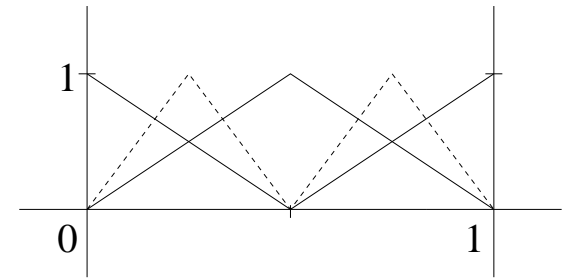
$$\phi_{j,k} = \frac{1}{\sqrt{2}} \left( \frac{1}{2}\phi_{j+1,2k-1} + \phi_{j+1,2k} + \frac{1}{2}\phi_{j+1,2k+1} \right)$$



## Hierarchical (complement) bases [Y]

$$\Psi_j := \{\psi_{j,k} := \phi_{j+1,2k+1} : k = 0, \dots, 2^j - 1\}$$

$$S_{j+1} = S_j \oplus \text{span}(\Psi_j)$$



Note  $\frac{d}{dx}\psi_{j,k}(x) = \frac{d}{dx}\phi_{j+1,2k+1}(x) = 2^{j+\frac{3}{2}}\psi_{j,k}^H(x) \rightsquigarrow$

$$\left\langle \frac{d}{dx}\psi_{j,k}, \frac{d}{dx}\psi_{l,m} \right\rangle = 2^{2j+3}\delta_{(j,k),(l,m)} \rightsquigarrow$$

**Diagonal scaling** produces uniformly bounded condition numbers!

## Summary

### Vanishing Moments – Cancellation Property (CP) $\leadsto$

- sparse representation of functions
- sparse representation of operators

### Norm Equivalence (NE) $\leadsto$

- Tight relation: function  $\leftrightarrow$  digits
- Well conditioned systems

### Possible Leeway?

- Is orthogonality of basis necessary?
- Is diagonalization of stiffness matrix necessary?

## Wavelet Bases - Functions $\leftrightarrow$ Sequences [Co, D3, D4]

**Goal:** Develop flexible concepts for the construction of multiscale bases for realistic domain geometries  $\Omega$  (bounded Euclidean domains, surfaces, manifolds) sharing in essence the features of the above examples

**General Format:**  $\Psi = \{\psi_\lambda : \lambda \in \mathcal{J}\} \subset L_2(\Omega)$  ,  $\|\psi_\lambda\|_{L_2} \sim 1$ ,  $\dim \Omega = d$ ,  $\mathcal{J} = \mathcal{J}_\phi \cup \mathcal{J}_\psi$  infinite index set where:

- $\#\mathcal{J}_\phi < \infty$ ,  $\leftrightarrow$  “scaling functions” (spanning polynomials);
- $\mathcal{J}_\psi \leftrightarrow$  “true” wavelets spanning complements between refinement levels;
- $\lambda \leftrightarrow (j = \text{scale}, e = \text{type}, k = \text{spatial location}), |\lambda| := j$

Example:  $\psi_\lambda(x, y) = 2^j \psi^{1,0}(2^j(x, y) - (k, l)) = 2^{j/2} \psi(2^j x - k) 2^{j/2} \phi(2^j y - l)$

## Notational Conventions:

- Diagonal scaling matrix:  $\mathbf{D}^{\mathbf{s}} := (\delta_{\lambda,\mu} 2^{\mathbf{s}|\lambda|})_{\lambda,\mu \in \mathcal{J}}$ ;  
scaled basis:  $\mathbf{D}^{-s}\Psi = \{2^{-s|\lambda|}\psi_\lambda\}$ ;
- Arrays of wavelet coefficients:  $\mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}$ ,  $\mathbf{d}, \mathbf{u}, \dots$ ,
- Wavelet expansions:  $\mathbf{d}^T \Psi := \sum_{\lambda \in \mathcal{J}} d_\lambda \psi_\lambda$
- Gramian matrices:  $c(\cdot, \cdot) : X \times Y \rightarrow \mathbb{R}$  bilinear form,  $\Xi \subset X, \Theta \subset Y \rightsquigarrow$

$$c(\Xi, \Theta) := (c(\xi, \theta))_{\xi \in \Xi, \theta \in \Theta}, \quad \langle \Psi, f \rangle = (\langle f, \psi_\lambda \rangle)_{\lambda \in \mathcal{J}}^T$$

