Multiscale and Wavelet Methods for Operator Equations

(4) Adaptive wavelet schemes (I)

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Adaptive Wavelet Schemes [CDD1, CDD2, DDU]

Goal: Adaptive tracking of the *significant* wavelet coefficients of $U = \mathbf{U}^T \mathbf{D}^{-1} \mathbf{\Psi}$

$$\mathbf{U} \to \mathbf{U}(\epsilon) \quad \text{s.t.} \quad \|\mathbf{U} - \mathbf{U}(\epsilon)\|_{\ell_2} \leq \epsilon$$

with $\operatorname{supp} \mathbf{U}(\epsilon)$ as small as possible

Main Theme: How to Extract Significant Information?

Explicitly given data:

images, signals, geometry

given functions

Implicitly given data:

PDEs, integral equations

solutions of operator eqs.

Nonlinear / Best N-Term – Approximation

practically feasible

idealized bench mark

for adaptive techniques

Adaptivity from Several Perspectives

- **Complexity Theory:** Adaptivity does **not** help! (Novak, Traub, Woczniakowski,...[N, TW])

- **Finite Elements:** Local mesh refinements – a-posteriori error estimators/indicators (Babuska/Rheinboldt, Bank et al, Verfürth, Johnson et al., Rannacher et al.,...Dörfler [Do, BR, BW, BEK, EEHJ])

- **Wavelet analysis:** Wavelet coefficients serve as error indicators (Averbuch/Beylkin/Coifman/Israeli, Bertoluzza/Maday, Canuto/Cravero, Dahlke/D./Hochmuth/Schneider,...[ABCI, Ber, BK, CM, DDHS, DHU])

  No convergence rates/proofs ! **d.o.f** $\overset{???}{\longleftarrow\!-\!\longrightarrow}$ **error**

Implicit Data – PDEs

- Hyperbolic Problems:
  Multiresolution of functionals of the solution - compression - perturbation analysis
  A. Harten, R. Abgral, F. Arrandiga, A. Cohen, W. D., R. DeVore, R. Donat, B. Lucier, O. Kaber, S. Müller, M. Postel,...

- Elliptic/Parabolic Systems: Primal multiresolution – convergence/complexity analysis
  Scope of problems:
  - elliptic boundary value problems
  - singular integral - boundary integral equations, transmission problems
  - noncoercive problems: saddle point problems (Stokes), multiple field problems

Main Issues

- **Work/Accuracy Balance:**
  A-priori estimates for quasi-uniform meshes:

$$\|u - u_h\|_{H^t} \lesssim h^\alpha \|u\|_{H^{t+\alpha}} = \epsilon \sim N^{-\alpha/d} \qquad N = \epsilon^{-d/\alpha}$$

Adaptive/nonlinear schemes: **d.o.f/work** N $\longleftarrow\overset{???}{-}\longrightarrow$ **accuracy** ϵ

$$u \in B^{t+\alpha} \implies \|u - u(\epsilon)\|_{H^t} \le \epsilon, \quad N = \epsilon^{-d/\alpha}$$

- **Stabilizing effect No LBB condition!**

Classical Approach

- Variational formulation $\longrightarrow$ discretization $\longrightarrow$ finite dimensional problem $\longrightarrow$ fast solvers

- Obstructions: Large systems, ill-conditioning, compatibility constraints (like LBB condition)

New Paradigm [CDD2]

(I) Variational formulation $\rightarrow$ mapping property

(II) Transformation into equivalent ∞ - dimensional well-posed ℓ_2 problem

(III) Convergent iteration for the ∞ - dimensional ℓ_2-problem

(IV) Adaptive application of operators

(III) Convergent Iteration for the ∞-Dimensional Problem

Objective:

$$\mathbf{LU} = \mathbf{F} \quad \Longleftrightarrow \mathbf{Mp} = \mathbf{G}$$

s.t. still

$$c_M \|\mathbf{q}\|_{\ell_2} \leq \|\mathbf{Mq}\|_{\ell_2} \leq C_M \|\mathbf{q}\|_{\ell_2}$$

and

$$\mathbf{p}^{n+1} = \mathbf{p}^n + \omega(\mathbf{G} - \mathbf{Mp}^n), \qquad \|\mathbf{I} - \omega\mathbf{M}\|_{\ell_2 \to \ell_2} \leq \rho < 1$$

