Well-balanced schemes for a class of models in chemotaxis

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We consider the quasi-linear hyperbolic model of chemotaxis in 1D, as presented in [3]. The system can be written as follows:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \varepsilon \rho^\gamma) = \chi \rho \partial_x \phi \end{cases} \tag{1}$$

where ρ is the density of cells, u is the velocity of the flow of cells, χ, ε and γ are constant parameters. The function ϕ is the concentration of chemoattractant and it is governed by the following diffusive equation:

$$\partial_t \phi = D \partial_{xx} \phi + a\rho - b\phi$$

If we assume that $\gamma > 1$, the steady states at rest associated with (1) are given by:

$$\begin{cases} u = 0 \\ \rho^{\gamma-1} - \frac{\gamma-1}{\gamma} \frac{\chi}{\varepsilon} \phi = \text{cst} \end{cases}$$

Our objective is to derive an accurate numerical scheme to approximate the weak solutions of (1) which is able to precisely capture the steady states at rest. From now on, let us underline that the fundamental discrepancy with the well-known shallow-water model stays in the nonlinear algebraic relation satisfied by the steady states at rest. This makes very difficult extensions of usual well-balanced techniques [1, 2].

We suggest to introduce a relevant approximate Riemann solver to get a Godunov-type scheme. Here, the main novelty stays in the introduction of the *source term* coming from the chemotaxis potential within the approximate Riemann solver. Such an introduction of the source term is deduced from the consistency result by Harten, Lax and van Leer and thus we get a generalized source term linearization. Next, the source term linearization is completed by enforcing the preservation of the steady states at rest. Finally, we obtain an approximate Riemann solver which is able to exactly capture the steady states at rest. As a consequence the resulting Godunov-type scheme turns out to be well-balanced. In addition, the scheme is proved to be positive density preserving.

References

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