## Main Features

- **Locality (L)**
- **Cancellation Properties (CP)**
- **Norm Equivalences (NE)**

**Locality (L):**

$$\Omega_\lambda := \text{supp } \psi_\lambda, \quad \text{diam } (\Omega_\lambda) \sim 2^{-|\lambda|}$$

**Cancellation Property (CP) of Order  $\tilde{m}$ :**

$$|\langle v, \psi_\lambda \rangle| \lesssim 2^{-|\lambda|(\tilde{m} + \frac{d}{2} - \frac{d}{p})} |v|_{W_p^{\tilde{m}}(\Omega_\lambda)}, \quad \lambda \in \mathcal{J}_\psi$$



$\Omega \subset \mathbb{R}^d$ , **vanishing moments** of order  $\tilde{m} \implies$  (CP):

$$\begin{aligned}
|\langle v, \psi_\lambda \rangle| &= \inf_{P \in \Pi_{\tilde{m}}} |\langle v - P, \psi_\lambda \rangle| \leq \inf_{P \in \mathbb{P}_{\tilde{m}}} \|v - P\|_{L_p(\Omega_\lambda)} \|\psi_\lambda\|_{L_{p'}}, \quad \frac{1}{p} + \frac{1}{p'} = 1 \\
&\lesssim 2^{-|\lambda|(\frac{d}{2} - \frac{d}{p})} \inf_{P \in \mathbb{P}_{\tilde{m}}} \|v - P\|_{L_p(\Omega_\lambda)} \\
&\quad (\|\psi_\lambda\|_{L_{p'}} \sim 2^{|\lambda|(\frac{d}{p'} - \frac{d}{2})} \sim 2^{|\lambda|(\frac{d}{2} - \frac{d}{p})})
\end{aligned}$$

## Local polynomial approximation

$$\inf_{P \in \mathbb{P}_k} \|v - P\|_{L_p(\Omega)} \lesssim (\text{diam } \Omega)^k |v|_{W_p^k(\Omega)}$$

**Norm Equivalences (NE):** For some  $\gamma, \tilde{\gamma} > 0$  and  $s \in (-\tilde{\gamma}, \gamma) \exists c_s, C_s$ , s.t.

$$c_s \|\mathbf{v}\|_{\ell_2} \leq \|\mathbf{v}^T \mathbf{D}^{-s} \Psi\|_{H^s} \leq C_s \|\mathbf{v}\|_{\ell_2}$$

where for  $s \geq 0$   $H_0^s(\Omega) \subset H^s \subset H^s(\Omega)$ ,  $H^s := (H^{-s})'$  when  $s < 0$ ,  
i.e.  $\mathbf{D}^{-s} \Psi$  is a **Riesz-basis** for  $H^s$ .

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**Remark 1 - A first consequence:** Let  $\|v\|_{\mathcal{H}_\epsilon}^2 := \epsilon \langle \nabla v, \nabla v \rangle + \langle v, v \rangle$ ,

$$\mathbf{D} := ((1 + \sqrt{\epsilon} 2^{|\lambda|}) \delta_{\lambda, \mu})_{\lambda, \mu}, \rightsquigarrow$$

$$(2(c_0^{-2} + c_1^{-1}))^{-1/2} \|\mathbf{v}\|_{\ell_2} \leq \|\mathbf{v}^T \mathbf{D}^{-1} \Psi\|_{\mathcal{H}_\epsilon} \leq (C_0^2 + C_1^2)^{1/2} \|\mathbf{v}\|_{\ell_2}$$