Choice of $\mathbf{M}$

- Wavelet least squares formulation: (A. Cohen, R. DeVore, A. Kunoth, R. Schneider, W.D., [CDD2, DKS])

$$\mathbf{M} := \mathbf{L}^T\mathbf{L}, \quad \mathbf{G} := \mathbf{L}^T\mathbf{F}, \quad \mathbf{q} = \mathbf{U}$$

$$\mathbf{M}\mathbf{U} = \mathbf{G} \quad \Longleftrightarrow \quad \|\mathcal{L}U - F\|_{\mathcal{H}'} = \min_{V\in\mathcal{H}} \|\mathcal{L}V - F\|_{\mathcal{H}'}$$

- Saddle point problems: ([DDU])

$$\mathbf{M} = \text{Schur complement}$$

...recall: in the saddle point case: $\mathcal{H} = X \times M$

$$X \leftrightarrow \Psi_X \qquad M \leftrightarrow \Psi_M$$

$$c_X \|\mathbf{v}\|_{\ell_2(\mathcal{J}_X)} \leq \|\mathbf{v}^T \mathbf{D}_X^{-1} \Psi_X\|_X \leq C_X \|\mathbf{v}\|_{\ell_2(\mathcal{J}_X)},$$

$$c_M \|\mathbf{q}\|_{\ell_2(\mathcal{J}_M)} \leq \|\mathbf{q}^T \mathbf{D}_M^{-1} \Psi_M\|_M \leq C_M \|\mathbf{q}\|_{\ell_2(\mathcal{J}_M)},$$

Stokes: $\rightsquigarrow$ $D_{X,\lambda} = 2^{|\lambda|}, \quad D_{M,\lambda} = 1$

Saddle Point Problem Continued

$$\boldsymbol{A} := \mathbf{D}_X^{-1} a(\Psi_X, \Psi_X) \mathbf{D}_X^{-1}, \quad \mathbf{B} := \mathbf{D}_M^{-1} b(\Psi_M, \Psi_X) \mathbf{D}_X^{-1}$$

$$\mathbf{f} := \mathbf{D}_X^{-1} \langle \Psi_X, f \rangle_X, \quad \mathbf{g} := \mathbf{D}_M^{-1} \langle \Psi_M, g \rangle_M, \qquad \leadsto$$

$$\mathbf{L}\mathbf{U} = \mathbf{F} \quad \Longleftrightarrow \quad \underbrace{\begin{pmatrix} \boldsymbol{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix}}_{\mathbf{L}} \underbrace{\begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix}}_{\mathbf{U}} = \underbrace{\begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}}_{\mathbf{F}} \quad \leadsto$$

$$c_L \|\mathbf{V}\|_{\ell_2} \leq \|\mathbf{L}\mathbf{V}\|_{\ell_2} \leq C_L \|\mathbf{V}\|_{\ell_2}, \quad \mathbf{V} \in \ell_2$$

$$\underbrace{\begin{pmatrix} \boldsymbol{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix}}_{\mathbf{L}} \underbrace{\begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix}}_{\mathbf{U}} = \underbrace{\begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}}_{\mathbf{F}} \quad \Longleftrightarrow \quad \begin{cases} \mathbf{M}\mathbf{p} := \mathbf{B}\boldsymbol{A}^{-1}\mathbf{B}^T\mathbf{p} & = \quad \mathbf{B}\boldsymbol{A}^{-1}\mathbf{f} - \mathbf{g} =: \mathbf{G} \\ \boldsymbol{A}\mathbf{u} & = \quad \mathbf{f} - \mathbf{B}^T\mathbf{p} \end{cases}$$

$$\mathbf{M} := \mathbf{B}\boldsymbol{A}^{-1}\mathbf{B}^T : \ell_2(\mathcal{J}_M) \to \ell_2(\mathcal{J}_M), \quad \|\mathbf{M}\mathbf{q}\|_{\ell_2(\mathcal{J}_M)} \sim \|\mathbf{q}\|_{\ell_2(\mathcal{J}_M)}$$

