Design and analysis of finite volumes scheme in the diffusion limit on distorted meshes.

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In this work we are interested by the discretization of hyperbolic system with stiff source terms:

$$\partial_t \mathbf{U} + \frac{1}{\varepsilon}\partial_x F(\mathbf{U}) + \frac{1}{\varepsilon}\partial_y G(\mathbf{U}) = -\frac{\sigma}{\varepsilon^2}R(\mathbf{U}),\ \mathbf{U} \in \mathbb{R}^n$$

with $\varepsilon \in]0,1]$ and $\sigma > 0$.
When the small parameter ε is close to zero the previous hyperbolic system can be approximate by a diffusion equation

$$\partial_t \mathbf{V} - \operatorname{div}\left(K(\nabla \mathbf{V}, \sigma)\right) = 0, \quad \mathbf{V} \in \operatorname{Ker} R.$$

This type of problem for hyperbolic or kinetic equations is present in plasma physics, fluids dynamics, biology, radiative transfer, neutronic.

It is known since some years that the Godunov-type schemes are not adapted for this problem since the parameters of discretization are constraint by ε. Indeed we can prove that the consistency error is $O\left(\frac{\Delta x}{\varepsilon} + \Delta t\right)$ and CFL stability is $\Delta t(\frac{1}{\Delta x \varepsilon} + \frac{\sigma}{\varepsilon^2}) \leq 1$. For these reasons the notion of asymptotic preserving schemes have been introduced some years ago. Since then, many methods have been proposed in 1D for kinetic ([CCL12]-[Goss1]-[LM07] for example) and hyperbolic problems ([BCT08]-[BLeFT11]-[BT10][BCMSS12]-[GT01]-[HLMc10] for example).

Recently we have introduce in [BDF11]-[FHSN11] the extension on unstructured meshes of these type of method for the hyperbolic heat equation. These extension are necessary to treat some problems as Lagrangian Radiative hydrodynamic which couple the Lagrangian methods for Euler equations ans stiff hyperbolic systems for neutronic or radiative transport problems. For the 2D extension, there is an additional difficulty. Indeed if we design an AP scheme using the classical formulation of finite volumes methods in 2D we observe that the limit diffusion scheme and consequently the AP scheme are not convergent on deformed meshes. In [BDF11] we propose to couple the classical method for well-balanced and AP schemes with a other formulation of finite volume methods where the fluxes are localized at the node.

After a quickly presentation of the scheme and the method introduced in [BDF11] we propose to develop two new aspects. The first aspect is the uniform convergence of the scheme. Using the principle of proof proposed in [JL91], some stability and convergence estimates on the diffusion scheme, the hyperbolic scheme and the models, we propose to prove that the scheme is convergent with the estimate

$$||\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon||_{L^2} = O(h^{\frac{1}{3}} + h)$$

with $\mathbf{V}_h^\varepsilon$ the numerical solution of hyperbolic heat equation and with $\mathbf{V}^\varepsilon$ the exact solution of hyperbolic heat equation. This proof is given on unstructured meshes and we can show that the scheme is convergent on a important class of distorted meshes. To my knowledge this proof is

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the completely new on non uniform grids. Secondly we propose some extension of the previous methods for the non linear case. Using Lagrange+remap nodal scheme and the classical AP methods we obtain a AP scheme with maximum principle for the M_1 model (radiative transfer) [BDF13]-[BDF13] and an AP, well-balanced, positive scheme for Euler equations with gravity and friction. To finish we will propose some considerations on the entropy property and the shallow water equations.

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