**Proof of Remark 1:** Let  $v = d^T \Psi$

$$\begin{aligned}
\|\{(1 + \sqrt{\epsilon}2^{|\lambda|})d_\lambda\}_{\lambda \in \mathcal{J}}\|_{\mathcal{J}}^2 &\leq 2 \sum_{\lambda \in \mathcal{J}} \left( |d_\lambda|^2 + \epsilon 2^{2|\lambda|} |d_\lambda|^2 \right) \\
&\stackrel{(NE)}{\leq} 2 \left( c_0^{-2} + c_1^{-2} \right) \left\{ \|v\|_{L_2}^2 + \epsilon |v|_{H^1}^2 \right\} \\
&= 2 \left( c_0^{-2} + c_1^{-2} \right) \|v\|_{\mathcal{H}_\epsilon}^2
\end{aligned}$$

$$\begin{aligned}
\|v\|_{L_2}^2 + \epsilon |v|_{H^1}^2 &\stackrel{(NE)}{\leq} C_0^2 \| \cdot \|_{\ell_2}^2 + \epsilon C_1^2 \| \mathbf{D}^1 \cdot \|_{\ell_2}^2 \leq (C_0^2 + C_1^2) \sum_{\lambda \in \mathcal{J}} (1 + \epsilon 2^{2|\lambda|}) |d_\lambda|^2 \\
&\leq (C_0^2 + C_1^2) \sum_{\lambda \in \mathcal{J}} (1 + \sqrt{\epsilon} 2^{|\lambda|})^2 |d_\lambda|^2
\end{aligned}$$

**Remark 2 - a second consequence - duality:** Let  $\mathcal{H}$  be a Hilbert space,  $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H}' \rightarrow \mathbb{R}$  and suppose that

$$c\|\mathbf{v}\|_{\ell_2} \leq \|\mathbf{v}^T \Theta\|_{\mathcal{H}} \leq C\|\mathbf{v}\|_{\ell_2}$$

$(\Theta \text{ is Riesz-basis for } \mathcal{H}) \implies \textcolor{red}{(\mathbf{NE})}'$ :

$$\textcolor{red}{C^{-1}\|\langle \Theta, v \rangle\|_{\ell_2} \leq \|v\|_{\mathcal{H}'} \leq c^{-1}\|\langle \Theta, v \rangle\|_{\ell_2}} \quad (1)$$

**Application:**  $\mathcal{H} = H_0^1(\Omega)$ ,  $\Theta := \mathbf{D}^{-1}\Psi$ ,  $\rightsquigarrow$

$$\textcolor{red}{C_1^{-1}\|\mathbf{D}^{-1}\langle \Psi, w \rangle\|_{\ell_2} \leq \|w\|_{H^{-1}(\Omega)} \leq c_1^{-1}\|\mathbf{D}^{-1}\langle \Psi, w \rangle\|_{\ell_2}}$$

## Proof of Remark 2:

1) Let  $F : \ell_2 \rightarrow \mathcal{H}$ ,  $F : \mathbf{v} \rightarrow \mathbf{v}^T \Theta$

$$c\|\mathbf{v}\|_{\ell_2} \leq \|\mathbf{v}^T \Theta\|_{\mathcal{H}} \leq C\|\mathbf{v}\|_{\ell_2} \implies \|F\|_{\ell_2 \rightarrow \mathcal{H}} = C, \quad \|F^{-1}\|_{\mathcal{H} \rightarrow \ell_2} = c^{-1}$$