$$\begin{aligned} \mathbf{p} \quad &= \quad \mathbf{p} + \omega \left((\mathbf{B}\boldsymbol{A}^{-1}\mathbf{f} - \mathbf{g}) - \mathbf{M}\mathbf{p} \right) \\ &= \quad \mathbf{p} + \omega \left(\mathbf{B}\, \underbrace{\boldsymbol{A}^{-1}(\mathbf{f} - \mathbf{B}^T\mathbf{p})}_{=\mathbf{u}} - \mathbf{g} \right) = \mathbf{p} + \omega(\mathbf{B}\mathbf{u} - \mathbf{g}) \quad \rightsquigarrow \end{aligned}$$

$$\rightsquigarrow \quad \mathbf{p}^{n+1} = \mathbf{p}^n + \omega(\mathbf{B}\mathbf{u}^n - \mathbf{g}), \quad \boldsymbol{A}\mathbf{u}^n = \mathbf{f} - \mathbf{B}^T\mathbf{p}^n$$

(IV) Approximate Iteration

$$\mathbf{MU} = \mathbf{G} \quad \rightsquigarrow \quad \mathbf{U}^{n+1} = \mathbf{U}^n + \omega(\mathbf{G} - \mathbf{MU}^n)$$

The Basic Routines: Need

RHS $[\eta, \mathbf{G}] \to \mathbf{G}_\eta$: s.t. $\|\mathbf{G} - \mathbf{G}_\eta\|_{\ell_2} \leq \eta$

APPLY $[\eta, \mathbf{M}, \mathbf{V}] \to \mathbf{W}_\eta$: s.t. $\|\mathbf{MV} - \mathbf{W}_\eta\|_{\ell_2} \leq \eta$

COARSE $[\eta, \mathbf{W}] \to \bar{\mathbf{W}}_\eta$: s.t. $\|\mathbf{W} - \bar{\mathbf{W}}_\eta\|_{\ell_2} \leq \eta$

The Adaptive Algorithm

SOLVE $[\epsilon, \mathrm{M}, \mathbf{G}] \to \bar{\mathbf{U}}(\epsilon)$

(i) *Set* $\bar{\mathbf{U}}^0 = \mathbf{0}, \quad \epsilon_0 := c_M^{-1}\|\mathbf{G}\|_{\ell_2}, \quad j = 0$

(ii) *If* $\epsilon_j \le \epsilon$, *stop* $\bar{\mathbf{U}}^j \to \mathbf{U}(\epsilon)$. *Else* $\mathbf{V}^0 := \bar{\mathbf{U}}^j$.

(ii.1) **RHS** $[\rho^l \epsilon_j, \mathbf{G}] \to \mathbf{G}_l$; **APPLY** $[\rho^l \epsilon_j, \mathrm{M}, \mathbf{V}^l] \to \mathbf{W}^l$, $l = 0, \ldots, K-1$

$$\mathbf{V}^{l+1} := \mathbf{V}^l + \alpha(\mathbf{G}_l - \mathbf{W}^l)$$

(ii.2) **COARSE** $[\mathbf{V}^K, 2\epsilon_j/5] \to \bar{\mathbf{U}}^{j+1}$, $\epsilon_{j+1} := \epsilon_j/2$, $j+1 \to j$ *go to* **(ii)**.

PROPOSITION: $\|\mathbf{U} - \bar{\mathbf{U}}^j\|_{\ell_2} \le \epsilon_j$, **Question:** $\#\mathbf{supp}\,\bar{\mathbf{U}}(\epsilon) \;\longleftarrow \overset{?}{-} \longrightarrow\; \epsilon$

Ideal Bench Mark - Best N-Term Approximation

$$\sigma_{N,\ell_2}(\mathbf{V}) := \|\mathbf{V} - \mathbf{V}_N\|_{\ell_2} = \min_{\#\mathrm{supp}\,\mathbf{W} \le N} \|\mathbf{V} - \mathbf{W}\|_{\ell_2}$$

$$(\mathbf{NE}) \implies \sigma_{N,\ell_2}(\mathbf{V}) \sim \inf_{\mathbf{W},\#\mathrm{supp}\,\mathbf{W} \le N} \|V - \mathbf{W}^T \mathbf{D}^{-1} \Psi\|_{\mathcal{H}}$$

Coarsening and best N-term approximation:

Let $\|\mathbf{v} - \mathbf{w}\|_{\ell_2} \le \eta/5$ with $\#\mathrm{supp}\,\mathbf{w} < \infty$, $\bar{\mathbf{w}}_\eta := \mathbf{COARSE}\,[\mathbf{w}, 4\eta/5]$.

