An all Mach scheme for the Euler–Korteweg model

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In this talk we describe an all Mach scheme for the Euler–Korteweg model. In non-dimensionalized form the Euler–Korteweg model reads

$$
\begin{aligned}
\rho_t + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \frac{1}{M^2} \nabla p(\rho) &= \frac{\gamma}{M^2} \rho \nabla \Delta \rho
\end{aligned}
\tag{1}
$$

where ρ is the density, $\mathbf{v}$ the velocity, $\gamma > 0$ is a capillarity coefficient, $M > 0$ is the Mach number and $p = p(\rho)$ is the pressure, given by a non–monotone constitutive relation such that the first order part of (1) is of hyperbolic-elliptic type. The model can be used to describe non-viscous, compressible multi-phase flows. Complemented with proper initial and boundary data solutions of (1) conserve the energy

$$
\int \frac{1}{M^2} W(\rho) + \frac{1}{2} \rho |\mathbf{v}|^2 + \frac{\gamma}{2M^2} |\nabla \rho|^2 dx \tag{2}
$$

where W and p are related via $p(\rho) = \rho W'(\rho) - W(\rho)$. The construction of all Mach schemes is particularly interesting for compressible multi-phase flows as the local Mach numbers in both phases differ strongly.

Our algorgorithm is related to the Algorithm presented in [1] for the Euler equation. The convection term in $(1)_1$ and the pressure and capillarity terms in $(1)_2$ are discretised implicitly. In each time step we need to solve an (implicit) time step of a Cahn-Hilliard equation with non-constant mobility to determine the new density distribution and an explicit equation to determine the new velocity distribution.

We obtain a fully discrete finite difference scheme which conserves mass and satisfies a discrete energy dissipation inequality, i.e. a discrete version of (2) can only grow by $\mathcal{O}((\Delta t)^2)$ per time step, for a time step restricted independently of M. In addition, we show that in the low Mach limit our scheme converges to a stable discretisation of the low Mach limit of (1), i.e. it has the 'asymptotic preserving' property. We complement our theoretical findings by numerical experiments in 1 and 2 space dimensions.

References

1. Pierre Degond and Min Tang. All speed scheme for the low Mach number limit of the isentropic Euler equations. *Commun. Comput. Phys.*, 10(1):1–31, 2011.