2) For the **adjoint**  $F^* : \mathcal{H}' \rightarrow \ell_2$  defined by  $\langle F\mathbf{v}, w \rangle = \mathbf{v}^T F^* w$  one has

$$\|F^* w\|_{\ell_2} = \sup_{\mathbf{v}} \frac{(F^* w)^T \mathbf{v}}{\|\mathbf{v}\|_{\ell_2}} = \sup_{\mathbf{v}} \frac{\langle F\mathbf{v}, w \rangle}{\|\mathbf{v}\|_{\ell_2}} \leq \sup_{\mathbf{v}} \frac{\|F\mathbf{v}\|_{\mathcal{H}} \|w\|_{\mathcal{H}'}}{\|\mathbf{v}\|_{\ell_2}} = \|F\|_{\ell_2 \rightarrow \mathcal{H}} \|w\|_{\mathcal{H}'}$$

$$\|F\mathbf{v}\|_{\mathcal{H}} = \sup_w \frac{\langle F\mathbf{v}, w \rangle}{\|w\|_{\mathcal{H}'}} \sup_w \frac{(F^* w)^T \mathbf{v}}{\|w\|_{\mathcal{H}'}} \leq \sup_w \frac{\|F^* w\|_{\ell_2} \|\mathbf{v}\|_{\ell_2}}{\|w\|_{\mathcal{H}'}} = \|F^*\|_{\mathcal{H}' \rightarrow \ell_2} \|\mathbf{v}\|_{\ell_2}$$

$$\implies \|F\|_{\ell_2 \rightarrow \mathcal{H}} = \|F^*\|_{\mathcal{H}' \rightarrow \ell_2} = C, \quad \|F^{-1}\|_{\mathcal{H} \rightarrow \ell_2} = \|(F^*)^{-1}\|_{\ell_2 \rightarrow \mathcal{H}'} = c^{-1}$$

3) Identify  $F^* w$ :  $(F^* w)_\lambda = (F^* w)^T \mathbf{e}_\lambda = \langle F\mathbf{e}_\lambda, w \rangle = \langle \theta_\lambda, w \rangle \rightsquigarrow$

$$F^* w = \langle \Theta, w \rangle \rightsquigarrow \|\langle \Theta, w \rangle\|_{\ell_2} \leq C\|w\|_{\mathcal{H}'}$$

## Riesz-Bases - Biorthogonality

**Note:**  $c\|\mathbf{v}\|_{\ell_2} \leq \|\mathbf{v}^T \Theta\|_{\mathcal{H}} \leq C\|\mathbf{v}\|_{\ell_2} \rightsquigarrow$

$F : \ell_2 \rightarrow \mathcal{H}$   $F : \mathbf{v} \rightarrow \mathbf{v}^T \Theta$ , adjoint  $F^* : \mathcal{H}' \rightarrow \ell_2$  defined by  $\langle F\mathbf{v}, w \rangle = \mathbf{v}^T F^* w$

are **topological isomorphisms** and  $F\mathbf{e}_\lambda = \theta_\lambda$  – Define:  $\tilde{\theta}_\lambda := (F^*)^{-1}\mathbf{e}_\lambda \rightsquigarrow$

$$\langle \theta_\lambda, \tilde{\theta}_\mu \rangle = \langle F\mathbf{e}_\lambda, (F^*)^{-1}\mathbf{e}_\mu \rangle = \mathbf{e}_\lambda^T \mathbf{e}_\mu = \delta_{\lambda,\mu} \rightsquigarrow \langle \Theta, \tilde{\Theta} \rangle = \mathbf{I} \rightsquigarrow$$

$$w = \langle w, \Theta \rangle \tilde{\Theta}, \quad (1) \rightsquigarrow \tilde{\Theta} \text{ is a Riesz basis for } \mathcal{H}'$$

**Corollary:**  $\mathcal{H} = L_2(\Omega) = \mathcal{H}' \rightsquigarrow$  for every Riesz basis  $\Psi$  for  $L_2$  there exists a **biorthogonal** basis  $\tilde{\Psi}$  which is also a Riesz basis for  $L_2$ .