- $\|\mathbf{v} - \mathbf{v}_N\|_{\ell_2} \lesssim N^{-s} \quad \overset{[\text{CDD1}]}{\Longrightarrow} \quad \#\mathrm{supp}\,\bar{\mathbf{w}}_\eta \lesssim \eta^{-1/s}, \quad \|\mathbf{v} - \bar{\mathbf{w}}_\eta\|_{\ell_2} \le \eta$

Compressible Matrices

Def.: $\mathbf{C} \in \mathcal{C}_{s^*}$ *if for any* $0 < s < s^*$ *and every* $j \in \mathbb{N}$ *there exists a matrix* $\mathbf{C}_j$ *obtained by replacing all but the order of* $\alpha_j 2^j$ $(\sum_j \alpha_j < \infty)$ *entries per row and column in* $\mathbf{C}$ *by zero while still*

$$\|\mathbf{C} - \mathbf{C}_j\| \leq \alpha_j 2^{-js}, \quad j \in \mathbb{N}, \quad \sum_j \alpha_j < \infty$$

$$(\textbf{Cancellation Properties}) \quad \Longrightarrow$$

Remark: The scaled wavelet representations $\mathbf{C} = \mathbf{D}^{-1} c(\Psi, \Psi) \mathbf{D}^{-1}$ of "all" differential and singular integral operators belong to $\mathcal{C}_{s^*}$ for some $s^* = s^*(\mathcal{L}, \Psi) > 0$.

Fast approximate matrix/vector multiplication [CDD1]

Define $\mathbf{v}_{[j]} := \mathbf{v}_{2^j},\ v_{[-1]} := 0$ and

$$\mathbf{w}_j := \boldsymbol{A}_j \mathbf{v}_{[0]} + \boldsymbol{A}_{j-1}(\mathbf{v}_{[1]} - \mathbf{v}_{[0]}) + \cdots + \boldsymbol{A}_0(\mathbf{v}_{[j]} - \mathbf{v}_{[j-1]}) \qquad (1)$$

$$\|\boldsymbol{A}\mathbf{v} - \mathbf{w}_j\|_{\ell_2} \quad \leq \quad c \underbrace{\|\mathbf{v} - \mathbf{v}_{[j]}\|_{\ell_2}}_{\sigma_{2^j,\ell_2}(\mathbf{v})} + \sum_{l=0}^{j} \alpha_l 2^{-ls} \underbrace{\|\mathbf{v}_{[j-l]} - \mathbf{v}_{[j-l-1]}\|_{\ell_2}}_{\lesssim \ \sigma_{2^{j-l-1},\ell_2}(\mathbf{v})} \quad \rightsquigarrow$$

MULT $[\eta, \boldsymbol{A}, \mathbf{v}] \to \mathbf{w}_\eta$ s.t.: $\|\boldsymbol{A}\mathbf{v} - \mathbf{w}_\eta\| \leq \eta$

Theorem: If $\boldsymbol{A} \in \mathcal{C}_{s^*}$ and $\|\mathbf{v} - \mathbf{v}_N\|_{\ell_2} \ \lesssim \ N^{-s}$, then

$$\#\mathrm{supp}\,\mathbf{w}_\eta,\quad \#\mathbf{flops} \ \lesssim \ \eta^{-1/s}, \quad (\#\mathbf{sort\ ops} \ \lesssim \ \eta^{-1/s}|\log \eta|)$$

$$\textsf{\textbf{APPLY}}\,[\eta, \mathbf{M}, \mathbf{V}], \quad \textsf{\textbf{RHS}}\,[\eta, \mathbf{G}]$$

Composition of **COARSE** $[c\eta, \mathbf{W}]$, **MULT** $[c\eta, \boldsymbol{A}, \mathbf{V}]$ or the adaptive solution of an elliptic problem with accuracy $c\eta$.

