A well-balanced relaxation scheme for the Euler equations with gravity

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The aim of this work is to derive well-balanced numerical schemes for the Euler equations with gravity. Unlike the shallow water equations where steady states are defined by an algebraic relation, the system under consideration involves steady states defined by a differential equation that cannot be integrated. This makes the derivation of well-balanced schemes more difficult.

The main idea to derive a relevant well-balanced numerical scheme comes from a suitable discretization of the equation governing the steady states at rest. We adopt a relaxation model of Suliciu-type where the gravity source term is "transported" at the speed of the fluid. The main benefit of this Suliciu-type relaxation model is that it makes the system under-determined because there is a missing Riemann invariant. This failure allows us to naturally introduce the discrete form of the differential equation governing the steady states to close the relaxation model.

To justify this choice of a closure approach, we can write a new relaxation model with a full set of Riemann invariants which admits the "same" Riemann solution. We obtain a well-balanced, robust and entropy preserving relaxation scheme.

Moreover this technique can be generalised to other systems of conservation laws with source terms like the shallow water equations or the Ripa model.

Next, we present a second-order MUSCL extension of this scheme. It raises the problem of the definition of second-order discrete steady states. A suitable definition is introduces and the MUSCL scheme is proved to be robust and well-balanced according to the chosen definition.