## Further Norm Equivalences - Besov-Spaces [BL, Co, DeV, DPJ, DP]

Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ . Renormalize:

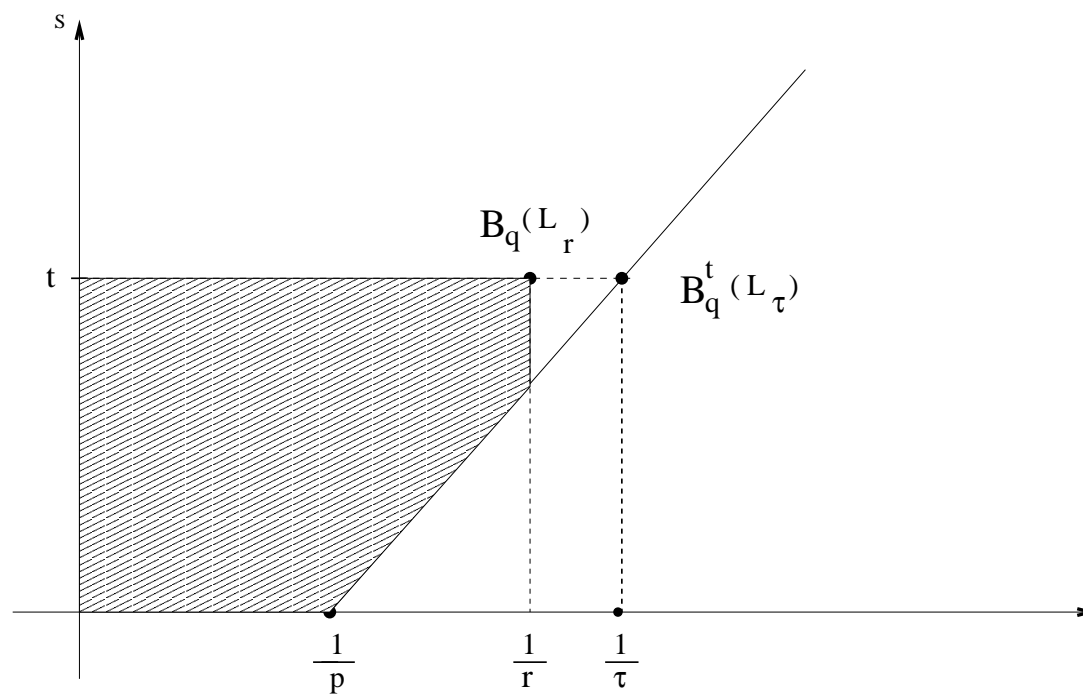
$$\psi_{\lambda,p} := 2^{d|\lambda|(\frac{1}{p}-\frac{1}{2})}\psi_{\lambda} \rightsquigarrow \|\psi_{\lambda,p}\|_{L_p} \sim 1, \quad \Psi_p := \{\psi_{\lambda,p} : \lambda \in \mathcal{J}\} \rightsquigarrow \langle \Psi_p, \tilde{\Psi}_{p'} \rangle = 1$$

Suppose  $B_q^t(L_{\tau}) \subset L_p$ . For some  $\gamma > 0$  and  $0 < s < \gamma$  one has

$$\|v\|_{B_q^t(L_{\tau})}^q \sim \|v\|_{L_{\tau}}^q + \sum_{j=0}^{\infty} \left( 2^{jd(\frac{t}{d} + \frac{1}{p} - \frac{1}{\tau})} \|\langle \tilde{\Psi}_{j,p'}, v \rangle\|_{\ell_{\tau}} \right)^q, \quad \tilde{\Psi}_{j,p'} := \{\tilde{\psi}_{\lambda,p'} : |\lambda| = j\}$$

## Embedding - the Critical Line:

$$\frac{t}{d} + \frac{1}{p} = \frac{1}{\tau} \quad \leadsto \quad \|v\|_{B_{\tau}^t(L_{\tau})} \sim \|v\|_{L_{\tau}} + \|\langle \tilde{\Psi}_{p'}, v \rangle\|_{\ell_{\tau}}$$