Least squares: $\mathbf{M} = \mathbf{L}^T\mathbf{L}, \ \ \mathbf{G} = \mathbf{L}^T\mathbf{F}$

$$\textsf{\textbf{RHS}}_{ls}[\eta, \mathbf{G}] := \mathbf{MULT}\left[\frac{\eta}{2}, \mathbf{L}^T, \mathbf{COARSE}\left[\frac{\eta}{2C_L}, \mathbf{F}\right]\right]$$

$$\textsf{\textbf{APPLY}}_{ls}[\eta, \mathbf{M}, \mathbf{V}] := \mathbf{MULT}\left[\frac{\eta}{2}, \mathbf{L}^T, \mathbf{MULT}\left[\frac{\eta}{2C_L}, \mathbf{L}, \mathbf{V}\right]\right]$$

Application through Uzawa Iteration [DDU]

IDEAL UZAWA: *Given any* $\mathbf{p}_0 \in \ell_2(\mathcal{J}_M)$, *compute for* $i = 1, 2, \ldots$

$$\begin{aligned} \boldsymbol{A}\mathbf{u}_i &= \mathbf{f} - \mathbf{B}^T \mathbf{p}_{i-1}, \qquad (\text{set } \boldsymbol{R} := \langle \tilde{\Psi}_M, \tilde{\Psi}_M \rangle_M) \\ \mathbf{p}_i &= \mathbf{p}_{i-1} + \omega \boldsymbol{R}(\mathbf{B}\mathbf{u}_i - \mathbf{g}), \qquad \|\mathbf{I} - \omega \boldsymbol{R}\mathbf{S}\|_{\ell_2 \to \ell_2} \leq \alpha < 1 \end{aligned}$$

REAL UZAWA: *Apply each step approximately*

Ingredients:

- $\mathbf{MULT}\,[\eta, \mathbf{C}, \mathbf{v}] \to \mathbf{w}_\eta$, s.t. $\|\mathbf{C}\mathbf{v} - \mathbf{w}_\eta\|_{\ell_2} \leq \eta \quad (\mathbf{C} \in \{\boldsymbol{A}, \mathbf{B}, \mathbf{B}^T\})$
- $\mathbf{ELLSOLVE}\,[\eta, \boldsymbol{A}, \mathbf{r}] \to \bar{\mathbf{u}}(\eta)$, s.t. $\|\mathbf{u} - \bar{\mathbf{u}}(\eta)\|_{\ell_2(\mathcal{J}_X)} \leq \eta$
- $\mathbf{COARSE}\,[\eta, \mathbf{v}] \to \mathbf{v}_\eta$, s.t. $\|\mathbf{v} - \mathbf{v}_\eta\|_{\ell_2} \leq \eta$

$\mathbf{M}$ = Schur complement: $\rightsquigarrow$

$\mathbf{APPLY}_{Uz}[\eta, \mathbf{M}, \mathbf{q}]$ consists of finitely many (uniformly bounded) applications of approximate Uzawa iterations [DDU]

Complexity of $\mathbf{APPLY} \in \{\mathbf{MULT},\ \mathbf{APPLY}_{ls},\ \mathbf{APPLY}_{Uz}\}$?

Main Result - Convergence/Complexity [CDD2]

Theorem: *Assume that* $\mathbf{L} \in \mathcal{C}_{s^*}$. *Then if the exact solution* $U = \mathbf{U}^T \mathbf{D}^{-1} \Psi$ *satisfies for some* $s < s^*$

$$\inf_{\#\mathrm{supp}\mathbf{V} \leq N} \|U - \mathbf{V}^T \mathbf{D}^{-1} \Psi\|_{\mathcal{H}} \lesssim N^{-s}$$

then, for any $\epsilon > 0$, *the approximations* $\bar{\mathbf{U}}(\epsilon)$ *produced by* **SOLVE** *satisfy*

$$\|U - \bar{\mathbf{U}}(\epsilon)^T \mathbf{D}^{-1} \Psi\|_{\mathcal{H}} \lesssim \epsilon.$$

and

$$\#\mathrm{supp}\, \bar{\mathbf{U}}(\epsilon), \ \text{comp. work} \lesssim \epsilon^{-1/s}.$$

- **No LBB condition**

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