## Function Spaces [A, BL, T]

**Sobolev spaces:**  $W_p^k(\Omega) := \{f : \partial^\alpha f \in L_p(\Omega), |\alpha| \leq k\}$

$$|f|_{W_p^k(\Omega)} := \left( \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L_p(\Omega)}^p \right)^{1/p}, \quad \|v\|_{W_p^k(\Omega)}^p := \sum_{m=0}^k |f|_{W_p^m(\Omega)}^p$$

**Fractional order spaces - An intrinsic norm:**  $k := \lfloor t \rfloor$

$$\|v\|_{W_p^t(\Omega)} = \left( \|v\|_{W_p^k(\Omega)}^p + \sum_{|\alpha|=k} \int_{\Omega} \int_{\Omega} \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^p}{|x - y|^{d+tp}} dx dy \right)^{1/p}$$

Interpolation  $\leadsto$  **Besov-spaces:**  $t < r$

$$|v|_{B_q^t(L_p(\Omega))} := \begin{cases} \left( \int_0^\infty [s^{-t} \omega_r(f, s, \Omega)_p]^q \frac{ds}{s} \right)^{1/q}, & 0 < q < \infty; \\ \sup_{s>0} s^{-t} \omega_r(f, s, \Omega)_p, & q = \infty, \end{cases}$$

$L_p$ -modulus of continuity:

$$\omega_r(f, t, \Omega)_p := \sup_{|h| \leq t} \|\Delta_h^r f\|_{L_p(\Omega_{r,h})}, \quad \Delta_h f := f(\cdot + h) - f(\cdot), \quad \Delta_h^k = \Delta_h \circ \Delta_h^{k-1}$$

where  $\Omega_{r,h} := \{x : x + sh \in \Omega, \ s \in [0, 1]\}$

**Special cases:**

- $W_p^t = B_p^t(L_p)$  for  $s > 0, s \notin \mathbb{N}$  ( $p \neq 2$ )
- $H^t := W_2^t = B_2^t(L_2)$  for  $s \in \mathbb{R}$  ( $H^{-t} := (H^t)'$ )

$$\|f\|_{X'} := \sup_{g \in X, g \neq 0} \frac{\langle f, g \rangle}{\|g\|_X}$$

## Local Polynomial Approximation, see e.g. [DS]

$$\inf_{P \in \mathbb{P}_k} \|v - P\|_{W_p^m(\Omega)} \lesssim (\text{diam } \Omega)^{k-m} |v|_{W_p^k(\Omega)}, \quad (m < k)$$

$$\inf_{P \in \mathbb{P}_k} \|v - P\|_{L_p(\Omega)} \lesssim (\text{diam } \Omega)^t |v|_{B_q^t(L_p(\Omega))}$$

$$\inf_{P \in \mathbb{P}_k} \|v - P\|_{L_p(\Omega)} \lesssim \omega_k(f, \text{diam } \Omega, \Omega)_p$$

**Idea of proof:** By rescaling it suffices to consider reference domain with **unit diameter** – suppose that  $\inf_{P \in \mathbb{P}_k} \|v_n - P\|_{W_p^m(\Omega)} \geq n |v_n|_{W_p^k(\Omega)}$  - rescale -  $\rightsquigarrow$

$$1 = \inf_{P \in \mathbb{P}_k} \|w_n - P\|_{W_p^m(\Omega)} = \|w_n\|_{W_p^m(\Omega)} \geq n |w_n|_{W_p^k(\Omega)} \rightsquigarrow$$

$\{w_n\}_n$  precompact in  $W_p^m \rightsquigarrow \exists w \in W_p^m$  s.t.  $|w|_{W_p^k(\Omega)} = \lim_{n \rightarrow \infty} |w_n|_{W_p^k(\Omega)} = 0 \rightsquigarrow w \in \mathbb{P}_k$ . On the other hand,  $\inf_{P \in \mathbb{P}_k} \|w - P\|_{W_p^m(\Omega)} = 1$  a contradiction  $\square$

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